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Properties of Delta-Matroids

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Dedication.
I would like to thank a few people that helped me a lot with writing this thesis.
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Title: Properties of Delta-Matroids

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Abstract: We investigate delta-matroids which are formed by families of subsets of a finite ground set such that the exchange axiom is satisfied. We deal with some natural classes of delta-matroids. The main result of this thesis establishes several relations between even, linear, and matching-realizable delta-matroids. Following up on the ideas due to Geelena, Iwatab, and Murota [2003], and applying the properties of field extensions from algebra, we prove that the class of strictly matching-realizable delta-matroids, the subclass of matching-realizable delta-matroids, is included in the class of linear delta-matroids. We also show that not every linear delta-matroid is matching-realizable by giving a skew-symmetric matrix representation to the non matching-realizable delta-matroid constructed by Kazda, Kolmogorov, and Rolínek [2019].

Keywords: delta-matroids, exchange axiom, constraint satisfaction

Contents

Introduction

Delta-matroids as a generalization of matroids are an important combinatorial structure that can be found in many different areas of mathematics, such as graph theory and linear algebra. Delta-matroids have also been used to study the *constraint satisfaction problem* (CSP), a classical topic in computer science.

The aim of our thesis is to describe the structure of delta-matroids and explore classes of even, linear and matching-realizable delta-matroids that have been introduced in articles regarding the edge Boolean CSP [Geelena et al., 2003, Dvořák and Kupec, 2015, Kazda et al., 2019]. We investigate these classes in more detail and point out their basic properties. As our main result, we show several relations between those classes. We recall that linear and matching-realizable delta-matroids are even. Following up on the ideas due to Geelena, Iwatab, and Murota [2003], and applying the properties of field extensions from algebra, we prove that every strictly matching-realizable delta-matroid is linear. We also show that not every linear delta-matroid is matching-realizable by giving a skew-symmetric matrix representation to the non matching-realizable delta-matroid constructed by Kazda, Kolmogorov, and Rolínek [2019]. In addition, we formulate explicitly several proofs that are omitted in the papers above. We also add a few examples and pictures that can give the reader more insight.

Our motivations thus issue from the *Boolean CSP* which is an important algorithmic problem of finding an evaluation of Boolean variables such that given constraints (each consisting of a subset of variables and a relation) are satisfied. This is an NP-complete problem in general. Given a language of constraints Γ, we may consider solving the instances of CSP that contain relations only from Γ. This problem is denoted by $CSP(\Gamma)$ and referred to as the *fixed template CSP*.

The connection between the fixed template CSP and delta-matroids is as follows. If all the variables occur in exactly two constraints, CSP has natural graph interpretation with edges being variables and vertices being constraints. This problem is referred to as the *edge CSP*. Feder [2001] provided the following striking theorem. If a constraint language Γ contains both unary constant relations (that means constant 0 and constant 1) and unless all the relations in Γ are delta-matroids, then the edge $CSP(\Gamma)$ has the same complexity as the unrestricted case $CSP(\Gamma)$.

It has been discovered that the Boolean edge CSP is polynomially solvable for many classes of delta-matroids. Geelena, Iwatab, and Murota [2003] solved the *delta matroid parity problem* (which is in fact equivalent to the Boolean edge CSP) for the class of *linear* delta-matroids which are represented by skew-symmetric matrices. Dvořák and Kupec [2015] defined the class of *matching-realizable* delta-matroids represented by graphs and showed their tractability. In a recent paper, Kazda, Kolmogorov, and Rolínek [2019] provided the tractability of *efficiently-coverable* delta-matroids. Not only this class contains all the classes that were known to be tractable before, but it also includes a large class of *even* delta-matroids.

The thesis is organized as follows. In Chapter 2, we introduce the key definition of delta-matroids and a few basic operations on them, such as twisting, deletion and contraction. We recall that the class of delta-matroids is closed under these operations.

Chapter 3 addresses the class of linear delta-matroids. We give a detailed description of their correspondence with perfect matchings which was mentioned by Geelena et al. [2003]. For this purpose, we introduce the term Pffafian from the book by Lovász and Plummer [2009]. In Section 3.3, we show that the complexity of algorithms for linear delta-matroids may significantly depend on their representation. We provide an example of a linear delta-matroid that has exponentially more feasible sets than the size of matrix representing it.

In Chapter 4, we look more closely on the class of matching-realizable delta-matroids. Matching-realizable delta-matroids are represented by a graph and a subset of its vertices that corresponds to the ground set of a delta-matroid. We call the remaining vertices *hidden* and we define the *hiddenness number* of a matching-realizable delta-matroid as the smallest number of hidden vertices needed to represent it. We come up with the definition of the class of *strictly matching-realizable* delta-matroids that contains exactly the matching-realizable delta-matroids with zero hiddenness number. We provide an example of matching-realizable delta-matroids with an arbitrary large hiddenness number.

In Chapter 5, our main results are stated and proven. We establish the relations between classes of delta-matroids introduced in previous sections. We show the linearity of the even delta-matroid that is not matching-realizable (mentioned in the article by Kazda et al. [2019, Appendix A]), implying that not every linear delta-matroid is matching-realizable. We use the correspondence of linear delta-matroids and perfect matchings, and the properties of field extensions to prove that strictly matching-realizable delta-matroids are linear. We state the conjecture that any matching-realizable delta-matroid is linear and verify it for small arities. We also give possible approaches of proving the conjecture, but it unfortunately remains open.

1. Preliminaries

1.1 Matrices

Let A be a square matrix of size n with a row set and a column set identified with V. We denote the determinant of a matrix A by $|A|$. A useful claim from linear algebra [Barto and Tůma, Claim 7.22, Page 236] states that A is a regular matrix if and only if $|A| \neq 0$. By $A_{i,j}$ we mean the submatrix of A obtained by deleting the i-th row and j-th column of A and by $A_{ij,kl}$ the submatrix of A obtained by deleting i-th and j-th rows and k-th and l-th columns of A. For $X \subseteq V$ let $A[X]$ be the principal submatrix of A on rows and columns from X only.

Let us mention the Laplace expansion of determinant, a fundamental lemma from linear algebra.

Lemma 1 (Laplace expansion)**.** *[Barto and Tůma, Theorem 7.32, Page 240] Let A be a square matrix of size n and $j \in \{1, \ldots, n\}$. Then the determinant of A can be computed as follows:*

$$|A| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}|A_{i,j}|,$$

Or alternatively we fix $i \in \{1, \ldots, n\}$:

$$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}|A_{i,j}|.$$

1.2 Fields

Let $(F, 0, 1, +, -, \cdot, ^{-1})$ be a field. We say that $(E, 0, 1, +, -, \cdot, ^{-1})$ is a *subfield* of F (denoted $E \subseteq F$), if E is a subset of F such that $0, 1 \in E$ and E is closed under all operations of the field. That means $\forall a, b \in E : a + b \in E, a \cdot b \in E$, $-a \in E$ and $\forall a \in E, a \neq 0 : a^{-1} \in E$. Note that a linear equation with coefficients in E has also its solution in E, because we can solve it using the operations of the field.

1.3 Permutations

Let V be a finite set. By a *permutation* of V, we mean any bijection $V \to V$. We denote by S_V the group of all permutations on V. Especially, for any natural n, we denote by S_n the group of permutations on $\{1, \ldots, n\}$.

We say that permutation $\sigma \in S_V$ is a *cycle* of length k, if we can find $\{v_1, \ldots, v_k\} \in V$ such that $\sigma(v_1) = v_2, \sigma(v_2) = v_3, \ldots, \sigma(v_{k-1}) = v_k, \sigma(v_k) = v_1$, and $\sigma(w) = w$ for any $w \in V \setminus \{v_1, \ldots, v_k\}$. By a *transposition*, we mean a cycle of length two.

Every permutation can be written as a composition of transpositions and the parity of the transposition count does not depend on the decomposition of the permutation [Barto and Tůma, Claims 7.6-7.8, Pages 227-228].

Hence, we can define the *sign* of a permutation σ (denoted by $sgn(\sigma)$) as 1 if σ is composed from an even number of transpositions and as -1 if σ is composed from an odd number of transpositions.

Note that the sign is multiplicative; for any two permutations $\sigma_1, \sigma_2 \in S_V$, we have $sgn(\sigma_1 \circ \sigma_2) = sgn(\sigma_1) \cdot sgn(\sigma_2)$.

1.4 Graphs and Perfect Matchings

Let G be a graph with a vertex set V and an edge set E. For $X \subseteq V$ we denote by $G[X]$ the subgraph of G induced by vertices from X.

By a *matching* M of a graph G, we mean a subset M of E such that no two edges from M share a vertex. We say that a matching M is *perfect* if all vertices from V are incident to M. Note that if a graph G has a perfect matching, the number of vertices of G has to be even.

2. Delta-Matroids

In this chapter, we introduce the key definition of a delta-matroid and we present a few basic operations on delta-matroids. Let us start with the definition of the symmetric difference of two sets.

Definition 2 (Symmetric difference)**.** *For two sets A and B, we define their* symmetric difference *as $(A \cup B) \setminus (A \cap B)$ and we denote it by $A \Delta B$.*

In other words, a symmetric difference of two sets contains elements that are members of exactly one of them. Note that this operation is commutative ($A\Delta B = B \Delta A$) and associative ($(A\Delta B)\Delta C = A\Delta(B\Delta C)$).

Definition 3 (Delta-matroid)**.** *A* delta-matroid *M is an ordered pair $M = (V, \mathcal{F})$, where V is a finite set called the* ground set *and $\mathcal{F}$ is a nonempty family of subsets of $V (\mathcal{F} \subseteq \mathcal{P}(V))$, called* feasible sets*, such that the following* exchange axiom *is satisfied:*

$$(\forall F, F' \in \mathcal{F})(\forall x \in F\Delta F')(\exists y \in F\Delta F') : F\Delta\{x, y\} \in \mathcal{F}.$$

Note that the x and y in the exchange axiom do not necessarily have to be distinct. For example, if two feasible sets have a symmetric difference of size 1 or 2, the exchange axiom is automatically satisfied.

Usually we without loss of generality assume that the ground set of a delta-matroid is identified with first natural numbers ($V = \{1, \dots, |V|\}$). We also sometimes prefer to list feasible sets of a delta-matroid by tuples containing 0 or 1, depending on whether the corresponding element belongs to the feasible set or not. For example, the delta-matroid $V = \{1, 2, 3\}, \mathcal{F} = \{\{\emptyset\}, \{1\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$ is represented by the following tuples:

$$000, 100, 011, 101, 111.$$

Besides brevity, another advantage of this representation is that the operation Δ on feasible sets is represented by an addition of tuples modulo 2.

Let us introduce the most important class of delta-matroids, called even delta-matroids. In this paper, we mostly discuss delta-matroids that are even.

Definition 4 (Even delta-matroid)**.** *We say that a delta-matroid is* even *if its feasible sets have the same cardinality modulo 2.*

We present several basic and useful operations on delta-matroids in the following definition.

Definition 5 (Operations on delta-matroids)**.** *Let $M = (V, \mathcal{F})$ be a delta-matroid and $X \subseteq V$ be a set. We define $M\Delta X \stackrel{\text{def}}{=} (V, \mathcal{F}\Delta X)$, where $\mathcal{F}\Delta X \stackrel{\text{def}}{=} \{F\Delta X | F \in \mathcal{F}\}$. This operation is called a* twisting *of M by X. By a* dual *of M, we mean $M\Delta V$.*
The second operation we define is a deletion *of X: $M \setminus X \stackrel{\text{def}}{=} (V \setminus X, \mathcal{F} \setminus X)$, where $\mathcal{F} \setminus X \stackrel{\text{def}}{=} \{F | F \in \mathcal{F}, F \cap X = \emptyset\}$.*
By a contraction *of X, we mean $M/X \stackrel{\text{def}}{=} (M\Delta X) \setminus X$.*

When we represent a delta-matroid by tuples, we can realize a deletion (or a contraction) by taking only the tuples that have 0's (or 1's) in specific positions (corresponding to X). Let us illustrate it on a simple example.

Example 6. *Let M be a delta-matroid with the ground set $V = \{1, 2, 3, 4\}$ and feasible sets represented by the following tuples:*

$$0000, 1100, 1001, 0110, 0101, 0011$$

and let $X = \{3, 4\}$.

Observe that every feasible set of delta-matroid M has an even cardinality, thus M is an even delta-matroid.

One can see that $M \setminus X$ contains two feasible sets represented by $00, 11$ and M/X has only one feasible set represented by 00. Note that both $M \setminus X$ and M/X are also even delta-matroids. That is not a coincidence, as the next claim shows.

Claim 7. *Let $M = (V, \mathcal{F})$ be a delta-matroid and $X \subseteq V$. Then $M\Delta X$ is a delta-matroid. $M \setminus X$ and M/X are also delta-matroids, provided that they contain at least one feasible set. In addition, the class of even delta-matroids is closed under twisting, deletion and contraction.*

Proof. We begin with showing that $M\Delta X$ is a delta-matroid. The non-emptiness of $\mathcal{F}\Delta X$ follows from its definition and from the non-emptiness of $\mathcal{F}$. To verify the exchange axiom, let us take arbitrary $F, F' \in \mathcal{F}\Delta X$. We can find $E, E' \in \mathcal{F}$ such that $F = E\Delta X$ and $F' = E'\Delta X$. Let x be any element of $F\Delta F'$. We have that $F\Delta F' = (E\Delta X)\Delta(E'\Delta X) = E\Delta E'$. Applying the exchange axiom for M on $E, E' \in \mathcal{F}$ and $x \in E\Delta E'$, we obtain $y \in E\Delta E'$ such that $E\Delta\{x, y\} \in \mathcal{F}$. Since $F\Delta\{x, y\} = (E\Delta X)\Delta\{x, y\} = (E\Delta\{x, y\})\Delta X$, the exchange axiom for $M\Delta X$ is verified.

Let us suppose that M is an even delta-matroid, we demonstrate that $M\Delta X$ is also an even delta-matroid. Let us take any $F \in \mathcal{F}\Delta X$ and corresponding $E \in \mathcal{F}$. We have (modulo 2):

$$|F| = |E\Delta X| = |E| + |X| - 2|E \cap X| \equiv |E| + |X|$$

and the evenness of $M\Delta X$ follows from the evenness of M.

Let us show that $M \setminus X$ is a delta-matroid. We have assumed the non-emptiness of $\mathcal{F} \setminus X$ in the statement of this claim, thus it suffices to verify the exchange axiom. Let us take arbitrary $F, F' \in \mathcal{F} \setminus X$ and $x \in F\Delta F'$. Since $F, F' \in \mathcal{F}$, we can apply the exchange axiom for M on F, F' and x. We obtain $y \in F\Delta F'$ such that $F\Delta\{x, y\} \in \mathcal{F}$. Since $F \cap X = \emptyset$, $F' \cap X = \emptyset$, and $x, y \in F\Delta F'$, we have that $(F\Delta\{x, y\}) \cap X = \emptyset$. Therefore, $(F\Delta\{x, y\}) \in \mathcal{F} \setminus X$ and the exchange axiom for $M \setminus X$ is verified.

Deletion preserves evenness because $\mathcal{F} \setminus X$ is a subset of $\mathcal{F}$.

Since the contraction is defined as the composition of twisting and deletion, we are done. □

Definition 8. *We say that two delta-matroids M, N on a same ground set V are* equivalent *if there exists a set $X \subseteq V$ such that $N = M\Delta X$.*

3. Linear Delta-Matroids

This chapter is devoted to study linear delta-matroids that form an important subclass of even delta-matroids. We briefly discuss basic properties of linear delta-matroids and we look more closely at the relation of linear delta-matroids with perfect matchings, which was introduced in the article by Geelena et al. [2003]. We conclude this chapter by showing that the complexity of algorithms for linear delta-matroids may significantly depend on their representation.

3.1 Definition and Basic Properties

Linear delta-matroids are represented by a skew-symmetric matrix; let us start with its definition.

Definition 9 (Skew-symmetric matrix)**.** *Let A be a square matrix over a field F. A is called* skew-symmetric *if $A^\top = -A$ and its diagonal entries are equal to zero.*

Note that the second condition is redundant for fields with a characteristic different from two (it is implied by the first condition).

Let A be a skew-symmetric matrix with a row set and a column set identified with V. Let $\mathcal{F}(A) \stackrel{\text{def}}{=} \{W \subseteq V,\ A[W] \text{ regular}\}$ and $M(A) \stackrel{\text{def}}{=} (V, \mathcal{F}(A))$, where A[W] is the principal submatrix of A; see preliminaries for more details. It is not difficult to observe that $A[W]$ is also a skew-symmetric matrix for any $W \subseteq V$.

Lemma 10. *[Bouchet, 1988, Claim 4.1, Page 172]*
M(A) forms a delta-matroid for any skew-symmetric matrix A.

Definition 11 (Linear delta-matroid)**.** *Delta-matroid M is called* linear *if it is equivalent to $M(A)$ for some skew-symmetric matrix A over an arbitrary field.*

Theorem 12 (Jacobi's Theorem)**.** *Let A be a skew-symmetric matrix of size n. If n is odd then A is singular.*

Proof. From the skew-symmetry of matrix A and applying properties of a determinant, we obtain:

$$|A| = |A^\top| = |-A| = (-1)^n|A|.$$

Since n is odd, we have $|A| = -|A|$, implying $|A| = 0$ and the singularity of matrix A. □

Claim 13. *Linear delta-matroids are even.*

Proof. Let $M = (V, \mathcal{F})$ be a linear delta-matroid, hence $M = M(A)\Delta X$ for some skew-symmetric matrix A and $X \subseteq V$. Since the class of even delta-matroids is closed under twisting (Claim 7), it suffices to show that $M(A)$ is an even delta-matroid. For an arbitrary $F \in \mathcal{F}(A)$, we have the regularity of $A[F]$ from the definition. As a consequence of Jacobi's Theorem applied on the skew-symmetric matrix $A[F]$, we have that $|F|$ has to be even and it follows that M(A) is even. □

3.2 Correspondence with Perfect Matchings

In this section, we explore the relation between linear delta-matroids and perfect matchings. To see this, let us introduce the definitions of a Pfaffian (from [Lovász and Plummer, 2009]) and of a support graph of a skew-symmetric matrix.

Let A be any $2n \times 2n$ skew-symmetric matrix. Let us without loss of generality suppose that its row and column set are indexed by $\{1, \ldots, 2n\}$. For a partition

$$P = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$$

of the set $\{1, \ldots, 2n\}$ into pairs, we define the expression A_P as follows:

$$A_P \stackrel{\text{def}}{=} sgn(\sigma_P) \prod_{i=1}^{n} a_{x_i y_i},$$

where σ_P is the following permutation:

$$\sigma_P \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ x_1 & y_1 & x_2 & y_2 & \ldots & x_n & y_n \end{pmatrix}.$$

The expression A_P is well defined, as we show in the following lemma.

Lemma 14. *The value of A_P defined above depends neither on the order of pairs in the partition P nor on the order of the two elements of a pair.*

Proof. It suffices to show that the value of A_P does not change if we swap any two pairs in the partition P or if we swap the elements in any pair. We discuss these two cases one by one.

Firstly, let us suppose that we swap the j-th and k-th pair in the partition P for some $j, k \in \{1, \ldots, n\}, j \neq k$. The value of the product $\prod_{i=1}^{n} a_{x_i y_i}$ does not change because we multiply the same elements of the matrix A (only in the different order). The sign of the corresponding permutation also stays the same because it differs from σ_P by two transpositions:

$$\begin{pmatrix} 1 \ldots & 2j-1 & 2j & \ldots & 2k-1 & 2k & \ldots & 2n \\ x_1 \ldots & x_k & y_k & \ldots & x_j & y_j & \ldots & y_n \end{pmatrix} = \sigma_P \circ \begin{pmatrix} x_j & x_k \end{pmatrix} \circ \begin{pmatrix} y_j & y_k \end{pmatrix}.$$

Now, let us suppose that we swap x_j and y_j for some $j \in \{1, \ldots, n\}$. From the skew-symmetry of matrix A, we have that $a_{y_j x_j} = -a_{x_j y_j}$. Hence the value of the product $\prod_{i=1}^{n} a_{x_i y_i}$ differs only by a sign. Sign of the corresponding permutation also changes because it differs from σ_P by one transposition:

$$\begin{pmatrix} 1 \ldots & 2j-1 & 2j & \ldots & 2n \\ x_1 \ldots & y_j & x_j & \ldots & y_n \end{pmatrix} = \sigma_P \circ \begin{pmatrix} x_j & y_j \end{pmatrix}.$$

Thus, the value of A_P does not change and we are done. □

Definition 15. *Let A be a skew-symmetric matrix of size $2n$. Then we define the* Pffafian *of A as follows:*

$$Pf(A) \stackrel{\text{def}}{=} \sum_P A_P = \sum_P (sgn(\sigma_P) \prod_{i=1}^{n} a_{x_i y_i}),$$

where the sum goes through all partitions P of the set $\{1, \ldots, 2n\}$ into pairs.

Example 16. *Let us compute the Pffafian of a general* 4×4 *skew-symmetric matrix* A*.*

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

We have three partitions of the set $\{1, 2, 3, 4\}$ into pairs:

$$P_1 = \{\{1, 2\}, \{3, 4\}\}, P_2 = \{\{1, 3\}, \{2, 4\}\}, \text{ and } P_3 = \{\{1, 4\}, \{2, 3\}\}.$$

Let us compute A_{P_1}, A_{P_2}, and A_{P_3}.

$$\begin{aligned} A_{P_1} &= sgn(id) \cdot a_{12} \cdot a_{34} = af \\ A_{P_2} &= sgn \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \cdot a_{13} \cdot a_{24} = -be \\ A_{P_3} &= sgn \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \cdot a_{14} \cdot a_{23} = cd \end{aligned}$$

Summing them up, we obtain that $PfA = af - be + cd$.

Definition 17 (Support graph)**.** *Let* A *be a skew-symmetric matrix with a vertex set and a row set identified with* V*. Let us define the* support graph *of* A *as* $G_A \stackrel{\text{def}}{=} (V, E)$*, such that there is an edge between* $i \in V$ *and* $j \in V$ *if and only if* $a_{ij} \neq 0$*.*

When computing the Pffafian of A, we sum through all partitions P of the set $\{1, \ldots, 2n\}$. Observe that partitions with nonzero values of A_P are in one-to-one correspondence with perfect matchings of support graph G_A. Using this fact and the Definition 15, we provide an alternative computation of Pffafian.

Lemma 18. *The Pfaffian of* $2n \times 2n$ *skew-symmetric matrix* A *can be computed as follows:*

$$Pf(A) = \sum_M (sgn(\sigma_M) \prod_{(i,j) \in M} a_{ij}).$$

where the sum goes through all perfect matchings M *of the support graph* G_A *of* A *and* σ_M *is the corresponding permutation to the partition* M *(defined in the beginning of this section).*

The following theorem shows the correspondence between the determinant and the Pffafian of a skew-symmetric matrix.

Theorem 19 (Cayley)**.** *[Lovász and Plummer, 2009, Lemma 8.2.2, Page 318] For a* $2n \times 2n$ *skew-symmetric matrix* A*,* $|A| = Pf^2(A)$*. As a simple consequence, we have that* $|A| = 0$ *if and only if* $Pf(A) = 0$*.*

Let $M = (V, \mathcal{F})$ be a linear delta-matroid represented by a skew-symmetric matrix A and let $W \subseteq V$, $|W|$ even. Applying Cayley's Theorem and Lemma 18 on the skew-symmetric matrix $A[W]$, we obtain a nice rule for determining whether W is a feasible set or not. If the support graph $G_{A[W]}$ of the matrix $A[W]$ has no perfect matching, the Pffafian of $A[W]$ is equal to zero and therefore $A[W]$ has to be singular. On the other hand, if $G_{A[W]}$ has exactly one perfect matching, A is regular. When $G_{A[W]}$ has more than one perfect matching, we can not decide the regularity of $A[W]$, because summands in the Pffafian may sum up to zero.

3.3 Algorithmic Considerations

In this section, we show that the complexity of algorithms for linear delta-matroids may significantly depend on their representation.

Kazda et al. [2019] introduced an algorithm that can efficiently solve edge Boolean CSP with all constraints being even delta-matroids. Since Kazda et al. represent delta-matroids by a list of tuples, the complexity of their algorithm is polynomial in the number of feasible sets of a given delta-matroid.

On the other hand, Geelena et al. [2003] presented an algorithm for linear delta-matroids that has polynomial complexity in terms of the dimension of the skew-symmetric matrix representing a given linear delta-matroid.

In the following claim, we provide an example of linear delta-matroids that have exponentially more feasible sets than is size of the matrix representing them, implying that the complexity of these algorithms can significantly differ.

Claim 20. *Let $n \in \mathbb{N}$ and V_n be a finite set with cardinality $2n$. Take $\mathcal{F}_n = \{F \subseteq V_n, |F| \text{ even}\}$. Then the delta-matroid $M_n = (V_n, \mathcal{F}_n)$ is linear.*

Proof. For any $n \in \mathbb{N}$, let us denote by A_n the following $2n \times 2n$ skew-symmetric matrix:

$$A_n = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & \dots & 1 & 1 \\ -1 & -1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{pmatrix}.$$

We claim that $M_n = M(A_n)$. As a result of Jacobi's Theorem (Theorem 12), we have that $\mathcal{F}(A_n)$ does not contain sets of an odd cardinality. We have to show that $\mathcal{F}(A_n)$ contains every subset of V_n that has an even size. Since every principal submatrix of size $2k$ of the matrix A_n is the matrix A_k, the proof is completed by showing that matrix A_k is regular for every natural k. We proceed by induction and we will prove a stronger statement that $|A_k| = 1$ for every k.

Case $k = 1$ is easy. For proving the induction step, let us suppose that $|A_{k-1}| = 1$ and let us denote $A = A_k$ and a_{ij} the element on i-th row and j-th column of the matrix A.

Applying the Laplace expansion of determinant (see Lemma 1) on matrix A and noting that $a_{11} = 0$, we get:

$$|A| = \sum_{i=1}^{2k} a_{i1}(-1)^{i+1}|A_{i,1}| = \sum_{i=2}^{2k} a_{i1}(-1)^{i+1}|A_{i,1}|. \tag{3.1}$$

By expanding the determinant of $A_{i,1}$ for $i \neq 1$:

$$|A_{i,1}| = \sum_{j=2}^{2k} a_{1j}(-1)^{j}|A_{1i,1j}|. \tag{3.2}$$

Plugging (3.2) into (3.1), we obtain:

$$|A| = \sum_{i=2}^{2k} \sum_{j=2}^{2k} a_{i1} a_{1j} (-1)^{i+j+1} |A_{1i,1j}|.$$

For each $i, j \in \{2, \dots, 2k\}$ we have $a_{i1} = -1, a_{1j} = 1$ from the definition of A_k. Therefore:

$$|A| = \sum_{i,j=2}^{2k} (-1)^{i+j} |A_{1i,1j}|.$$

Noting that $A_{1i,1j}$ is the matrix A_{k-1}, we can apply the induction hypothesis:

$$|A| = \sum_{i,j=2}^{2k} (-1)^{i+j}.$$

The last sum contains $(2k-1)^2 = 4k^2 - 4k + 1$ summands and their value depends on the parity of $i + j$. We have that $(-1)^{i+j} = 1$ if and only if i and j have the same parity. We obtain k^2 summands with both i and j even and $(k-1)^2$ with both of them odd. In total, we have $2k^2 - 2k + 1$ summands equal to 1 and the rest $(2k^2 - 2k)$ equal to -1. Altogether, it sums up to 1 and we are done.

$\square$

4. Matching-Realizable Delta-Matroids

4.1 Definition and Basic Properties

In this section, we investigate the class of matching-realizable delta-matroids that was introduced by Dvořák and Kupec [2015] in their article regarding the planar Boolean CSP. Matching-realizable delta-matroids are represented by graphs.

Let $G = (V, E)$ be a finite graph and $W = \{w_1, \ldots, w_n\}$ its pairwise distinct vertices. For $X \subseteq V$, we denote by $G[X]$ the induced subgraph of G on vertices from X.

We define:

$$\mathcal{F}(G, W) \stackrel{\text{def}}{=} \{F \subseteq W : G[(V \setminus W) \cup F] \text{ has a perfect matching}\}$$
$$M(G, W) \stackrel{\text{def}}{=} (W, \mathcal{F}(G, W)).$$

We call vertices from $V \setminus W$ *hidden*. Note that the sets in $\mathcal{F}(G, W)$ correspond to subgraphs of G that contain all hidden vertices and have a perfect matching.

Claim 21. *Let $G = (V, E)$ be a graph, $W \subseteq V$, and $\mathcal{F}(G, W) \neq \emptyset$. Then $M(G, W)$ is an even delta-matroid.*

Proof. We are going to start with verifying the exchange axiom. Let $F_1, F_2 \in \mathcal{F}(G, W)$ and let M_1, M_2 be the corresponding perfect matchings of $G[(W \setminus V) \cup F_1]$ and $G[(W \setminus V) \cup F_2]$ respectively. Let x be an arbitrary element of $F_1 \Delta F_2$.

We construct the auxiliary graph $H = (V, M_1 \Delta M_2)$. Since the degree of every vertex of H is clearly at most two, H is a union of paths and cycles. Vertex x belongs to the symmetric difference of F_1 and F_2, hence it is an endpoint of a path P in H. Let us denote by y the second endpoint of the path P. Observe that the vertex y also lies in $F_1 \Delta F_2$.

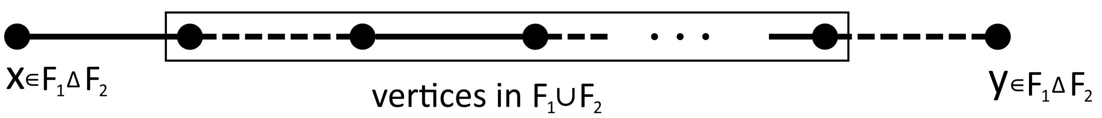

Figure 4.1: Alternating path P.

We claim that $F_1 \Delta \{x, y\} \in \mathcal{F}(G, W)$. One can see that the subgraph $G[(W \setminus V) \cup (F_1 \Delta \{x, y\})]$ is perfectly matched by the matching $M_1 \Delta P$.

To show that $M(G, W)$ is even, we take an arbitrary $F \in \mathcal{F}(G, W)$. The induced subgraph $G[(V \setminus W) \cup F]$ has a perfect matching, implying $|V \setminus W \cup F|$ has to be even. Since $|F| = |(V \setminus W) \cup F| - |V \setminus W|$ and $|V \setminus W|$ does not depend on a choice of F, all feasible sets of $M(G, W)$ have the same parity. $\square$

Definition 22 (Matching-realizable delta-matroids)**.** *A delta-matroid M is said to be* matching-realizable *if there exists a graph $G = (V, E)$ and $W \subseteq V$, such that $M = M(G, W)$.*

In Section 5.2 of this thesis, we discuss a relation between linear and matching-realizable delta-matroids. In order to keep the notation compatible, our definition of matching-realizable delta-matroids differs by twisting from the definition introduced in the article from Dvořák and Kupec [2015] and in the article from Kazda et al. [2019]. This difference is insignificant, since the class of matching-realizable delta-matroids is closed under twisting, as the following claim states.

Claim 23. *The class of matching-realizable delta-matroids is closed under twisting. In other words, for a matching-realizable delta-matroid $M = (W, \mathcal{F})$ and $X \subseteq W, M\Delta X$ is also a matching-realizable delta-matroid.*

Proof. Since M is matching-realizable, we can find a graph $G = (V, E)$ and $W \subseteq V$ such that $M = M(G, W)$. We construct a graph G' from G as follows. We replace each vertex x_i from X with a new vertex y_i. Then we add a new vertex x_i and an edge between x_i and y_i. To make the construction more clear, we give a picture below.

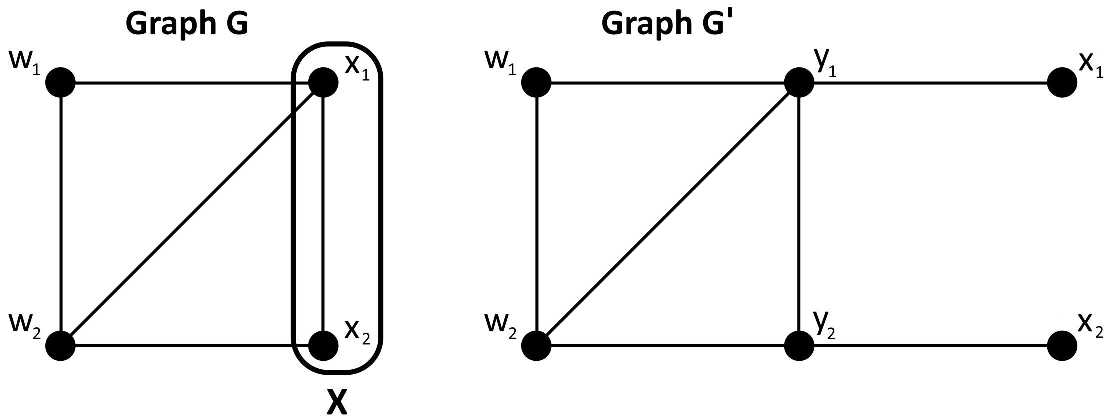

Figure 4.2: The construction of the graph G' from the graph G.

We claim that $M(G, W)\Delta X = M(G', W)$. Let F be any subset of W. We are reduced to proving that $F \in \mathcal{F}(G, W)$ implies that $(F\Delta X) \in \mathcal{F}(G', W)$, since the second inclusion can be shown similarly. Let us suppose that the subgraph $G[(V \setminus W) \cup F]$ has a perfect matching N. Note that N forms a matching in the graph G. We will construct a matching N' in the graph G' from the matching N. For each vertex x_i from X, we make the following changes. If x_i occurs in a matching N, we replace it by y_i, otherwise we add an edge $\{y_i, x_i\}$ to N. Observing that N' is a perfect matching of $G'[(V' \setminus W) \cup (F\Delta X)]$, we are done. $\square$

4.2 Strictly Matching-Realizable Delta-Matroids

Definition 24 (Strictly matching-realizable delta-matroid)**.** *Let $M = (W, \mathcal{F})$ be a matching-realizable delta-matroid represented by a graph $G = (V, E)$ and $W \subseteq V$. We say that a vertex of G is* hidden *if it belongs to $V \setminus W$. We define the* hiddenness number *of a graph G representing M as the number of hidden vertices of G. The* hiddenness number *of a delta-matroid M is the lowest hiddenness number among all graphs that represent M. We say that M is* strictly matching-realizable *if its hiddenness number is equal to zero.*

Strictly matching-realizable delta-matroids are an important subclass of matching-realizable delta-matroids. Note that for any matching-realizable delta-matroid M, the hiddenness number of graphs representing M has the same parity. Hence, matching-realizable delta-matroids that contain feasible sets with odd cardinality have an odd hiddenness number and can not be strictly matching-realizable. As the following claim shows, we can even find a matching-realizable delta-matroid with sufficiently large hiddenness number.

Claim 25. *For any natural n, there exists a matching-realizable delta-matroid M whose hiddenness number is n.*

Proof. We shall construct such a delta-matroid. For every natural n, let us define a graph $G_n = (V_n, E_n)$, where $V_n = \{1, \dots, n, h_1, \dots h_n\}$ and $E_n = \{\{i, h_i\}, i \in \{1, \dots, n\}\}$ and let $M_n = M(G_n, \{1, \dots, n\})$.

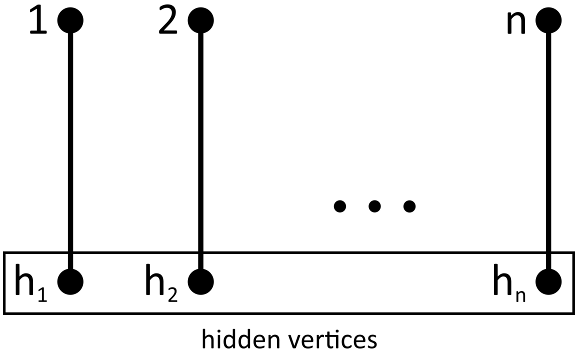

Figure 4.3: Graph G_n representing the delta-matroid M_n.

Since G_n is a union of n isolated edges, the only induced subgraph that contains all hidden vertices and has a perfect matching is the graph G_n itself. Hence, M_n has exactly one feasible set: $\{1, \dots, n\}$. It is clear from the construction that the hiddenness number of M_n is at most n.

We are going to show that the hiddenness number of M_n is at least n. For a contradiction, let us consider a graph $G' = (V', E')$ that represents M_n and has a smaller hiddenness number m. We have that $M_n = M(G', \{1, \dots, n\})$ for some vertices $\{1, \dots, n\} \subseteq V'$. Since $\{1, \dots, n\}$ is a feasible set of M_n, we have that G' has a perfect matching, let us denote it by M. Because $\{1, \dots, n\}$ is strictly larger than the number of hidden vertices of G', we are able to find a pair of vertices $i, j \in \{1, \dots, n\}$ that are matched in M. Therefore $G'[V' \setminus \{i, j\}]$ has a perfect matching, implying that $\{1, \dots, n\} \setminus \{i, j\}$ is another feasible set of M_n which is a contradiction. □

Note that the structure of the delta matroids from the previous claim is simple; they all contain only one feasible set. And we also have $M_n = (\{1, \dots, n\}, \{\emptyset\}) \Delta \{1, \dots, n\}$. Since the delta-matroid $(\{1, \dots, n\}, \{\emptyset\})$ can be represented by a graph containing n vertices and no edges, delta-matroids M_n are indeed equivalent to a delta-matroid that has hiddenness number zero. We are often able to reduce the hiddenness number by twisting the delta-matroid.

Even though the hiddenness number of constructed M_n can be arbitrarily large, it is still in a linear correspondence with its arity. Can we find matching-realizable delta-matroids with hiddenness numbers exponentially larger than their arity? We leave this question as an impulse for further study.

5. Relations Between Classes of Delta-Matroids

In this chapter, we discuss the relations between previously introduced classes of delta-matroids.

5.1 Not Every Linear Delta-Matroid Is Matching-Realizable

In the section 4.1, we have pointed out that matching-realizable delta-matroids are even (Claim 21). On the other hand, is every even delta-matroid matching realizable? Dvořák and Kupec [2015, Page 442] provided a table of representative even delta-matroids and graphs by that they are matching-realized. Using a program, they examined that every even delta-matroid of arity at most 5 differs from one listed in the table by twisting and/or by permuting the ground set. Since the class of matching-realizable delta-matroids is closed under those operations (see Claim 23), every even delta-matroid of arity at most 5 is matching-realizable.

However, Kazda et al. [2019, Appendix A] provided an example of an even delta-matroid of arity 6 (let us denote it by R_6) that is not matching realizable.

R_6 contains following feasible sets:

000000	100100	001111	111111
	100010	011011	
	011000	100111	
	010001	101011	
	001100	101101	
	001010	110011	
	001001	110101	
	000101	111010	
		111100	

Kazda et al. verified by a computer that R_6 is indeed an even delta-matroid. Even though the list of tuples above may seem irregular, we provide the representation of R_6 by a skew-symmetric matrix in the following claim.

Claim 26. *R_6 is a linear delta-matroid.*

Proof. We would like to find a 6×6 skew-symmetric matrix A and a set X such that $R_6 = M(A)\Delta X$. We choose $X = \emptyset$. Since $|A[\{i,j\}]| = (a_{ij})^2$, we want $a_{ij} = 0$ if and only if $\{i,j\} \notin R_6$. Therefore, A has a form below (the star stands for some nonzero element):

$$A = \begin{pmatrix} 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & * & 0 & 0 & * \\ 0 & * & 0 & * & * & * \\ * & 0 & * & 0 & 0 & * \\ * & 0 & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \end{pmatrix}.$$

Now we have to choose appropriate nonzero elements. One of the possibilities is shown below:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \end{pmatrix}.$$

With enough patience, one can check that R_6 is linearly represented by the matrix A. □

Not only does Claim 26 give R_6 the structure of an even delta-matroid, but it is also an example of a linear delta-matroid that is not matching realizable.

5.2 Linearity of Strictly Matching-Realizable Delta-Matroids

In Chapter 3 concerning linear delta-matroids, we have pointed out the relation between linear delta-matroids and perfect matchings of their support graphs. We will use this relation to prove the following theorem.

Theorem 27. *Strictly matching realizable delta-matroids are linear.*

Proof. Let us suppose that M is strictly matching-realizable by a graph $G = (V, E)$. We can without loss of generality suppose that V={1,...,n}. Let $m = |E|$ and we denote the edges of G by $\{e_1, \dots, e_m\}$ in any order. We are going to construct a skew-symmetric matrix A such that $M = M(A)$.

Let $K_0 \subsetneqq K_1 \subsetneqq K_2 \subsetneqq \cdots \subsetneqq K_m$ be a chain of fields. Let us denote by k_{i+1} one arbitrary element that belongs to $K_{i+1} \setminus K_i$. For example we can choose $K_0 = \mathbb{Q}, K_i = \mathbb{Q}(\sqrt[2^i]{2}), k_i = \sqrt[2^i]{2}, i \in \{1, \dots, m\}$ and observe it has the properties above.

Let A be a square matrix of size n whose elements are defined as follows:

$$a_{ij} = \begin{cases} 0 & \{i,j\} \notin E \\ k_l & \{i,j\} \in E, \{i,j\} = e_l, i < j \\ -k_l & \{i,j\} \in E, \{i,j\} = e_l, i > j \end{cases}$$

The matrix A is clearly skew-symmetric and we claim that $M(G, V) = M(A)$. In other words, we would like to prove that $F \in \mathcal{F}(G, V)$ if and only if $F \in \mathcal{F}(A)$ for any $F \subseteq V$. Noting that G is identical to the support graph G_A of A, we obtain:

$$F \in \mathcal{F}(G, V) \iff G[F] \text{ has a perf.m.} \iff G_A[F] = G_{A[F]} \text{ has a perf.m.}$$

As a consequence of Theorem 19, we have:

$$F \in \mathcal{F}(A) \iff |A[F]| \neq 0 \iff Pf(A[F]) \neq 0.$$

We finish the proof by showing that $Pf(A[F]) \neq 0$ if and only if $G_{A[F]}$ has a perfect matching.

Firstly, let us suppose that $Pf(A[F]) \neq 0$. Since the Pfaffian can be written as a sum over perfect matchings of its support graph (see Lemma 18), there has to be at least one summand, hence $G_{A[F]}$ has a perfect matching.

We prove the converse by contradiction. Let us suppose $G_{A[F]}$ has at least one perfect matching and $Pf(A[F])$ is zero. From Lemma 18 we have:

$$0 = Pf(A[F]) = \sum_M (\sigma_M \prod_{(i,j)\in M} a_{ij}),$$

where M goes through all perfect matchings of $G_{A[F]}$ and $\sigma_M \in \{-1, 1\}$.

Since the sum above contains at least one summand, let l be the largest natural number such that k_l occurs in the sum above. Let us solve this linear equation for k_l and convert it to the standard form $a \cdot k_l = b$, where a, b have to belong to K_{l-1}. We are going to distinguish two cases.

If $a \neq 0$, then $k_l = ba^{-1}$. But $k_l \notin K_{l-1}$ from the definition, however $ba^{-1} \in K_{l-1}$, contradiction.

When $a = 0$, we choose $l_2 < l$ largest such that k_{l_2} occurs in the expression of a and we apply the same argument for the linear equation $a = 0$ and k_{l_2}. Iterating the process above, we either find l_n such that the first case occurs, or we construct an infinite strictly decreasing sequence of natural numbers, which is a contradiction. □

In order to make the construction in the proof above more clear, we provide a simple example. Let M be a strictly matching-realizable delta-matroid represented by a graph G below. In the first step, we denote the edges of G by $\{e_1, \ldots, e_m\}$ in any order.

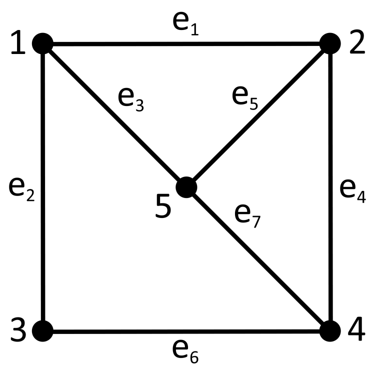

Figure 5.1: Graph G that represents matching-realizable delta-matroid M.

Now let $K_i = \mathbb{Q}(\sqrt[2^i]{2}), k_i = \sqrt[2^i]{2}$ for every $i \in \{1, \ldots, 7\}$. The matrix A constructed in the theorem above is:

$$A = \begin{pmatrix} 0 & \sqrt{2} & \sqrt[4]{2} & 0 & \sqrt[8]{2} \\ -\sqrt{2} & 0 & 0 & \sqrt[16]{2} & \sqrt[32]{2} \\ -\sqrt[4]{2} & 0 & 0 & \sqrt[64]{2} & 0 \\ 0 & -\sqrt[16]{2} & -\sqrt[64]{2} & 0 & \sqrt[128]{2} \\ -\sqrt[8]{2} & -\sqrt[32]{2} & 0 & -\sqrt[128]{2} & 0 \end{pmatrix}$$

and we proved that $M(G, \{1, \ldots, 5\}) = M(A)$.

5.3 Linearity of Matching-Realizable Delta-Matroids

In the previous section, we proved that the subclass of strictly matching-realizable delta-matroids belongs to the class of linear delta-matroids. But are we able to linearly represent a general matching-realizable delta-matroid? In this section, we provide a few approaches and ideas about this problem but it unfortunately remains open. Let us start with examining delta-matroids of small arities. If we narrow down our interest to delta-matroids of arity at most 5, we obtain the following:

Claim 28. *Every even delta-matroid of arity at most 5 is linear.*

Proof. We proceed by case consideration; we went through the table of representatives of even delta-matroids of arity at most 5 that was introduced in the article by Dvořák and Kupec [2015, Page 442] (for more details, see the beginning of the section 5.1. We showed that these representatives are linear delta-matroids (the most of them are equivalent to a strictly matching-realizable delta-matroid). Because the class of linear delta-matroids is clearly closed under twisting and permuting the ground set, we are done. □

Matching-realizable delta-matroids are even (see Claim 21). As a consequence of this fact and the claim above, we have the partial result that matching-realizable delta-matroids of arity at most 5 are linear.

We already know that strictly matching-realizable delta-matroids are linear. The natural approach to finding linear representation of a matching-realizable delta-matroid is to reduce its hiddenness number. One of the ways is to find an equivalent delta-matroid with a lower hiddenness number. Even though this approach works for many matching-realizable delta-matroids, it fails in general, as the next example shows.

Example 29. *Matching-realizable delta-matroid M represented by the graph G below has hiddenness number one and is not equivalent to a strictly matching-realizable delta-matroid.*

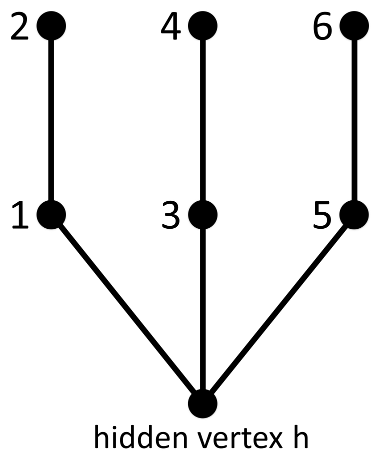

Figure 5.2: Graph G representing delta-matroid M.

Proof. For a contradiction, let us suppose that we can find a graph $H = (V, E)$, where $V = \{1, \ldots, 6\}$, and a subset X of V such that $M = M(H, V)\Delta X$. Since $\emptyset \in \mathcal{F}(H, V)$, X has to be chosen as one of the feasible sets of M. Since G is symmetric, we can without loss of generality assume that $X \in \{\{1\}, \{1, 2, 3\}, \{1, 2, 3, 4, 5\}\}$ and distinguish these three options.
Case 1, $X = \{1\}$.

We have that $\{1, 3\} \in \mathcal{F}(H, V)$ because $\{3, h\}$ is an edge of G and $\{2, 5\} \in \mathcal{F}(H, V)$ because $G[\{1, 2, 5, h\}]$ has a perfect matching. Therefore $\{1, 3\}$ and $\{2, 5\}$ are edges of the graph H, implying that $\{1, 2, 3, 5\}$ has a perfect matching in H. On the other hand $\{1, 2, 3, 5\} \notin \mathcal{F}(H, V)$ because the graph $G[\{2, 3, 5, h\}]$ does not have a perfect matching, which is a contradiction.
Case 2, $X = \{1, 2, 3\}$.

The argument is similar as in the first case. We have that $\{2, 4\}$ and $\{3, 5\}$ are edges of the graph H, because subgraphs induced in G by $\{1, 3, 4, h\}$ and $\{1, 2, 5, h\}$ have a perfect matching. But $\{2, 3, 4, 5\} \notin \mathcal{F}(H, V)$ because the graph $G[\{1, 4, 5, h\}]$ has no perfect matching.
Case 3, $X = \{1, 2, 3, 4, 5\}$.

As in previous cases, we have $\{2, 6\} \in \mathcal{F}(H, V), \{4, 5\} \in \mathcal{F}(H, V)$ but also $\{2, 4, 5, 6\} \notin \mathcal{F}(H, V)$ which is a contradiction.

□

However, the delta-matroid M from the example above is not a counterexample to our conjecture, because we can represent M linearly. We have that $M = M(A)\Delta\{1\}$, where A is the following skew-symmetric matrix:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Unfortunately, we are not always able to reduce the hiddenness number by twisting. Let us present another approach to deal with the hiddenness number. Let M be a matching-realizable delta-matroid with a nonzero hiddenness number that is represented by the graph $G = (V, E)$ and $W \subsetneq V$. We define the strictly matching-realizable delta-matroid M' as $M(G, V)$. As a result of Theorem 27, M' is linear. It is not difficult to observe that $M = M'/(V \setminus W)$. Our problem reduces to showing that the class of linear delta-matroids in closed under contraction. Unfortunately, we do not know if that holds.

Conclusion

Throughout this thesis, we have pointed out several connections between classes of delta-matroids. Most importantly, we have proven that strictly matching-realizable delta-matroids are linear and showed an example of a linear delta-matroid that is not matching-realizable. To summarize and give a better understanding of how the classes of even, linear, and (strictly) matching-realizable delta-matroids interact, we provide the following diagram.

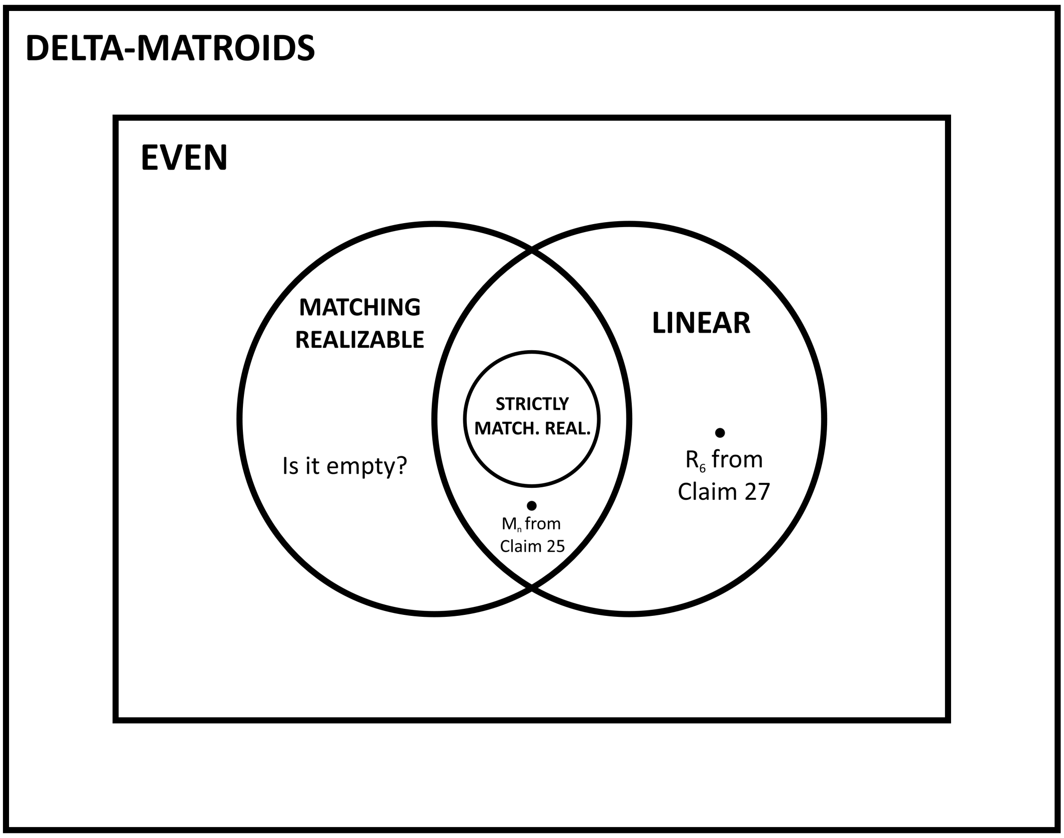

Figure 5.3: Diagram of relations between classes of delta-matroids.

If we confine ourselves to the arity at most 5, we obtain that the classes of even, linear, and matching-realizable delta-matroids coincide. Studying the delta-matroids of small arities led us to believe that the class of matching-realizable delta-matroids is included in the class of linear delta-matroids, but this conjecture remains open for further studies.

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