Abstract evolution systems*

WIESŁAW KUBIŚ
PAULINA RADECKA
Institute of Mathematics, Czech Academy of Sciences (CZECH REPUBLIC)

We introduce the concept of an abstract evolution system, which provides a convenient framework for studying generic mathematical structures and their properties. Roughly speaking, an *evolution system* is a category endowed with a selected class of morphisms called *transitions*, satisfying certain natural conditions. It can also be viewed as a generalization of abstract rewriting systems, where the partially ordered set is replaced by a category. In our setting, the *process* of rewriting plays a nontrivial role, whereas in rewriting systems only the result of a reduction/rewriting is relevant. An analogue of Newman's Lemma holds in our setting, although the proof is a bit more delicate, nevertheless, still based on Huet's idea using well founded induction.

Formally, an *evolution system* is a structure of the form $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object (called the *origin*) and $\mathscr{T}$ is a class of $\mathfrak{V}$-arrows (its elements are called *transitions*). An *evolutions* is a sequence of the form

$$\Theta \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition. The category $\mathfrak{V}$ serves as the universe of discourse. Given a $\mathfrak{V}$-object X, we denote $\mathscr{T}(X) = \{f \in \mathscr{T} \colon \operatorname{dom}(f) = X\}$, that is, the set of all transitions with domain X. Two transitions $f, g \in \mathscr{T}(X)$ are *isomorphic* if there is an isomorphism h in $\mathfrak{V}$ such that $g = h \circ f$. The system is *regular* if transitions commute with isomorphisms, that is, $f \circ h$ is a transition whenever f is a transition and h is an isomorphism. An object X will be called *finite* if there exist transitions $f_0, \dots, f_{n-1}$ such that $f_i \colon X_i \to X_{i+1}$ for $i < n$, $X_0 = \Theta$ and $X_n = X$. We say $\mathscr{E}$ has the *finite amalgamation property* if for every finite object C, for every transitions $f \colon C \to A$, $g \colon C \to B$ there are paths $f' \colon A \to D$, $g' \colon B \to D$ with $f' \circ f = g' \circ g$. An evolution system $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$ is *essentially countable* if for every finite object X there is a countable set of transitions $\mathscr{F}(X) \subseteq \mathscr{T}(X)$ such that every transition in $\mathscr{T}(X)$ is isomorphic to a transition in $\mathscr{F}(X)$.

Below are two natural motivating examples of evolution systems.

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Example 1. Let $\mathscr{F}$ be a class of finite structures in a fixed first-order language consisting of relations only. It is convenient to assume $\mathscr{F}$ is closed under isomorphisms. Let $\sigma\mathscr{F}$ denote the class of all structures of the form $\bigcup_{n\in\omega} X_n$, where $\{X_n\}_{n\in\omega}$ is a chain in $\mathscr{F}$. Let $\mathfrak{V}$ be the category of all embeddings between structures in $\sigma\mathscr{F}$. Let $\mathscr{T}$ consist of all embeddings of the form $f\colon X \to Y$, where $Y \setminus f[X]$ is a singleton or the empty set. In other words, transitions are one-point extensions and isomorphisms. Finally, Θ might be the empty structure. Clearly, $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$ is an evolution system.

Example 2. Let $\mathscr{F}$ be a fixed class of finite nonempty relational structures and consider it as a category where the arrows are epimorphisms. Define transitions to be epimorphisms $f\colon X \to Y$ such that either f is an isomorphism (a bijection) or else there is a unique $y \in Y$ with a nontrivial f-fiber and moreover $f^{-1}(y)$ consists of precisely two points. Define $\mathfrak{V}$ to be the opposite category, so that $f \in \mathfrak{V}$ is an arrow from Y to X if it is an epimorphism from X onto Y. Then $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$ is an evolution system, where Θ is a prescribed finite structure in $\mathscr{F}$.

We say that an evolution $\vec{u}$ has the *absorption property* if for every $n \in \omega$, for every transition $t\colon U_n \to Y$ there are $m > n$ and a path $g\colon Y \to U_m$ such that $g \circ t = u_n^m$.

Theorem 3. *Assume $\mathscr{E}$ is an essentially countable evolution system that has the finite amalgamation property. Then there exists a unique, up to isomorphism, evolution with the absorption property.*

A system is *terminating* if every evolution is eventually trivial, namely, from some point on all transitions are isomorphisms. The following result is an extension of Newman's Lemma [3]; the proof is based on the idea of Huet [1], using well founded induction.

Theorem 4. *A locally confluent regular terminating evolution system is confluent.*

Confluent terminating systems provide a good framework for studying finite homogeneous structures.

References

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[3] M.H.A. Newman, *On theories with a combinatorial definition of "equivalence."*, Ann. of Math. (2) 43 (1942) 223–243.