Arbitrary Abstraction and Logicality

Ludovica Conti

In this talk, I will discuss a criterion (*general weak invariance*) that has been recently suggested in order to argue for the logicality of abstraction operators, when they are understood as arbitrary expressions (cf. Boccuni Woods 2020).

Abstractionist theories are systems composed by a logical theory augmented with one or more abstraction principles (AP), of form: $f_R\alpha = f_R\beta \leftrightarrow R(\alpha, \beta)$ – that introduce, namely rule and implicitly define, the corresponding term-forming operators f_R. Thus, the logicality of these theories plainly depends on the logicality of the abstraction principles. This issue was originally raised into the seminal abstractionist program, Frege's Logicism – proposed with the foundational purpose to derive arithmetical laws as logical theorems and to define arithmetical expressions by logical terms. The inconsistency of this project (i.e. a theory equivalent to second-order logic augmented with Basic Law V) seemed to determine the inconsistency and, then (in a classical logic) the non-logicality of Basic Law V and – *a fortiori* – of any other abstraction principle[1].

Recently, the issue of the logicality has been resumed regarding the consistent abstraction principles, in order to clarify that conclusion in light of the intervening studies about logicality and represents, still today, an open question of the abstractionist debate. Briefly, a standard account of logicality has been provided, in semantical terms, by means of the Tarskian notions of invariance under permutation and isomorphism (cfr. [7]). In order to apply these criteria to abstraction principles, we can specify at least three different subjects to be examined: the whole abstraction principle [2], the abstraction

[1]We will describe a relation between the abstraction principles based on the *finesse* of their equivalence relations. Cfr. [1].

[2]Regarding the abstraction principle, the more informative criterion consists of *contextual invariance*: an abstraction principle AP is *contextually invariant* if and only if , for any abstraction function f_R: $D_2 \to D_1$ and permutation π, $\pi(f_R)$ satisfies AP whenever f_R does (cfr. [1]).

relation[3] and the abstraction operator[4]. Different results has been already proved (cf. [7], [6], [1], [4], [2], [5]) but a new dilemma appeared. More precisely, given a semantical definition of logicality as permutation and/or isomorphism invariance, we are able to prove that some abstraction principles (like Hume's Principle) are logical ([4])[6] but their implicit *definienda* are not ([1])[7] – so preventing a full achievement of Logicist goal.

My preliminary aim will consists in showing that this unfortunate situation closely depends on the (unjustified) adoption of a same notion of reference for all the expressions of a same syntactical category (e.g. singular terms as always referential and denoting singular, knowable and standard objects). On the contrary, a less demanding reading of the abstractionist vocabulary – namely, a reading that renounces to the semantical assumption mentioned above – is available; furthermore, such a reading, by admitting a different evaluation of primitive an defined expressions, is able to focus on the only information actually provided by the APs and turns out to be preferable because it is more faithful to the theory. Thus, chosen this reading of the APs and, particularly, an arbitrary interpretation (cf. [3]) of the abstractionist vocabulary, my main aim will consist in inquiring its consequences on the logicality of abstractionist theories.

Particularly, given such an interpretation of the APs, we can rephrase the main criterion of logicality for abstraction operators (*objectual invariance*, cf. [1]), obtaining a weaker one (*general objectual invariance*[8], GWI, cf. [8], [2]) and proving that it is satisfied not not only by cardinal operator but also by many other second-order ones, including those implicitly defined by consistent weakenings of Fregean Basic Law V. So, we will note that, given (what I

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[3]Regarding the abstraction relation, we can distinguish, at least, four kind of invariance: *weak invariance*, *double invariance*, *internal invariance* and *double weak invariance* (cfr. [1], [4], [6].).

[4]Regarding the canonical reading of the abstraction operator, logicality is usually spelled out in terms of *objectual invariance*[5] (cf. [1]).

[6]More precisely, some abstraction principles (like Hume's Principle) satisfy the criterion of *contextual invariance* and their abstraction relations (e.g. equinumerosity) satisfy many logicality criteria, like *weak invariance*, *internal invariance*, *double internal invariance*. Cf. [1], [6], [4].

[7]More precisely, the corresponding abstraction operators (e.g. cardinal operators) do not satisfy the criterion of *objectual invariance*. Furthermore, such criterion fails precisely in case of operators related to internal (and,*a fortiori* double internal) invariant relation (cfr. [1]). So, operators fail to be *logical* though – just in case – they are implicitly defined by *logical* AP.

[8]An expression ϕ is *generally weak invariant* just in case, for all domains D, D' and bijections ι from D to D', the set of candidate denotations of ϕ on D (ϕ^{*D}) $= \{\gamma : \gamma$ is a candidate denotation for ϕ on $D\}$ is such that $\iota(\phi^{*D}) = \phi^{*D'} = \{\gamma : \gamma$ is a candidate denotation for ϕ on $D'\}$.

argued as) a preferable reading of the APs, both main strategies pursued in the last century to save Fregean project – Neologicism and consistent revisions of *Grundgesetze* – are able to achieve the desirable logicality objective. Further generalising, I will prove that the logicality criterion could be satisfied by a large range of APs and is apparently liable to a triviality objection – e.g. it is not able to distinguish between HP and some of its Bad Companions (like Nuisance Principle). I will answer to such a potential objection by showing that GWI however introduces interesting differences. More precisely, I will discuss the controversial case of Ordinal Abstraction and I will prove that GWI is not satisfied by any first-order abstraction principles (cf. [7], [8]). So, by comparing respective schemas of first-order and second-order APs[9], we will note that logicality (in the chosen meaning) mirrors a relevant distinction between same-order and different-order abstraction principles.

References

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[9] More precisely, a schematic second-order abstraction principle – of form $\S(RF) = \S(RG) \leftrightarrow R(F,G)$, where § is a binary abstraction operator and E any isomorphism invariant equivalence relation – defines an abstraction function from $\wp(\wp(D) \times \wp(D)) \times \wp(D) \to D$ that satisfies the criterion of GWI and – differently from the specific unary operators – is total ([8]). On the other side, a schematic first-order abstraction principle – of form $\S(Ra) = \S(Rb) \leftrightarrow R(a,b)$, where § is a binary abstraction operator and E any first-order equivalence relation – defines an abstraction function from $\wp(D \times D) \times D \to D$ that is – differently from the corresponding unary operators – total, but however does not satisfy GWI.

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