Solutions to Some Open Problems on Totally Ordered Monoids

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Abstract

In this paper solutions to three open problems on ordered commutative monoids posed in [4] are presented. By an ordered monoid we always mean a totally ordered monoid. All the problems are related to the class of ordered commutative monoids which are homomorphic images of ordered free commutative monoids.

1 Introduction

Various classes of residuated lattices form equivalent algebraic semantics for many non-classical logics [6]. Those which are representable (i.e. subdirect products of chains) correspond to fuzzy logics [2, 3, 7]. In order to characterize the structure of such algebras we have to understand those which are totally ordered. Since the ℓ-monoidal reduct of any totally ordered residuated lattice forms an ordered monoid, it is clear that a better understanding of ordered monoids will lead to a better understanding of representable residuated lattices. The paper [4] posed several important open questions concerning the structure of ordered monoids. All the questions concern an interesting class of ordered monoids whose structure seems to be more accessible. Our main aim in this paper is to answer those questions.

All monoids considered in this paper are commutative. A *lattice ordered monoid* (ℓ-monoid) is an algebra $\mathbf{S} = (S, +, \wedge, \vee, 0)$, where $(S, +, 0)$ is a monoid, $(S, \wedge, \vee)$ is a lattice and the identity $a + (b \vee c) = (a + b) \vee (a + c)$ is valid in $\mathbf{S}$. An ℓ-monoid $\mathbf{S} = (S, +, \wedge, \vee, 0)$ is called *positive* if $0 \leq s$ for all $s \in S$. On the

other hand, if $s \leq 0$ for all $s \in S$ then $\mathbf{S}$ is called *negative*. If the lattice reduct forms a chain then we call $\mathbf{S}$ an *ordered monoid*.

The commutative free monoid over the set of generators I will be denoted by $F_{\mathbb{N}}(I)$. Similarly we denote the free Abelian group $F_{\mathbb{Z}}(I)$ and free vector space $F_{\mathbb{Q}}(I)$ generated by I. Elements of $F_{\mathbb{N}}(I)$ (resp. $F_{\mathbb{Z}}(I)$, $F_{\mathbb{Q}}(I)$) are mappings from I to $\mathbb{N}$ (resp. $\mathbb{Z}$, $\mathbb{Q}$) whose supports are finite. We have clearly $F_{\mathbb{N}}(I) \subseteq F_{\mathbb{Z}}(I) \subseteq F_{\mathbb{Q}}(I)$. Whenever we say that $\leq$ is a total order on $F_{\mathbb{N}}(I)$, $F_{\mathbb{Z}}(I)$, $F_{\mathbb{Q}}(I)$ respectively, we always mean a total order which makes the underlying monoid an ordered monoid.

Now we can define the above-mentioned interesting class of ordered monoids introduced in [4]. An ordered monoid $\mathbf{S}$ is called *formally integral* if there is an ordered monoid $\mathbf{M}$ whose monoidal reduct is free (i.e. $M = F_{\mathbb{N}}(I)$) such that $\mathbf{S}$ is a homomorphic image (as an ℓ-monoid) of $\mathbf{M}$.

There is a useful criterion of formal integrality. It is based on the fact that any total order on $F_{\mathbb{N}}(I)$ can be uniquely extended to a total order on $F_{\mathbb{Q}}(I)$ and each total order on $F_{\mathbb{Q}}(I)$ can be described by means of a convex cone (for details see [4]). A subset $C \subseteq F_{\mathbb{Q}}(I)$ is called a *convex cone* if for all non-negative rational numbers α, β we have $\alpha C + \beta C \subseteq C$. In addition, if C contains no proper subspace then C is said to be *pointed*. Let $\mathbf{S}$ be an ordered monoid and $\phi : F_{\mathbb{N}}(I) \to \mathbf{S}$ a monoidal homomorphism. Define the following set

$$D(\phi) = \{y - x \in F_{\mathbb{Z}}(I) \mid x, y \in F_{\mathbb{N}}(I) \text{ and } \phi(x) < \phi(y)\}.$$

The convex cone in $F_{\mathbb{Q}}(I)$ generated by $D(\phi)$ will be denoted $C(\phi)$. The following theorem describes the criterion of formal integrality.

Theorem 1.1 ([4]) *For any ordered monoid $\mathbf{S}$, the following are equivalent:*

1. $\mathbf{S}$ *is formally integral,*

2. *for some surjective monoidal homomorphism $\phi \colon F_{\mathbb{N}}(I) \to \mathbf{S}$, $C(\phi) \subseteq F_{\mathbb{Q}}(I)$ is pointed,*

3. *for all monoidal homomorphisms $\phi \colon F_{\mathbb{N}}(I) \to \mathbf{S}$, $C(\phi) \subseteq F_{\mathbb{Q}}(I)$ is pointed,*

Further, recall other useful results from [4].

Theorem 1.2 ([4]) *First, the class of formally integral ordered monoids is closed under taking subalgebras and homomorphic images. Second, each cancellative ordered monoid is formally integral.*

Finally, we also present an example of ordered monoid which is not formally integral because we will need it later on. Let $C \subseteq \mathbb{N}$ and $d \in \mathbb{N}$. By symbol $\langle C \rangle$ we denote the sub-ℓ-monoid of $(\mathbb{N}, +, \leq, 0)$ generated by C. Moreover, $\langle C \rangle / d$ denotes its quotient where all elements greater than or equal to d are identified to one element denoted ∞. If $C = \{a_1, \ldots, a_n\}$ then we write $\langle a_1, \ldots, a_n \rangle$ instead of $\langle \{a_1, \ldots, a_n\} \rangle$.

Example 1.3 ([4]) Let $S = \{32^*\} \cup \langle 9, 12, 16 \rangle / 30$ denote the totally ordered monoid obtained from $\langle 9, 12, 16 \rangle / 30$ by adding one additional element, denoted by 32^*. This element satisfies $16 + 16 = 32^*$, $32^* + z = \infty$ for $z \neq 0$, and the whole monoid is to be ordered as follows:

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty \,.$$

All the relations that do not involve 32^* are as in $\langle 9, 12, 16 \rangle / 30$. This ordered monoid is not formally integral (for the proof see [4]).

2 Solutions to open problems

As we already mentioned in the introduction, the paper [4] posed several open questions. The first one was presented as possibly one of the most basic problems in the structure theory of ordered monoids. Before we formulate the question we have to recall the notion of an Archimedean class. We say that a positive ordered monoid $\mathbf{S}$ is *Archimedean* if for each $0 < x \leq y$ there is $n \in \mathbb{N}$ such that $nx \geq y$. If a positive ordered monoid $\mathbf{S}$ is not Archimedean we can define an equivalence splitting $\mathbf{S}$ into Archimedean ordered monoids. Let $\sim$ be the equivalence defined by

$$x \sim y \text{ iff there is } n \in \mathbb{N} \text{ such that } x \leq y \leq nx \text{ or } y \leq x \leq ny.$$

The equivalence classes are called *Archimedean classes* of $\mathbf{S}$. Observe that if C is an Archimedean class of $\mathbf{S}$ then $C \cup \{0\}$ is an Archimedean sub-ℓ-monoid of $\mathbf{S}$. Dually, we can do the same for negative ordered monoids. Let $\mathbf{S}$ be a positive ordered monoid and C one of its Archimedean classes. We say that C is formally integral if the sub-ℓ-monoid $C \cup \{0\}$ of $\mathbf{S}$ is formally integral.

Problem 1 If $\mathbf{S}$ is a positive ordered monoid and each Archimedean class of $\mathbf{S}$ is formally integral, does it follow that $\mathbf{S}$ is formally integral?

We construct an example of a finite ordered monoid which is not formally integral. Thus we obtain the following negative answer to Problem 1.

Theorem 2.1 *There is a positive ordered monoid $\mathbf{S}$ whose Archimedean classes are formally integral but $\mathbf{S}$ fails to be formally integral.*

PROOF: Let $\mathbf{A}_1 = (A_1, +_1, 0, \leq_1)$ be the ordered monoid $\langle 9, 12, 16 \rangle / 30$ and $\mathbf{A}_2 = (A_2, +_2, 0, \leq_2)$ the ordered monoid $\langle 9, 12, 16 \rangle / 30 \cup \{32^*\}$ from Example 1.3. Observe that $A_1 = A_2 \setminus \{32^*\}$. We define

$$S = \left(\{0\} \times A_1 \right) \cup \left(\{1\} \times (A_2 \setminus \{0\}) \right) .$$

The elements of S are ordered lexicographically where the order on $\{0, 1\}$ is the usual one, i.e. $\langle a, b \rangle \leq \langle x, y \rangle$ iff $a < x$ or $a = x$ and $b \leq_1 y$ (resp. $b \leq_2 y$) for

$a = 0$ (resp. $a = 1$). The monoidal operation is defined as follows:

$$\langle a, b\rangle + \langle x, y\rangle = \begin{cases} \langle 0, b +_1 y\rangle & \text{if } a = x = 0\,, \\ \langle 1, \infty\rangle & \text{if } a = x = 1\,, \\ \langle 1, b +_2 y\rangle & \text{otherwise.} \end{cases}$$

The operation $+$ is clearly commutative. It is easy to see that $\langle 0, 0\rangle$ is a neutral element. We check that $+$ is associative. Let $\langle a, b\rangle, \langle c, d\rangle, \langle x, y\rangle \in S$. If at least two of a, c, x equal 1, then

$$(\langle a, b\rangle + \langle c, d\rangle) + \langle x, y\rangle = \langle a, b\rangle + (\langle c, d\rangle + \langle x, y\rangle) = \langle 1, \infty\rangle\,.$$

If $a = c = x = 0$, then all the elements are from a subuniverse isomorphic to $\mathbf{A}_1$. Thus associativity holds, because it holds in $\mathbf{A}_1$.

Suppose that $a = x = 0$ and $c = 1$. Then we have

$$(\langle a, b\rangle + \langle c, d\rangle) + \langle x, y\rangle = \langle 1, b +_2 d +_2 y\rangle = \langle 0, b\rangle + \langle 1, d +_2 y\rangle = \langle a, b\rangle + (\langle c, d\rangle + \langle x, y\rangle)\,.$$

Finally, assume that $a = 1$ and $c = x = 0$. Then $b \neq 0$. Observe that if $d +_2 y \neq 32^*$, then $d +_2 y = d +_1 y$. Suppose first that $d +_2 y \neq 32^*$. Then we have

$$(\langle a, b\rangle + \langle c, d\rangle) + \langle x, y\rangle = \langle 1, b +_2 d +_2 y\rangle$$
$$= \langle 1, b\rangle + \langle 0, d +_2 y\rangle = \langle 1, b\rangle + \langle 0, d +_1 y\rangle = \langle a, b\rangle + (\langle c, d\rangle + \langle x, y\rangle)\,.$$

Now assume that $d +_2 y = 32^*$. Then $d = y = 16$ because $d, y \in A_1$ and $b +_2 d +_2 y = \infty$ since $b \neq 0$. Thus we have

$$(\langle a, b\rangle + \langle c, d\rangle) + \langle x, y\rangle = (\langle 1, b\rangle + \langle 0, 16\rangle) + \langle 0, 16\rangle = \langle 1, \infty\rangle$$
$$= \langle 1, b\rangle + \langle 0, \infty\rangle = \langle 1, b\rangle + (\langle 0, 16\rangle + \langle 0, 16\rangle) = \langle a, b\rangle + (\langle c, d\rangle + \langle x, y\rangle)\,.$$

The case when $a = c = 0$ and $x = 1$ can be proved by the previous case and commutativity of $+$.

Next we have to show that $+$ is monotone. Let $\langle a, b\rangle \leq \langle c, d\rangle$ and $\langle x, y\rangle \in S$. The only interesting cases are $a = c = 0$, $x = 1$ and $a = c = 1$, $x = 0$. We will check the first one. The second one is similar. Assume that $a = c = 0$ and $x = 1$. Then $b \leq_1 d$ which implies also $b \leq_2 d$. Thus we have

$$\langle 0, b\rangle + \langle 1, y\rangle = \langle 1, b +_2 y\rangle \leq \langle 1, d +_2 y\rangle = \langle 0, d\rangle + \langle 1, y\rangle\,.$$

We have proved that $\mathbf{S} = (S, +, \leq, \langle 0, 0\rangle)$ is an ordered monoid. Observe that $\mathbf{S}$ has three Archimedean classes, namely $C_1 = \{\langle 0, 0\rangle\}$, $C_2 = \{0\} \times (A_1 \setminus \{0\})$, and $C_3 = \{1\} \times (A_2 \setminus \{0\})$. The first one C_1 forms obviously a formally integral ordered monoid. The second one C_2 is formally integral since $\mathbf{A}_1$ is a formally integral ordered monoid (it is a quotient of a sub-ℓ-monoid of $\mathbb{N}$; see Theorem 1.2). We show that C_3 is formally integral. To see this let $f \colon C_3 \to \mathbb{N}$ be any order-preserving one-to-one mapping such that $f(C_3) \subseteq [n, 2n]$ for

some sufficiently large $n \in \mathbb{N}$. Then $C_3 \cup \{\langle 0, 0 \rangle\}$ is isomorphic to the quotient $\langle f(C_3) \rangle / f(1, \infty)$ of the sub-ℓ-monoid of $\mathbb{N}$ generated by $f(C_3)$ hence formally integral again by Theorem 1.2.

Finally, we will show that $\mathbf{S}$ is not formally integral. Let $\phi \colon \mathbb{N}^6 \to S$ be the monoidal homomorphism mapping the generators of $\mathbb{N}^6$ respectively to $\langle 0, 9 \rangle$, $\langle 0, 12 \rangle$, $\langle 0, 16 \rangle$, $\langle 1, 9 \rangle$, $\langle 1, 12 \rangle$, $\langle 1, 16 \rangle$. Then we have the following relations:

$$\begin{aligned}
\phi(0,1,0,0,1,0) = \langle 1, 24 \rangle &< \langle 1, 25 \rangle = \phi(0,0,1,1,0,0) \\
\phi(2,0,0,1,0,0) = \langle 1, 27 \rangle &< \langle 1, 28 \rangle = \phi(0,1,0,0,0,1) \\
\phi(0,0,1,0,0,1) = \langle 1, 32^* \rangle &< \langle 1, \infty \rangle = \phi(2,0,0,0,1,0)
\end{aligned}$$

Thus $\langle 0, -1, 1, 1, -1, 0 \rangle, \langle -2, 1, 0, -1, 0, 1 \rangle, \langle 2, 0, -1, 0, 1, -1 \rangle \in D(\phi)$. If we sum the first two tuples, we obtain $\langle -2, 0, 1, 0, -1, 1 \rangle \in C(\phi)$ which is the opposite of the third one showing that $C(\phi)$ is not pointed. Hence $\mathbf{S}$ is not formally integral by Theorem 1.1. □

The second problem concerns the structure of positive Archimedean formally integral ordered monoids. Let $\mathbf{R}_{\geq 0}$ denote the ordered monoid of non-negative reals with addition.

Problem 2 Is every positive formally integral Archimedean ordered monoid a quotient of a sub-ℓ-monoid of $\mathbf{R}_{\geq 0}$?

Consider an ordered Abelian group constructed as the lexicographic product of two copies of the additive group of real numbers $\mathbf{R}^2_{\text{lex}}$. Since $\mathbf{R}^2_{\text{lex}}$ is cancellative as a monoid, each sub-ℓ-monoid of $\mathbf{R}^2_{\text{lex}}$ is formally integral. Then the sub-ℓ-monoid $\mathbf{S}$ generated by $\langle 1, 0 \rangle, \langle 1, 1 \rangle$ is Archimedean, positive, and formally integral. Moreover, the pair $\langle 1, 0 \rangle, \langle 1, 1 \rangle$ forms a so-called anomalous pair[1] because $\langle n, 0 \rangle <_{\text{lex}} \langle n+1, n+1 \rangle$ and $\langle n, n \rangle <_{\text{lex}} \langle n+1, 0 \rangle$ for all $n > 0$. Assume that $\mathbf{S}$ is a quotient of a sub-ℓ-monoid $\mathbf{M}$ of $\mathbf{R}_{\geq 0}$ (i.e. $\mathbf{S} = \mathbf{M}/\theta$). Let $a/\theta = \langle 1, 0 \rangle$ and $b/\theta = \langle 1, 1 \rangle$ for some $a, b \in M$. Then $\mathbf{M}$ contains an anomalous pair as well, namely a, b, since θ is a lattice congruence, i.e. $x/\theta < y/\theta$ implies $x < y$. As every sub-ℓ-monoid of $\mathbf{R}_{\geq 0}$ contains no anomalous pairs (see [5, Page 167, Theorem 4]), we obtain a contradiction. Summing up we obtain the following statement.

Theorem 2.2 *The ordered monoid* $\mathbf{S}$ *is Archimedean, positive, and formally integral but it is not a quotient of a sub-ℓ-monoid of* $\mathbf{R}_{\geq 0}$.

The last question concerns so-called divisible negative ordered monoids. A negative ordered monoid $\mathbf{S}$ is called *divisible* if $x \leq y$ implies $x = y + z$ for some $z \in S$.

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[1] Let $\mathbf{S} = (S, +, \leq, 0)$ be a positive ordered monoid. Two distinct elements $a, b \in S \setminus \{0\}$ are said to form an anomalous pair if $na < (n+1)b$ and $nb < (n+1)a$ for all $n > 0$ (see [5]).

Problem 3 Is every divisible negative ordered monoid formally integral?[2]

We are not able to answer the question completely. Nevertheless, we can do it in the case when the considered ordered monoids are residuated. From the point of view of the theory of residuated lattices, this is in fact the only interesting case. An ordered monoid is said to be *residuated* if each inequality $a + x \leq b$ has a maximal solution for x. The structure of divisible negative residuated ordered monoids can be described by means of the ordinal sum which is defined as follows.

Definition 2.3 Let $\langle J, \prec \rangle$ be a chain. For all $j \in J$, let $\mathbf{A}_j = (A_j, +_{A_j}, \leq_{A_j}, 0)$ be a negative ordered monoid such that for $j \neq k$, $A_j \cap A_k = \{0\}$. Then $\mathbf{A} = \bigoplus_{j \in J} \mathbf{A}_j$ (the *ordinal sum* of the family $\{\mathbf{A}_j \mid j \in J\}$) is a negative ordered monoid whose universe is $A = \bigcup_{j \in J} A_j$, the monoidal operation is defined by

$$
x +_A y = \begin{cases} x +_{A_j} y & \text{if } x, y \in A_j, \\ y & \text{if } x \in A_j \text{ and } y \in A_k \setminus \{0\} \text{ with } j \succ k, \\ x & \text{if } x \in A_j \setminus \{0\} \text{ and } y \in A_k \text{ with } j \prec k, \end{cases}
$$

and the order is defined as follows

$$
x \leq_A y \text{ iff } x \leq_{A_j} y \text{ for } x, y \in A_j \text{ or } x \in A_j \setminus \{0\}, \ y \in A_k \text{ and } j \prec k.
$$

For every $j \in J$, $\mathbf{A}_j$ is called a *component* of the ordinal sum.

Let $\mathbf{G}$ be an ordered Abelian group and $\mathbf{G}^-$ the ordered monoid on its negative cone. Given $a \in G^-$, we define an ordered monoid $\mathbf{G}^-/a$ as the homomorphic image of $\mathbf{G}^-$ by the homomorphism $x \mapsto a \vee x$. The resulting monoid $\mathbf{G}^-/a$ is clearly formally integral by Theorem 1.2. In order to answer Problem 3, we employ here the following crucial result.

Theorem 2.4 ([1]) *Every divisible negative residuated ordered monoid[3] is isomorphic to $\bigoplus_{j \in J} \mathbf{A}_j$ and for each $\mathbf{A}_j$ there is an ordered Abelian group $\mathbf{G}_j$ such that $\mathbf{A}_j$ is isomorphic either to $\mathbf{G}_j^-$ or to $\mathbf{G}_j^-/a$ for some $a \in G_j^- \setminus \{0\}$.*

In the light of Theorem 2.4 it is obvious that if we prove that any ordinal sum of formally integral negative ordered monoids is formally integral, we will obtain as a corollary the partial affirmative answer to Problem 3 mentioned above.

Theorem 2.5 *Let $(J, \prec)$ be a chain and $\mathbf{A} = \bigoplus_{j \in J} \mathbf{A}_j$ an ordinal sum of negative ordered monoids. Then $\mathbf{A}$ is formally integral if, and only if, all components $\mathbf{A}_j$ are formally integral.*

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[2]We formulate here in fact a dual question to that of given in [4]. Both formulations are equivalent. The reason for this is that we use some results from the theory of residuated lattices which are usually formulated in the dual setting.

[3]Divisible negative residuated ordered monoids are called totally ordered basic hoops in [1].

PROOF: One direction is trivial since formal integrality is inherited by sub-ℓ-monoids. Conversely, suppose that all components $\mathbf{A}_j$ are formally integral. Then for each $j \in J$ there is an ordered monoid $\mathbf{M}_j = (F_\mathbb{N}(I_j), +, \leq_j, 0)$ whose monoidal reduct is a free monoid $F_\mathbb{N}(I_j)$ and a surjective homomorphism $\phi_j \colon F_\mathbb{N}(I_J) \to A_j$. Without any loss of generality we can assume that all $\mathbf{M}_j$'s are negative because all generators of $F_\mathbb{N}(I_j)$ greater than 0 have to be mapped to 0 by ϕ_j. Let I be the disjoint union of I_j's. We will prove that $\mathbf{A}$ is formally integral by showing that there is a total order $\leq$ on $F_\mathbb{N}(I)$ and a surjective monoidal homomorphism $\phi \colon F_\mathbb{N}(I) \to A$ preserving the order, i.e. $x \leq y$ implies $\phi(x) \leq_A \phi(y)$.

Consider the direct product $\prod_{j \in J} F_\mathbb{N}(I_j)$ of the monoids $F_\mathbb{N}(I_j)$. For each $k \in J$ we have the projection homomorphism $\pi_k \colon \prod_{j \in J} F_\mathbb{N}(I_j) \to F_\mathbb{N}(I_k)$. The free monoid $F_\mathbb{N}(I)$ can be viewed as a subalgebra of $\prod_{j \in J} F_\mathbb{N}(I_j)$ such that $a \in F_\mathbb{N}(I)$ iff $a(j) \neq 0$ only for finitely many $j \in J$. Due to this fact it is possible to order $F_\mathbb{N}(I)$ lexicographically, namely

$a < b$ iff $\pi_k(a) <_k \pi_k(b)$, where $k \in J$ is the least element such that
$$\pi_k(a) \neq \pi_k(b).$$

It is easy to see that the relation $\leq$ is a total order making the monoid $F_\mathbb{N}(I)$ a negative ordered monoid.

For each $a \in F_\mathbb{N}(I) \setminus \{0\}$, let $j_a \in J$ be the least element such that $a(j) \neq 0$. Now we define the mapping $\phi \colon F_\mathbb{N}(I) \to A$ by letting $\phi(a) = \phi_{j_a}(\pi_{j_a}(a))$ for $a \neq 0$ and $\phi(0) = 0$. First, we prove that ϕ is a surjective monoidal homomorphism. Observe that $j_{a+b} = \min\{j_a, j_b\}$. Let $a, b \in F_\mathbb{N}(I) \setminus \{0\}$ such that $a \leq b$, i.e. $j_{a+b} = j_a$. Then

$$\phi(a + b) = \phi_{j_{a+b}}(\pi_{j_{a+b}}(a + b)) =$$
$$= \phi_{j_{a+b}}(\pi_{j_{a+b}}(a)) +_{A_{j_{a+b}}} \phi_{j_{a+b}}(\pi_{j_{a+b}}(b)) = \phi_{j_a}(\pi_{j_a}(a)) +_A \phi_{j_a}(\pi_{j_a}(b)),$$

since $\phi_{j_{a+b}}$ and $\pi_{j_{a+b}}$ are monoidal homomorphisms. Now, suppose that $j_a = j_b$. Then $\phi(a)$ and $\phi(b)$ are in the same component and

$$\phi_{j_a}(\pi_{j_a}(a)) +_A \phi_{j_a}(\pi_{j_a}(b)) = \phi_{j_a}(\pi_{j_a}(a)) +_A \phi_{j_b}(\pi_{j_b}(b)) = \phi(a) +_A \phi(b).$$

Second, assume that $j_a < j_b$. Then $\phi(a)$ and $\phi(b)$ are in different components. Then
$$\phi_{j_a}(\pi_{j_a}(a)) +_A \phi_{j_a}(\pi_{j_a}(b)) = \phi(a) +_A 0 = \phi(a) +_A \phi(b).$$

The fact that ϕ is surjective follows easily from surjectivity of ϕ_j's.

Finally, we show that ϕ is order-preserving. Let $a, b \in F_\mathbb{N}(I) \setminus \{0\}$ and $a \leq b$. There are two cases. First, suppose that $j_a \prec j_b$. Then $\phi(a) \in A_{j_a}$ and $\phi(b) \in A_{j_b}$, i.e. $\phi(a) <_A \phi(b)$ by the definition of the order in the ordinal sum. Second, assume that $j_a = j_b$. Then $\pi_{j_a}(a) \leq_{j_a} \pi_{j_a}(b)$ and $\phi(a), \phi(b) \in A_{j_a}$. Consequently,

$$\phi(a) = \phi_{j_a}(\pi_{j_a}(a)) \leq_{A_{j_a}} \phi_{j_a}(\pi_{j_a}(b)) = \phi(b),$$

i.e. $\phi(a) \leq_A \phi(b)$. $\qquad \square$

Corollary 2.6 *Every divisible negative residuated ordered monoid is formally integral.*

Acknowledgement

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