Cut Languages in Rational Bases

Jiří Šíma, Petr Savický

Institute of Computer Science
The Czech Academy of Sciences

Non-Standard Positional Numeral Systems

- real base (radix) β such that $|\beta| > 1$
- finite set $A \neq \emptyset$ of real digits

word $a = a_1 \ldots a_n \in A^*$ over alphabet A is a finite β-expansion (base-β representation) of real number x if

$$x = (a)_\beta = (a_1 \ldots a_n)_\beta = \sum_{k=1}^{n} a_k \beta^{-k}$$

generalization of the usual representations of numbers in an integer base β:

- decimal expansions: $\beta = 10$ and $A = \{0, 1, 2, \ldots, 9\}$
  e.g. $\frac{3}{4} = (75)_{10} = 7 \cdot 10^{-1} + 5 \cdot 10^{-2}$
- binary expansions: $\beta = 2$ and $A = \{0, 1\}$
  e.g. $\frac{3}{4} = (11)_2 = 1 \cdot 2^{-1} + 1 \cdot 2^{-2}$

Cut Languages

cut language $L_{<c} \subseteq A^*$ over alphabet A contains all the finite β-expansions of numbers that are less than a given real threshold c

$$L_{<c} = \{a \in A^* \mid (a)_\beta < c\} = \left\{ a_1 \ldots a_n \in A^* \;\middle|\; \sum_{k=1}^{n} a_k \beta^{-k} < c \right\}$$

- $L_{<c}$ contains finite base-β representations of a Dedekind cut
- similarly for $L_{>c}$
- $L_{<c}$ can be defined over any alphabet Γ by using a bijection $\sigma : \Gamma \longrightarrow A$ so that each symbol $u \in \Sigma$ represents a distinct digit $\sigma(u) \in A$.

Motivation:

refining the analysis of the computational power of neural network models (NNs) between integer and rational weights

The Computational Power of Neural Networks

depends on the information contents of weight parameters:

1. integer weights: finite automaton (Minsky, 1967)

2. rational weights: Turing machine (Siegelmann, Sontag, 1995)
   polynomial time $\equiv$ complexity class P

3. arbitrary real weights: "super-Turing" computation (Siegelmann, Sontag, 1994)
   polynomial time $\equiv$ nonuniform complexity class P/poly
   exponential time $\equiv$ any I/O mapping

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   polynomial time & increasing Kolmogorov complexity of real weights $\equiv$
   a proper hierarchy of nonuniform complexity classes between P and P/poly
   (Balcázar, Gavaldà, Siegelmann, 1997)

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   a gap between integer a rational weights w.r.t. the Chomsky hierarchy

   regular (Type-3) $\times$ recursively enumerable (Type-0) languages

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   polynomial time $\equiv$ complexity class P

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Integer-Weight NNs with an Extra Analog Neuron (NN1A)

TWO analog neurons with rational weights (plus a few integer-weight neurons) can implement a 2-stack pushdown automaton $\equiv$ Turing machine

$\longrightarrow$ **What is the computational power of ONE extra analog neuron ?**

Representation Theorem (Šíma, IJCNN 2017): a language $L \subset \Sigma^*$ over alphabet Σ, that is accepted by a NN1A, can be written in the form such as

$$L = h\left(\left(\left(\bigcup_{r=1}^{p-1}\left(\overline{L_{<c_r}} \cap L_{<c_{r+1}}\right)\cdot\Gamma_r\right)^{Pref} \cap R_0\right)^* \cap R\right)$$

where

- $L_{<c_r}$ are **cut languages** for **rational** β, A, $c_1 \leq c_2 \leq \cdots \leq c_p$
- $\Gamma_1, \ldots, \Gamma_p$ is a partition of alphabet Γ
- S^{Pref} denotes the largest prefix-closed subset of S
- $R\,, R_0 \subseteq \Gamma^*$ are regular languages
- $h : \Gamma^* \longrightarrow \Sigma^*$ is a letter-to-letter morphism

Infinite β-Expansions (Rényi, 1957; Parry, 1960)

word $a = a_1 a_2 a_3 \cdots \in A^\omega$ is an infinite β-expansion of real number x if

$$x = (a)_\beta = (a_1 a_2 a_3 \cdots)_\beta = \sum_{k=1}^{\infty} a_k \beta^{-k}$$

usual simplistic assumptions (most results can be generalized to arbitrary β and A):

1. $\beta > 1$
2. $A = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

Existence: for every $x \in \overline{D_\beta}$ where $\overline{D_\beta}$ is the closure of an open real interval

$$D_\beta = \left(0\,,\ \frac{\lceil \beta \rceil - 1}{\beta - 1}\right)\,,$$

there exists a β-expansion $a \in A^\omega$ of $x = (a)_\beta$

Eventually Periodic β-Expansions

β-expansion $a \in A^\omega$ is eventually periodic if

$$a = a_1 a_2 \dots a_{k_1} (a_{k_1+1} a_{k_1+2} \dots a_{k_2})^\omega$$

- k_1 is the length of preperiodic part $a_1 a_2 \dots a_{k_1} \in A^{k_1}$
- if $k_1 = 0$, then a is a periodic β-expansion
- $m = k_2 - k_1 > 0$ is the length of repetend $a_{k_1+1} a_{k_1+2} \dots a_{k_2} \in A^m$
- the minimum of m is called the period of a

any eventually periodic β-expansion can be evaluated as

$$(a_1 a_2 \dots a_{k_1} (a_{k_1+1} a_{k_1+2} \dots a_{k_2})^\omega)_\beta = (a_1 \dots a_{k_1})_\beta + \beta^{-k_1} \varrho$$

where $\varrho \in \mathbb{R}$ is a periodic point satisfying

$$(a_{k_1+1} a_{k_1+2} \dots a_{k_2})_\beta = \sum_{k=1}^{m} a_{k_1+k} \beta^{-k} = \varrho \left(1 - \beta^{-m}\right)$$

Uniqueness of β-Expansions for Integer β

for integer base $\beta \in \mathbb{Z}$,

we have $\overline{D_\beta} = [0,1]$ and $A = \{0, 1, \ldots, \beta - 1\}$, and it is well known that

- the endpoints 0 and 1 have trivial unique periodic β-expansions 0^ω and $(\beta - 1)^\omega$, e.g. $1 = (999\ldots)_{10} = (111\ldots)_2$
- irrational $x \in D_\beta \cap (\mathbb{R} \setminus \mathbb{Q})$ has a unique non-periodic infinite β-expansion
- rational $x = (a_1 a_2 \ldots a_n)_\beta \in D_\beta \cap \mathbb{Q}$ with finite β-expansion $a_1 a_2 \ldots a_n$ has exactly two distinct eventually periodic β-expansions $a_1 a_2 \ldots a_n 0^\omega$ and $a_1 a_2 \ldots a_{n-1}(a_n - 1)(\beta - 1)^\omega$, e.g. $\frac{3}{4} = (75)_{10} = (75000\ldots)_{10} = (74999\ldots)_{10}$
- rational $x \in D_\beta \cap \mathbb{Q}$ with no finite β-expansion has a unique eventually periodic β-expansion

Uniqueness of β-Expansions for Non-Integer β

for non-integer base $\boldsymbol{\beta}$, almost every $\boldsymbol{x} \in \overline{\boldsymbol{D_\beta}}$ has a continuum of distinct $\boldsymbol{\beta}$-expansions (Sidorov, 2003)

particularly for $\mathbf{1} < \boldsymbol{\beta} < \mathbf{2}$, we have $\boldsymbol{A} = \{0, 1\}$, $\boldsymbol{D_\beta} = (0\,, 1/(\boldsymbol{\beta}-1))$, and

- $\mathbf{1} < \boldsymbol{\beta} < \boldsymbol{\varphi}$ where $\boldsymbol{\varphi} = (1+\sqrt{5})/2 \approx \mathbf{1.618034}$ is the golden ratio: every $\boldsymbol{x} \in \boldsymbol{D_\beta}$ has a continuum of distinct $\boldsymbol{\beta}$-expansions (Erdös et al., 1990)
- $\boldsymbol{\varphi} \leq \boldsymbol{\beta} < \boldsymbol{q}$ where $\boldsymbol{q} \approx \mathbf{1.787232}$ is the Komornik-Loreti constant (i.e. the unique solution of equation $\sum_{k=1}^{\infty} t_k q^{-k} = 1$ where $(t_k)_{k=1}^{\infty}$ is the Thue-Morse sequence in which $t_k \in \{0, 1\}$ is the parity of the number of 1's in the binary representation of $\boldsymbol{k}$):

  countably many $\boldsymbol{x} \in \boldsymbol{D_\beta}$ have unique (eventually periodic) $\boldsymbol{\beta}$-expansions (Glendinning, Sidorov, 2001),

  examples: $\boldsymbol{x} = (0^n(10)^\omega)_\beta$ or $\boldsymbol{x} = (1^n(01)^\omega)_\beta \quad (n \geq 0)$

  vs. $\boldsymbol{\varphi}$-expansions of $\boldsymbol{x} = 1$: $(10)^n 110^\omega$, $(10)^\omega$, $(10)^n 01^\omega \quad (n \geq 0)$
- $\boldsymbol{q} \leq \boldsymbol{\beta} < \mathbf{2}$: a continuum (Cantor-like set) of $\boldsymbol{x} \in \boldsymbol{D_\beta}$ with unique $\boldsymbol{\beta}$-expansions
- $\boldsymbol{q_2} \leq \boldsymbol{\beta} < \mathbf{2}$ where $\boldsymbol{q_2} \approx \mathbf{1.839287}$ is the real root of $q_2^3 - q_2^2 - q_2 - 1 = 0$: there is $\boldsymbol{x} \in \boldsymbol{D_\beta}$ with exactly two $\boldsymbol{\beta}$-expansions etc. (Sidorov, 2009)

Uniqueness of β-Expansions for Arbitrary A

alphabet A can contain non-integer digits

Baker, 2015: there exist two critical bases φ_A and q_A, $1 < \varphi_A \leq q_A$, such that

$$\text{the number of unique } \beta\text{-expansions is } \begin{cases} \text{finite} & \text{if } 1 < \beta < \varphi_A \\ \text{countable} & \text{if } \varphi_A < \beta < q_A \\ \text{uncountable} & \text{if } \beta > q_A \end{cases}$$

$\times$ the determination of φ_A and q_A for arbitrary A is still not complete even for three digits (Komornik, Pedicini, 2016)

Eventually Periodic Greedy β-Expansions

the lexicographically maximal (resp. minimal) β-expansion of x is called greedy (resp. lazy), e.g. a unique β-expansion is simultaneously greedy and lazy

Per(β) is the set of numbers with eventually periodic greedy β-expansions:

- for integer $\beta \in \mathbb{Z}$ it is well known that Per$(\beta) = \mathbb{Q} \cap [0, 1)$
- for non-integer β, we have Per$(\beta) \subseteq \mathbb{Q}(\beta) \cap D_\beta^0$ where $\mathbb{Q}(\beta)$ is the smallest field extension of $\mathbb{Q}$ including β, and $D_\beta^0 = D_\beta \cup \{0\}$
- if $\mathbb{Q} \cap [0, 1) \subset$ Per(β), then β must be a Pisot or Salem number (Schmidt,1980)

  where a Pisot (resp. Salem) number is a real algebraic integer (a root of some monic polynomial with integer coefficients) greater than 1 such that all its Galois conjugates (other roots of such a unique monic polynomial with minimal degree) are in absolute value less than 1 (resp. less or equal to 1 and at least one equals 1)
- for Pisot β, we have Per$(\beta) = \mathbb{Q}(\beta) \cap D_\beta^0$ (open for Salem β) (Schmidt,1980)
- for rational non-integer $\beta \in \mathbb{Q} \backslash \mathbb{Z}$ (i.e. β is not Pisot nor Salem by the integral root theorem), there exists rational $x \in D_\beta \cap \mathbb{Q}$ such that $x \notin$ Per(β)

Eventually Quasi-Periodic β-Expansions

β-expansion $a = a_1 a_2 a_3 \ldots \in A^\omega$ is eventually quasi-periodic if there is an infinite sequence of indices, $0 \le k_1 < k_2 < \cdots$, such that for every $i \ge 1$,

$$(a_{k_i+1} \ldots a_{k_{i+1}})_\beta = \sum_{k=1}^{m_i} a_{k_i+k} \beta^{-k} = \varrho \left(1 - \beta^{-m_i}\right)$$

- $m_i = k_{i+1} - k_i > 0$ is the length of quasi-repetend $a_{k_i+1} \ldots a_{k_{i+1}} \in A^{m_i}$
- $\varrho \in \mathbb{R}$ is a periodic point
- k_1 is the length of preperiodic part $a_1 a_2 \ldots a_{k_1} \in A^{k_1}$
- if $k_1 = 0$, then a is a quasi-periodic β-expansion

any eventually quasi-periodic β-expansion can be evaluated as

$$(a_1 a_2 a_3 \ldots)_\beta = \sum_{k=1}^{\infty} a_k \beta^{-k} = (a_1 \ldots a_{k_1})_\beta + \beta^{-k_1} \varrho$$

$\longrightarrow$ an arbitrary sequence of quasi-repetends yields a β-expansion of the same number

generalization of periodic β-expansions,
e.g. if greedy β-expansion of x is quasi-periodic, then $x \in \mathrm{Per}(\beta)$

An Example of Quasi-Periodic β-Expansions

choose any base β such that $|\beta| > 1$, and periodic point $\varrho \neq 0$

define a set of digits $A = \{\alpha_1, \alpha_2, \alpha_3\}$ as

$$\alpha_1 = \frac{\varrho(\beta^2 - 1)}{\beta} \qquad \alpha_2 = \frac{\varrho(\beta - 1)}{\beta} \qquad \alpha_3 = 0$$

for every $n \geq 0$, $\alpha_1 \alpha_2^n \alpha_3$ is a proper quasi-repetend of length $n+2$:

$$(\alpha_1 \alpha_2^n \alpha_3)_\beta = \alpha_1 \beta^{-1} + \sum_{k=2}^{n+1} \alpha_2 \beta^{-k} + \alpha_3 \beta^{-n-2} = \varrho \left(1 - \beta^{-n-2}\right)$$

whereas

$$(\alpha_1 \alpha_2^r)_\beta = \alpha_1 \beta^{-1} + \sum_{k=2}^{r+1} \alpha_2 \beta^{-k} \neq \varrho \left(1 - \beta^{-r-1}\right) \text{ for every } r \in \{0, \dots, n\}$$

$\longrightarrow$ number ϱ has uncountably many distinct quasi-periodic β-expansions:

$$(\alpha_1 \alpha_2^{n_1} \alpha_3 \alpha_1 \alpha_2^{n_2} \alpha_3 \alpha_1 \alpha_2^{n_3} \alpha_3 \dots)_\beta = \varrho$$

where $(n_i)_{i=1}^{\infty}$ is any infinite sequence of nonnegative integers

Eventually Quasi-Periodic β-Expansions and Tail Sequences

$(r_n)_{n=0}^{\infty}$ is a tail sequence of β-expansion $a = a_1 a_2 a_3 \ldots \in A^{\omega}$ if

$$r_n = (a_{n+1} a_{n+2} \ldots)_{\beta} = \sum_{k=1}^{\infty} a_{n+k} \beta^{-k} \qquad \text{for every } n \geq 0$$

denote

$$R(a) = \{ r_n \mid n \geq 0 \} = \left\{ \sum_{k=1}^{\infty} a_{n+k} \beta^{-k} \;\middle|\; n \geq 0 \right\}$$

Lemma 1 *β-expansion $a \in A^{\omega}$ is eventually quasi-periodic with a periodic point ϱ* **iff** *its tail sequence $(r_n)_{n=0}^{\infty}$ contains a constant infinite subsequence $(r_{k_i})_{i=1}^{\infty}$ such that $r_{k_i} = \varrho$ for every $i \geq 1$. Thus, if $R(a)$ is finite, then a is eventually quasi-periodic.*

Theorem 1 *Let $\beta \in \mathbb{Q}$ be a rational base and $A \subset \mathbb{Q}$ be a set of rational digits. Then β-expansion $a \in A^{\omega}$ is eventually quasi-periodic* **iff** *$R(a)$ is finite.*

$\times$ there is eventually quasi-periodic β-expansion $a \in A^{\omega}$ for some $\beta \in \mathbb{R} \setminus \mathbb{Q}$ & $R(a)$ is infinite

Quasi-Periodic Numbers

a real number c is β-quasi-periodic within A if every infinite β-expansion of c is eventually quasi-periodic

Note: numbers with no β-expansion are formally quasi-periodic (e.g. numbers from the complement of the Cantor set are 3-quasi-periodic within $\{0, 2\}$)

Examples:

- for $\beta > 2$, any $\varrho > 0$ is β-quasi-periodic within $A = \{\alpha_1, \alpha_2, \alpha_3\}$ where

$$\alpha_1 = \frac{\varrho(\beta^2 - 1)}{\beta} \qquad \alpha_2 = \frac{\varrho(\beta - 1)}{\beta} \qquad \alpha_3 = 0$$

  (all the uncountably many β-expansions of ϱ are eventually quasi-periodic)

- greedy $\frac{3}{2}$-expansion $100000001\ldots$ of

$$c = (0(011)^{\omega})_{\frac{3}{2}} = \frac{40}{57} \qquad \text{is not quasi-periodic within } A = \{0, 1\}$$

Quasi-Periodic Numbers and Tail Values

for $c \in \mathbb{R}$, denote the set of tail values of all the β-expansions of c as

$$\mathcal{R}_c = \bigcup_{a \in A^\omega \,:\, (a)_\beta = c} R(a)$$

Theorem 2 *The following three conditions are equivalent*

(i) c *is* β*-quasi-periodic within* A

(ii) $\mathcal{R}_c$ *is finite*

(iii) $\mathcal{R}'_c = \{r_c(a) \mid I \leq r_c(a) \leq S\,,\ a \in A^*\}$ *is finite*

where $\quad r_c(a) = \beta^{|a|}(c - (a)_\beta)\,, \quad I = \inf\limits_{a \in A^*} (a)_\beta\,, \quad S = \sup\limits_{a \in A^*} (a)_\beta\,.$

In addition, if c is not β-quasi-periodic within A, then there exists an infinite β-expansion of c whose tail sequence contains pair-wise different values.

Note: Theorem 2 is valid for arbitrary $\beta \in \mathbb{R}$

$\times$ Theorem 1 for single β-expansions holds only if $\beta \in \mathbb{Q}$ and $A \subset \mathbb{Q}$

Regular and Context-Sensitive Cut Languages

Theorem 3 *A cut language $L_{<c}$ is regular* **iff** *c is β-quasi-periodic within A.*

proof: by Myhill-Nerode theorem

Example: any regular language $L \subset A^*$ where $\{\alpha_1, \alpha_2\} \subseteq A$ such that $L \cap \{\alpha_1, \alpha_2\}^2 = \{\alpha_1\alpha_2, \alpha_2\alpha_1\}$, is not a cut language

Theorem 4 *Let $\beta \in \mathbb{Q}$ and $A \subset \mathbb{Q}$. Every cut language $L_{<c}$ with threshold $c \in \mathbb{Q}$ is context-sensitive.*

proof: linear bounded automaton that accepts $L_{<c}$, evaluates $s_n = \sum_{k=1}^{n} a_k \beta^{-k}$ and tests whether $s_n < c$

Non-Context-Free Cut Languages

Theorem 5 *If c is not β-quasi-periodic within A, then the cut language $L_{<c}$ is not context-free.*

Proof by a pumping lemma:

infinite word $a \in A^\omega$ is approximable in a language $L \subseteq A^*$, if for every finite prefix $u \in A^*$ of a, there is $x \in A^*$ such that $ux \in L$.

Lemma 2 *Let $a \in A^\omega$ be approximable in a context-free language $L \subseteq A^*$. Then there is a decomposition $a = uvw$ where $u, v \in A^*$ and $w \in A^\omega$, such that $|v| > 0$ is even and for every integer $i \geq 0$, word uv^iw is approximable in L.*

Corollary 1 *Any cut language $L_{<c}$ is either regular or non-context-free (depending on whether c is a β-quasi-periodic number within A).*

Neural Networks Between Integer and Rational Weights

(Šíma, IJCNN 2017)

present results on cut languages + representation theorem for NN1A

$$L = h(((\bigcup_{r=1}^{p-1} \left(L_{\geq c_r} \cap L_{<c_{r+1}}\right) \cdot \Gamma_r)^{Pref} \cap R_0)^* \cap R)$$

- the languages accepted by NN1A are context-sensitive
- a sufficient condition when NN1A accepts a regular language, which is based on quasi-periodicity of weight parameters
- examples of non-context-free languages accepted by NN1A

Conclusions

- motivated by the analysis of NNs, we have introduced the class of cut languages and classified them within the Chomsky hierarchy
- we have shown an interesting link to active research on β-expansions in non-integer bases
- we have introduced new concepts of eventually quasi-periodic β-expansions and quasi-periodic numbers which generalize eventually periodic (greedy) β-expansions
- open problems:
  - generalization of results to arbitrary real bases $\beta \in \mathbb{R}$ is not complete
  - characterization of quasi-periodic numbers vs. $\mathrm{Per}(\beta)$