A category-theoretic language for metric Fraïssé theory

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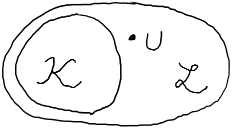
Abstract Convergence Schemes And Their Complexities

Classical Fraïssé theory

- Let L be a first-order language.
- Let $\mathcal{K}$ be a class of finitely generated L-structures.
- Let $\mathcal{L} = \sigma\mathcal{K}$ be the class of all L-structures that are unions of countable increasing chains of structures from $\mathcal{K}$.
- If $\mathcal{K}$ is sufficiently nice, then there is a special structure U in $\mathcal{L}$ that is universal/cofinal and (ultra)homogeneous over $\mathcal{K}$.
- Such structure U is unique, and it is cofinal in the whole $\mathcal{L}$.



- Alternatively, we may start with U, put $\mathcal{K} = \mathrm{Age}(U)$, and ask whether U is homogeneous over its age.
- Classical Fraïssé theorem asserts that there is a one-to-one correspondence between hereditary Fraïssé classes $\mathcal{K}$ and (ultra)homogeneous countably generated structures U.

Projective Fraïssé theory

- In classical (injective) Fraïssé theory, extensions of structures are embeddings and we consider direct limits of sequences:

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_\infty$$

- In projective Fraïssé theory, extensions are opposite quotients and consider inverse limits of sequences:

$$X_0 \twoheadleftarrow X_1 \twoheadleftarrow X_2 \twoheadleftarrow \cdots \twoheadleftarrow X_\infty$$

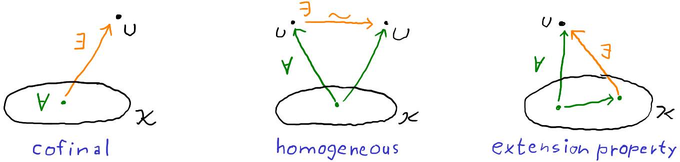
- Irwin and Solecki (2006) introduced projective Fraïssé theory for topological structures, i.e. L-structures endowed with a compact zero-dimensional metrizable topology such that the L-operations are continuous and L-relations are closed.
- As morphisms they consider (opposite) quotient maps (quotient both the sense of L-structures and topological spaces).

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
embeddings	finite graphs	countable graphs	Rado/random graph
embeddings	finite groups	locally finite countable groups	Hall's universal group
embeddings	finite rational metric spaces	countable rational metric spaces	rational Urysohn space
quotients	finite discrete spaces	zero-dimensional metrizable compacta	Cantor space
quotients	finite discrete linear graphs	zero-dimensional metrizable compacta with a special closed symmetric relation	pseudo-arc prespace [Irwin–Solecki]

Abstract discrete Fraïssé theory

- It is possible to naturally express all the relevant notions in the language of category theory.
- We fix a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ and say that an $\mathcal{L}$-object U
  - is cofinal if for every $\mathcal{K}$-object X there is an $\mathcal{L}$-map $X \to U$,
  - is homogeneous if for every $\mathcal{K}$-object X and $\mathcal{L}$-maps $f, g \colon X \to U$ there is an automorphism $h \colon U \to U$ such that $h \circ g = f$,
  - has the extension property if for every $\mathcal{K}$-object X, $\mathcal{L}$-map $f \colon X \to U$, and $\mathcal{K}$-map $g \colon X \to Y$ there is an $\mathcal{L}$-map $h \colon Y \to U$ such that $h \circ g = f$.



- Unions of countable chains are replaced by category-theoretical (co)limits of sequences.
- The injective and the projective case can be treated uniformly via dualization in category theory.

Free completion

For the general correspondence to hold, we need some assumptions on the pair $\mathcal{K} \subseteq \mathcal{L}$.

Definition

A pair of categories $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion if

(L1) Every $\mathcal{K}$-sequence has a limit in $\mathcal{L}$,

(L2) Every $\mathcal{L}$-object is a limit of a $\mathcal{K}$-sequence,

(F1) For every $\mathcal{K}$-sequence $\langle X_*, f_* \rangle$ with $\mathcal{L}$-limit $\langle X_\infty, f_*^\infty \rangle$ and every $\mathcal{L}$-map from a $\mathcal{K}$-object $h\colon Y \to X_\infty$ there is a $\mathcal{K}$-map $g\colon Y \to X_n$ such that $f_n^\infty \circ g = h$.

(F2) For every $\mathcal{K}$-sequence $\langle X_*, f_* \rangle$ with $\mathcal{L}$-limit $\langle X_\infty, f_*^\infty \rangle$ and every $\mathcal{K}$-maps $g, h\colon Y \to X_n$ such that $f_n^\infty \circ g = f_n^\infty \circ h$ there is $m \geq n$ such that $f_n^m \circ g = f_n^m \circ h$.

- Informally, $\mathcal{L}$ arises by freely adding limits of sequences to $\mathcal{K}$.
- For every $\mathcal{K}$ there is essentially one free completion.
- In the classical and projective model-theoretic situations the conditions are automatically true.

Fraïssé limit

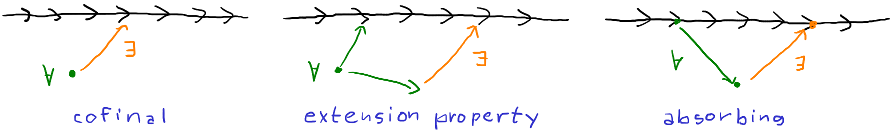
Theorem

Let $\langle\mathcal{K},\mathcal{L}\rangle$ be a free completion and let U be an $\mathcal{L}$-object. The following conditions are equivalent.

1. U is homogeneous and cofinal in $\langle\mathcal{K},\mathcal{L}\rangle$.
2. U has the extension property and is cofinal in $\langle\mathcal{K},\mathcal{L}\rangle$.
3. U is an L-limit of a *Fraïssé sequence* in $\mathcal{K}$.

Such object U is unique up to isomorphism and is cofinal in $\mathcal{L}$. It is called the Fraïssé limit in $\langle\mathcal{K},\mathcal{L}\rangle$.

Fraïssé sequence: a cofinal sequence in $\mathcal{K}$ with the extension property (typically equivalent to absorption)



Fraïssé category

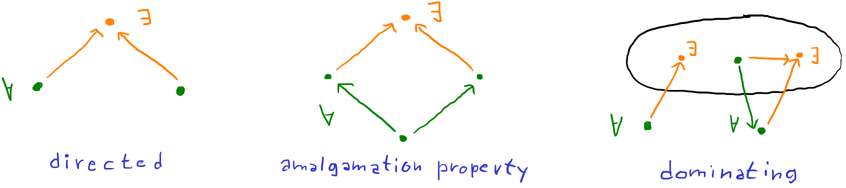
The notion of Fraïssé sequence depends only on $\mathcal{K}$.

Theorem

A category $\mathcal{K} \neq \emptyset$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is *directed*,
2. $\mathcal{K}$ has the *amalgamation property*,
3. $\mathcal{K}$ is *dominated* by a countable subcategory.

Such $\mathcal{K}$ is called a Fraïssé category.



Approximate Fraïssé theory

- Irwin and Solecki showed that $\mathcal{I}^\Delta$, the category of finite linear graphs and quotient maps, is a Fraïssé category.
- The limit in $\langle \mathcal{I}^\Delta, \sigma\mathcal{I}^\Delta \rangle$ is $\langle 2^\omega, R \rangle$, where R is a closed equivalence relation such that the topological quotient $2^\omega / R$ is the pseudo-arc, a well-known object in continuum theory.
- Irwin and Solecki characterized the pseudo-arc as a unique arc-like continuum $\mathbb{P}$ such that for every continuous surjections $f, g \colon \mathbb{P} \to Y$ and $\varepsilon > 0$ there is a homeomorphism $h \colon \mathbb{P} \to \mathbb{P}$ such that $\sup_{x \in \mathbb{P}} d(f(x), g(h(x))) < \varepsilon$ "$f \approx_\varepsilon g \circ h$".
- With a suitable generalization of our framework, we are able to view the pseudo-arc directly as Fraïssé limit in $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$, where $\mathcal{I}$ is the category of all continuous surjections on the unit interval and $\sigma\mathcal{I}$ is the category of all arc-like continua.
- The characterization above becomes a general homogeneity notion of our framework.

Approximate Fraïssé theory

We use projective view, so "bigger objects" correspond to domains.

Definition

Recall that a metric-enriched category is a category $\mathcal{K}$ endowed with distance maps $d\colon \mathcal{K}(X, Y) \to [0, \infty]$ such that

1. $d(g \circ f, h \circ f) \leq d(g, h)$ for every $f\colon X \to Y$, $g, h\colon Y \to Z$,
2. $d(f \circ g, f \circ h) \leq d(g, h)$ for every $f\colon X \to Y$, $g, h\colon Z \to X$.

$\mathcal{K}$ is an MU-category if 2 is replaced by

2'. For every $f\colon X \to Y$ and $\varepsilon > 0$ there is $\delta > 0$ such that for every $g, h\colon Z \to X$ we have $g \approx_\delta h \implies f \circ g \approx_\varepsilon f \circ h$.

- Every category can be viewed as a discrete MU-category.
- $\mathbf{Met_u}$, the category of metric spaces and uniformly continuous maps with supremum distance, is an MU-category.
- **MCpt**, the category of metrizable compacta, can be viewed as an MU-category in essentially unique way.

Approximate Fraïssé theory

- For all Fraïssé-theoretic notions there is an appropriate definition for MU-categories, for example,
  - an $\mathcal{L}$-object U is homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object X, $\mathcal{L}$-maps $f, g \colon X \leftarrow U$ and $\varepsilon > 0$ there is an $\mathcal{L}$-automorphism $h \colon U \to U$ such that $f \approx_\varepsilon g \circ h$.
- If $\mathcal{K}$ is a discrete MU-category, then the homogeneity condition is equivalent to the discrete one, and similarly for other conditions.
- The definition of a free completion of MU-categories includes, besides the approximate versions of (L1), (L1), (F1), (F2), also a condition (C), which results in even nicer theory.
- If a pair $\langle \mathcal{K}, \mathcal{L} \rangle$ is a discrete free completion, then it is typically not a free MU-completion, but $\mathcal{L}$ can be endowed with an MU-structure (instead of taking the discrete one) such that it becomes the free MU-completion.

Approximate Fraïssé theory

- In other words, the free MU-completion of a discrete category is not discrete.
- The free completion MU-structure is appropriate even in the discrete setting.

Theorem

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion of MU-categories and let U be an $\mathcal{L}$-object. The following conditions are equivalent.

1. U is homogeneous and cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$.
2. U has the extension property and is cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$.
3. U is an L-limit of a *Fraïssé sequence* in $\mathcal{K}$.

1'. U is homogeneous and cofinal in $\mathcal{L}$.

2'. U has the extension property and is cofinal in $\mathcal{L}$.

Such object U is unique up to isomorphism. It is called the Fraïssé limit in $\langle \mathcal{K}, \mathcal{L} \rangle$.

Applications

- We consider MU-categories metrizable compacta and continuous surjections.
- For every such $\mathcal{K}$ let $\sigma\mathcal{K}$ denote the MU-category of all inverse limits of $\mathcal{K}$-sequences as objects and all continuous surjections “sufficiently approximated” by $\mathcal{K}$ as morphisms.
- For every class $\mathcal{P}$ of connected polyhedra, with all continuous surjections as morphisms, $\langle \mathcal{P}, \sigma\mathcal{P} \rangle$ is a free MU-completion, and $\sigma\mathcal{P} \subseteq \mathbf{MCpt_s}$ is full.
- In particular, for $\mathcal{I} = \{[0,1]\}$, $\sigma\mathcal{I}$ consists of all arc-like continua and the pseudo-arc is the Fraïssé limit in $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$.
- For $\mathcal{S} = \{\text{circle}\}$, $\sigma\mathcal{S}$ consists of all circle-like continua, but there is no Fraïssé limit since $\mathcal{S}$ does not have the amalgamation property.
- For every set of primes P, $\mathcal{S}_P \subseteq \mathcal{S}$ consisting of maps whose degree uses only primes from P is a Fraïssé category, and the Fraïssé limit of $\langle \mathcal{S}_P, \sigma\mathcal{S}_P \rangle$ is the P-adic pseudo-solenoid.

Thank you.