PRODUCT ŁUKASIEWICZ LOGIC

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Abstract. Łukasiewicz logic plays a fundamental role among many-valued logics. However, the expressive power of this logic is restricted to piecewise linear functions. In this paper we enrich the language of Łukasiewicz logic by adding a new connective which expresses multiplication. The resulting logic, PŁ, is defined, developed, and put into the context of other well-known many-valued logics. We also deal with several extensions of this propositional logic. A predicate version of PŁ logic is introduced and developed too.

1. Introduction

Łukasiewicz logic [16, 11] is one of the most important logics in the broad family of many-valued logics. Its corresponding algebraic structures of truth values (MV-algebras) are well-known and deeply studied. Mundici's famous result [2] established an important correspondence between MV-algebras and Abelian l-groups with strong unit. There is an obvious question if there is a logic, whose corresponding algebras of truth values are in the analogous correspondence with l-rings.

There are several papers dealing with so-called product MV-algebras. Montagna's papers [17, 18, 20] are fundamental to our aims. There is also a paper by Di Nola and Dvurečenskij [5]. A product MV-algebra (PMV-algebra for short) is an MV-algebra enriched by a product operation in such a way that the resulting structures correspond to the f-rings with strong unit. In [17], Montagna proved the subdirect representation theorem for PMV-algebras and established a correspondence between linearly ordered f-rings with strong unit and linearly ordered PMV-algebras. Later in [18], he introduced PMV$_\triangle$-algebras (PMV-algebras enriched by the 0-1 projector $\triangle$) and proved the categorical equivalence between PMV$_\triangle$-algebras and certain extension of f-rings (so-called δ-f-rings). Finally in [20], it was shown by Montagna and Panti that the variety of PMV$_\triangle$-algebras is generated by the standard PMV$_\triangle$-algebra (over the real unit interval).

In the forthcoming paper [19], Montagna introduced a quasi-variety $\mathbf{PMV}^+$ containing only the PMV-algebras without non-trivial zero-divisors and showed that $\mathbf{PMV}^+$ is generated by the standard PMV-algebra (over the real unit interval).

However, so far there is no logic corresponding to the all above-mentioned algebras. The main aim of this paper is to define and develop such a logic. Our logic, which corresponds to PMV-algebras, is called PŁ logic. Further, we introduce PŁ$'$ logic corresponding to the algebras from $\mathbf{PMV}^+$. We also study extensions of PŁ and PŁ$'$ logics by Baaz's $\triangle$ (PŁ$_\triangle$ and PŁ$'_\triangle$ logics). The algebras of truth values of PŁ$'_\triangle$ logic correspond to PMV$_\triangle$-algebras. This, together with the fact that there are also several other different algebraic structures called PMV-algebras, is the reason why we call PŁ-algebras the algebras of truth values corresponding to PŁ logic. Analogously, we introduce PŁ$'$-algebras, PŁ$_\triangle$-algebras, and PŁ$'_\triangle$-algebras.

We use the above-mentioned algebraic results to obtain completeness of all of these logics and standard completeness for PŁ$'$ and PŁ$'_\triangle$ logic. Further, we show an example of the PŁ-algebra which demonstrates that PŁ logic is not standard complete.

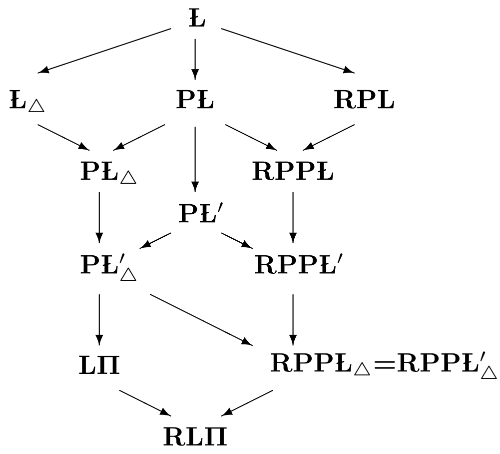

Figure 1. Relations between logics of this paper.

Then we show a relation of our logics to the well-known ŁΠ logic which was defined in [7]. Roughly speaking, the logic ŁΠ is the extension of PŁ′ by the product residuum.

Furthermore, we extend these logics by rational constants in the same way as the Rational Pavelka's logic (RPL) extends Łukasiewicz logic (see [22], [21], and [11, Section 3.3]). We obtain RPPŁ, RPPŁ′, RPPŁ$_\triangle$, and RPPŁ$'_\triangle$ logics. We prove Pavelka-style completeness of these logics and show that the logics RPPŁ$_\triangle$ and RPPŁ$'_\triangle$ coincide. Further, we prove standard completeness of RPPŁ$_\triangle$ and show the relation of these logics to RŁΠ (the extension of ŁΠ by rational constants; see [7]) and RPL.

Then we investigate the predicate versions of all logics mentioned above with the exception of PŁ′ (the problem is that we can prove only completeness of this logic w.r.t. all PŁ′-algebras, but we are not able to prove it w.r.t. linearly ordered PŁ′-algebras). We prove completeness for PŁ∀, PŁ$_\triangle$∀ and PŁ$'_\triangle$∀, Pavelka style completeness for RPPŁ∀, RPPŁ$_\triangle$∀ and even standard completeness for RPPŁ$_\triangle$∀. Then we deal with the arithmetical complexity of the set of tautologies of these logics which entails that PŁ∀, PŁ$_\triangle$∀ and PŁ$'_\triangle$∀ logics do not have the standard completeness property. Finally, we show a relation of these logics to the predicate versions of ŁΠ and RŁΠ logics (which were introduced in [3]) and the well-known logic of Takeuti and Titani [23].

All logics considered in this paper lie between Łukasiewicz logic and RŁΠ logic [3, 7]. Their mutual relations are depicted in Figure 1.

2. Preliminaries

In this section we summarize the basic notions and results from several systems of propositional fuzzy logic that will be used throughout this paper.

2.1. **Łukasiewicz logic and Łukasiewicz logic with △.** Łukasiewicz logic was introduced in [16]. In this paper we will understand this logic as an extension of the basic logic BL introduced by Hájek in [11]. As usual, the language of BL contains a set of propositional variables, a conjunction $\otimes$, an implication $\to$, and the constant $\overline{0}$. Further connectives are defined as follows:

$$\begin{array}{lll} \varphi \wedge \psi & \text{is} & \varphi \otimes (\varphi \to \psi), \\ \varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \neg\varphi & \text{is} & \varphi \to \bar{0}, \\ \varphi \equiv \psi & \text{is} & (\varphi \to \psi) \otimes (\psi \to \varphi), \\ \varphi \oplus \psi & \text{is} & \neg\varphi \to \psi, \\ \varphi \ominus \psi & \text{is} & \varphi \otimes \neg\psi, \\ \bar{1} & \text{is} & \neg\bar{0}. \end{array}$$

The following formulas are the *axioms* of BL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(A2) $\varphi \otimes \psi \to \varphi$,
(A3) $\varphi \otimes \psi \to \psi \otimes \varphi$,
(A4) $\varphi \otimes (\varphi \to \psi) \to \psi \otimes (\psi \to \varphi)$,
(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \otimes \psi \to \chi)$,
(A5b) $(\varphi \otimes \psi \to \chi) \to (\varphi \to (\psi \to \chi))$,
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,
(A7) $\bar{0} \to \varphi$.

The only *deduction rule* of BL is modus ponens.

The notions of *theory, proof, provability,* and *theorem* are defined as usual. We use also the notion of *complete theory* which is defined as follows: theory T is *complete* if for each pair φ, ψ of formulas, $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$. These notions will be used in the same meaning through out the paper up to several exceptions which will be mentioned explicitly.

In [11] it was shown that the Łukasiewicz logic, denoted by Ł, is the extension of BL by the axiom

(Ł) $\neg\neg\varphi \to \varphi$.

Now we recall several theorems of Łukasiewicz logic used in the sequel (see [11]).

PROPOSITION 2.1. *In Ł the following formulas are provable:*

(H1) $\varphi \to (\psi \to \varphi \otimes \psi)$,
(H2) $(\varphi \to \psi) \to (\varphi \otimes \chi \to \psi \otimes \chi)$,
(H3) $((\varphi_1 \to \psi_1) \otimes (\varphi_2 \to \psi_2)) \to (\varphi_1 \otimes \varphi_2 \to \psi_1 \otimes \psi_2)$,
(H4) $\varphi \to \varphi \vee \psi$, $\varphi \vee \psi \to \psi \vee \varphi$,
(H5) $\varphi \wedge \psi \to \varphi$, $\varphi \wedge \psi \to \psi \wedge \varphi$, $\varphi \otimes \psi \to \varphi \wedge \psi$,
(H6) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to (\varphi \vee \psi \to \chi)$,
(H7) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$,
(H8) $(\bar{1} \to \varphi) \equiv \varphi$,
(H9) $(\varphi \to \psi) \to (\varphi \to \varphi \wedge \psi)$.

In [11], Hájek also studies the extension of Łukasiewicz logic by the unary connective $\triangle$ (0-1 projector). The *axioms* of the extended Łukasiewicz logic $Ł_\triangle$ are those of Łukasiewicz logic Ł plus:

(Ł$\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$,
(Ł$\triangle$2) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$,
(Ł$\triangle$3) $\triangle\varphi \to \varphi$,
(Ł$\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$,
(Ł$\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$.

Deduction rules of $Ł_\triangle$ are modus ponens and *necessitation* for $\triangle$: from φ derive $\triangle\varphi$.

2.2. **MV-algebras and MV$_\triangle$-algebras.** Here we introduce the definition of MV-algebras. They represent the semantics of Łukasiewicz logic. By abuse of language, we use the same symbols to denote logical connectives and the corresponding algebraic operations.

DEFINITION 2.2. *An MV-algebra is a structure* $\mathbf{L} = (L, \oplus, \neg, \mathbf{0})$ *such that, letting* $x \ominus y = \neg(\neg x \oplus y)$*, and* $\mathbf{1} = \neg\mathbf{0}$ *the following conditionas are satisfied:*

(MV1) $(L, \oplus, \mathbf{0})$ *is a commutative monoid,*
(MV2) $x \oplus \mathbf{1} = \mathbf{1}$,
(MV3) $\neg\neg x = x$,
(MV4) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$.

In each MV-algebra, we define the additional connectives: $x \otimes y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$, $x \vee y = (x \ominus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$. We also define the relation $\leq$ as a natural lattice order with top element $\mathbf{1}$ and bottom element $\mathbf{0}$ w.r.t. the lattice operations $\vee$ and $\wedge$ (because $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ forms a lattice).

Now we list several useful claims (see [2]).

PROPOSITION 2.3. *In every MV-algebra, the following conditions hold:*

(1) $x \ominus \mathbf{0} = x$,
(2) $x \ominus x = \mathbf{0}$,
(3) $\mathbf{0} \ominus x = \mathbf{0}$,
(4) *the following conditions are equivalent:* $x \leq y$, $x \ominus y = \mathbf{0}$, $x \to y = \mathbf{1}$,
(5) *if* $x \leq y$, *then* $x \oplus z \leq y \oplus z$, $x \ominus z \leq y \ominus z$, *and* $z \ominus y \leq z \ominus x$,
(6) $(x \ominus y) \wedge (y \ominus x) = \mathbf{0}$,
(7) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,
(8) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
(9) $\mathbf{1} \ominus x = \neg x$,
(10) $a = (a \oplus \neg a) \ominus \neg a$.

The algebraic structures corresponding to Ł$_\triangle$ logic are so-called MV$_\triangle$-algebras (for details see [11]).

DEFINITION 2.4. *An* MV$_\triangle$*-algebra is a structure* $\mathbf{L} = (L, \oplus, \neg, \mathbf{0}, \mathbf{1}, \triangle)$ *such that* $(L, \oplus, \neg, \mathbf{0}, \mathbf{1})$ *is an MV-algebra and the unary operation* $\triangle$ *satisfies the following conditions:*

(△1) $\triangle x \vee \neg\triangle x = \mathbf{1}$,
(△2) $\triangle(x \vee y) \leq \triangle x \vee \triangle y$,
(△3) $\triangle x \leq x$,
(△4) $\triangle x \leq \triangle\triangle x$,
(△5) $\triangle(x \to y) \leq \triangle x \to \triangle y$,
(△6) $\triangle\mathbf{1} = \mathbf{1}$.

PROPOSITION 2.5. *In each* MV$_\triangle$*-algebra, we have:* $\triangle(x \otimes y) = \triangle x \wedge \triangle y$.

2.3. **ŁΠ-algebras and the ŁΠ logic.** In this subsection we recall the definitions of ŁΠ-algebras and ŁΠ logic. This logic was introduced by Esteva, Godo and Montagna in [7].

DEFINITION 2.6. *An* ŁΠ*-algebra is a structure:* $\mathbf{L} = (L, \oplus, \neg, \to_\Pi, \odot, \mathbf{0}, \mathbf{1})$ *such that:*

(1) $(L, \oplus, \neg, \triangle, \mathbf{0})$ is an MV$_\triangle$-algebra,
(2) $(L, \odot, \mathbf{1})$ is a commutative monoid,
(3) $x \odot (y \ominus z) = (x \odot y) \ominus (x \odot z)$,
(4) $\triangle(x \equiv y) \wedge \triangle(z \equiv t) \leq ((x * z) \equiv (y * t))$, where $* \in \{\to_\Pi, \odot\}$,
(5) $x \wedge (x \to_\Pi \mathbf{0}) = \mathbf{0}$,
(6) $\triangle(x \to y) \leq (x \to_\Pi y)$,
(7) $\triangle(y \to x) \leq (x \odot (x \to_\Pi y) \equiv y)$.

The operation $\triangle$ is defined by $\neg x \to_\Pi \mathbf{0}$ and $\ominus$, $\equiv$, $\to$, $\wedge$, $\otimes$ are defined as in the MV-algebra.

Now we define ŁΠ logic. The basic connectives of this logic are $\oplus$, $\neg$, $\odot$, $\to_\Pi$. The derived connectives are defined in the same way as the corresponding ŁΠ-algebraic operations.

DEFINITION 2.7. *The logic* ŁΠ *is given by the following axioms and deduction rules:*

(*Ł*Π1) Axioms of the Łukasiewicz logic with $\triangle$,
(*Ł*Π2) $\triangle(\varphi \equiv \psi) \wedge \triangle(\chi \equiv \delta) \to ((\varphi * \chi) \equiv (\psi * \delta))$, for $* \in \{\to_\Pi, \odot\}$,
(*Ł*Π3) $(\varphi \odot \psi) \to (\psi \odot \varphi)$,
(*Ł*Π4) $(\varphi \odot \psi) \odot \chi \equiv \varphi \odot (\psi \odot \chi)$,
(*Ł*Π5) $\varphi \wedge (\varphi \to_\Pi \overline{0}) \to \overline{0}$,
(*Ł*Π6) $\varphi \odot (\psi \ominus \chi) \equiv (\varphi \odot \psi) \ominus (\varphi \odot \chi)$,
(*Ł*Π7) $\triangle(\varphi \to \psi) \to (\varphi \to_\Pi \psi)$,
(*Ł*Π8) $\triangle(\psi \to \varphi) \to (\varphi \odot (\varphi \to_\Pi \psi) \equiv \psi)$.

The deduction rules are modus ponens and the necessitation of $\triangle$ (from φ infer $\triangle\varphi$).

The RŁΠ logic is given by the rules and axioms of the ŁΠ logic plus the axiom $\overline{\tfrac{\mathbf{1}}{\mathbf{2}}} \equiv \neg\overline{\tfrac{\mathbf{1}}{\mathbf{2}}}$ *and the infinitary deduction rule (IR): from $\varphi \to \overline{r}$ or each $r < 1$ infer φ, where $\overline{\tfrac{\mathbf{1}}{\mathbf{2}}}$ is a new basic nullary connective and for each $r \in \mathbb{Q} \cap [0,1]$, $\overline{r}$ is a derived nullary connective.*

The ŁΠ logic is sound and complete w.r.t. ŁΠ-algebras. The completeness of the ŁΠ logic was shown in [7]. A simplification of this axiomatic system was shown by one of the authors in [4].

THEOREM 2.8. *Let φ be a formula of ŁΠ. Then φ is a theorem of the ŁΠ logic iff φ is an $\mathbf{L}$-tautology w.r.t. each ŁΠ-algebra $\mathbf{L}$.*

THEOREM 2.9. *Each ŁΠ-algebra is subdirect product of linearly ordered ŁΠ-algebras.*

2.4. **Groups and rings.** In this section, we introduce *o*-monoids, *o*-groups, and *o*-rings which are the totally ordered *l*-monoids, *l*-groups, and *l*-rings respectively (see [9]).

DEFINITION 2.10. *The structure $(S,+,0,\leq)$ is an* *o*-monoid *if $(S,+,0)$ is commutative linearly ordered monoid with neutral element* 0 *and $x \leq y$ implies $x+z \leq y+z$.*

DEFINITION 2.11. *A linearly ordered Abelian group (*o*-group for short) is a structure $(G,+,0,-,\leq)$ such that $(G,+,0,-)$ is an Abelian group and the following is satisfied:*
(i) *$(G,\leq)$ is a linearly ordered lattice,*
(ii) *if $x \leq y$, then $x+z \leq y+z$ for all $z \in G$.*

DEFINITION 2.12. *A linearly ordered commutative ring with strong unit (*o*-ring for short) is a structure $(R,+,-,\times,0,1,\leq)$ such that $(R,+,0,-,\leq)$ is an o-group, $(R,+,-,\times,0,1)$ is a commutative ring with strong unit, and the following is satisfied: if $x \geq 0$ and $y \geq 0$, then $x \times y \geq 0$.*

DEFINITION 2.13. *Let $\mathcal{R} = (R,+,-,\times,0,1,\leq)$ be an o-ring and let $L = \{x \in R \mid 0 \leq x \leq 1\}$. For all $x,y \in L$ define $x \oplus y = \min\{1,(x+y)\}$ and $\neg x = 1-x$. By $\odot$ we denote the operation $\times$ restricted to L. Then the algebra $(L,\oplus,\odot,\neg,0,1)$ is called the* interval algebra *of $\mathcal{R}$.*

3. PŁ and PŁ′ logics

3.1. **Syntax – PŁ logic and PŁ′ logics.** In this section, we introduce the PŁ logic (PŁ for short), an extension of Łukasiewicz logic by a new binary connective

$\odot$. This connective plays the role of multiplication. Thus the basic connectives are $\otimes, \rightarrow, \odot, \bar{0}$. Additional connectives $\oplus, \ominus, \neg, \wedge, \vee, \equiv, \bar{1}$ are defined as in Łukasiewicz logic. We also introduce the PŁ$'$ logic (PŁ$'$ for short), an extension of PŁ by one additional deduction rule. The reason for this extension is that the PŁ logic does not possess the standard completeness property.

DEFINITION 3.1. *The axioms of PŁ logic are the axioms of Łukasiewicz logic, i.e. (A1)–(A7), (Ł), and the following axioms:*

(P1) $(\chi \odot \varphi) \ominus (\chi \odot \psi) \equiv \chi \odot (\varphi \ominus \psi)\,,$
(P2) $\varphi \odot (\psi \odot \chi) \equiv (\varphi \odot \psi) \odot \chi\,,$
(P3) $\varphi \rightarrow \varphi \odot \bar{1}\,,$
(P4) $\varphi \odot \psi \rightarrow \varphi\,,$
(P5) $\varphi \odot \psi \rightarrow \psi \odot \varphi\,.$

The only deduction rule is modus ponens.

The PŁ$'$ logic is obtained from PŁ by adding a new deduction rule (ZD): from $\neg(\varphi \odot \varphi)$ *infer* $\neg\varphi$.

It is obvious that all theorems of Łukasiewicz logic are also theorems of PŁ and all theorems of PŁ are also theorems of PŁ$'$.

Further, we show several useful theorems of PŁ logic. The most important one is theorem (TP4) stating that the connective $\equiv$ is a congruence w.r.t. the product $\odot$ (the fact that $\equiv$ is a congruence w.r.t. the other connectives is known from [11]).

LEMMA 3.2. *The following are theorems of PŁ logic:*

(TP1) $(\varphi \rightarrow \psi) \rightarrow (\varphi \odot \chi \rightarrow \psi \odot \chi)\,,$
(TP2) $(\varphi \equiv \psi) \rightarrow (\varphi \odot \chi \equiv \psi \odot \chi)\,,$
(TP3) $(\varphi_1 \rightarrow \psi_1) \otimes (\varphi_2 \rightarrow \psi_2) \rightarrow (\varphi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \psi_2)\,,$
(TP4) $(\varphi_1 \equiv \psi_1) \otimes (\varphi_2 \equiv \psi_2) \rightarrow (\varphi_1 \odot \varphi_2 \equiv \psi_1 \odot \psi_2)\,,$
(TP5) $\varphi \otimes \psi \rightarrow \varphi \odot \psi\,,$
(TP6) $(\varphi \wedge \psi) \odot \chi \equiv (\varphi \odot \chi) \wedge (\psi \odot \chi)\,.$

Proof. (***TP1***): We start with one direction of equivalence (P1) $(\chi \odot \varphi) \ominus (\chi \odot \psi) \rightarrow \chi \odot (\varphi \ominus \psi)$. By (P4) and (A1) we get $(\chi \odot \varphi) \ominus (\chi \odot \psi) \rightarrow (\varphi \ominus \psi)$. Using (H7) we obtain $\neg(\varphi \ominus \psi) \rightarrow \neg((\chi \odot \varphi) \ominus (\chi \odot \psi))$. This is what we want to prove (because $\neg(\varphi \ominus \psi) \equiv (\varphi \rightarrow \psi)$).

(***TP2***): We use (TP1) and (TP1) with φ, ψ exchanged. Using (H3) we get $(\varphi \rightarrow \psi) \otimes (\psi \rightarrow \varphi) \rightarrow (\varphi \odot \chi \rightarrow \psi \odot \chi) \otimes (\psi \odot \chi \rightarrow \varphi \odot \chi)$.

(***TP3***): Let us start with (TP1) $(\varphi_1 \rightarrow \psi_1) \rightarrow (\varphi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \varphi_2)$ and (TP1) again $(\varphi_2 \rightarrow \psi_2) \rightarrow (\psi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \psi_2)$. Now using (H3) we get $(\varphi_1 \rightarrow \psi_1) \otimes (\varphi_2 \rightarrow \psi_2) \rightarrow (\varphi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \varphi_2) \otimes (\psi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \psi_2)$. By axiom (A1) (after applying axiom (A5)) we get $(\varphi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \varphi_2) \otimes (\psi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \psi_2) \rightarrow (\varphi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \psi_2)$. Axiom (A1) completes the proof.

(***TP4***): This is an analogy of the proof (TP2), except that we use (TP3) instead of (TP1).

(***TP5***): We start with (TP1) in the form $(\bar{1} \rightarrow \varphi) \rightarrow (\bar{1} \odot \psi \rightarrow \varphi \odot \psi)$. By (H8) and (A5) we obtain $\varphi \otimes (\bar{1} \odot \psi) \rightarrow \varphi \odot \psi$. The rest is obvious.

(***TP6***): We start with the first direction: by (H9) and (TP1) we obtain $(\varphi \rightarrow \psi) \rightarrow (\varphi \odot \chi \rightarrow (\varphi \wedge \psi) \odot \chi)$. By (H5) and (A1) we get $(\varphi \rightarrow \psi) \rightarrow ((\varphi \odot \chi) \wedge (\psi \odot \chi) \rightarrow (\varphi \wedge \psi) \odot \chi)$. Analogously we get $(\psi \rightarrow \varphi) \rightarrow ((\varphi \odot \chi) \wedge (\psi \odot \chi) \rightarrow (\varphi \wedge \psi) \odot \chi)$. Axiom (A6) completes the proof of this direction.

Reverse direction: by (T1) and (H9) we get $(\varphi \to \psi) \to (\varphi \odot \chi \to (\varphi \odot \chi) \wedge (\psi \odot \chi))$. By (H5) and (TP1) we get $(\varphi \wedge \psi) \odot \chi \to \varphi \odot \chi$. By (A1) we get $(\varphi \to \psi) \to ((\varphi \wedge \psi) \odot \varphi \to (\varphi \odot \chi) \wedge (\psi \odot \chi))$. The rest of the proof is analogous to the first part. □ □

Since we have the same deduction rules in PŁ as in BL, we obtain also the same deduction theorem for PŁ. See [11, Theorem 2.2.18] for details.

However, in the PŁ$'$ logic the situation is quite different. After we introduce the semantics we show that the deduction in the same form as in PŁ does not hold.

THEOREM 3.3 (Deduction theorem). *Let T be a theory over PŁ and φ, ψ be formulas. Then $T \cup \{\varphi\} \vdash \psi$ iff there is an n such that $T \vdash \varphi^n \to \psi$, where $\varphi^n = \varphi \otimes \ldots \otimes \varphi$.*

Due to the same reasons as in the previous theorem, we may reread the proof of [11, Lemma 2.4.2] and get the following theorem for PŁ. We will prove an analogy of this theorem for PŁ$'$ at the end of the next section (cf. Corollary 3.19).

THEOREM 3.4. *Let T be a theory over PŁ and φ a formula such that $T \not\vdash \varphi$. Then there is a complete extension T' of T such that $T' \not\vdash \varphi$.*

3.2. **Semantics – PŁ-algebras.** Now we define the algebras corresponding to PŁ and PŁ$'$ logics – the PŁ-algebras. They coincide with PMV-algebras which were introduced in Montagna's paper [17]. Furthermore, PŁ-algebras are also a subclass of more general algebras introduced by Dvurečenskij and Di Nola in paper [5] (they do not require $\mathbf{1}$ to be a neutral element for the product $\odot$ and commutativity of $\odot$). However, we decided to use the name PŁ-algebras, because some authors use the name PMV-algebras for the different structures (e.g. pseudo MV-algebras).

The PŁ$'$-algebras coincide with PMV$^+$-algebras introduced in forthcoming Montagna's paper [19]. These are the subreducts of ŁΠ-algebras.

DEFINITION 3.5. *A PŁ-algebra is a structure $\mathbf{L} = (L, \oplus, \neg, \odot, \mathbf{0}, \mathbf{1})$, where the reduct $\mathbf{L}^* = (L, \oplus, \neg, \mathbf{0}, \mathbf{1})$ is an MV-algebra and the following identities hold:*

(1) $(a \odot b) \ominus (a \odot c) = a \odot (b \ominus c)\,,$
(2) $a \odot (b \odot c) = (a \odot b) \odot c\,,$
(3) $a \odot \mathbf{1} = a\,,$
(4) $a \odot b = b \odot a\,.$

where $a \ominus b = \neg(\neg a \oplus b) = a \otimes \neg b$ and $a \otimes b = \neg(\neg a \oplus \neg b)$. Moreover, we say that $\mathbf{L}$ is a PŁ$'$-algebra if it fulfills the following quasi-identity:

(5) *if $a \odot a = \mathbf{0}$ then $a = \mathbf{0}$.*

Observe that the PŁ-algebras form a variety and the PŁ$'$-algebras form a quasi-variety.

EXAMPLE 3.6. *If $([0,1], \oplus, \neg, 0, 1)$ is the standard MV-algebra (i.e. $x \oplus y = min(1, x + y)$ and $\neg x = 1 - x$) and $\odot$ is the usual algebraic product of reals then $[0,1]_S = ([0,1], \oplus, \neg, \odot, 0, 1)$ is called the* standard PŁ-algebra. *The standard PŁ-algebra $[0,1]_S$ is also PŁ$'$-algebra, thus we will also call it the* standard PŁ$'$-algebra.

Notice that a linearly ordered PŁ-algebra is PŁ$'$-algebra iff it has only trivial zero-divisors.

Now we recall several known results about PŁ and PŁ$'$-algebras. In [17, Lemma 4.3 (a)], Montagna showed that there is a correspondence between linearly ordered PŁ-algebras and o-rings. The analogous result for PŁ$'$-algebras is a consequence of [19, Corollary 4.3].

THEOREM 3.7. *An algebra $\mathbf{L}$ is a linearly ordered PŁ-algebra if and only if $\mathbf{L}$ is isomorphic to the interval algebra of some o-ring $R_{\mathbf{L}}$. Furthermore, $\mathbf{L}$ is PŁ$'$-algebra iff $R_{\mathbf{L}}$ is a domain of integrity.*

The following fact is a corollary of the previous theorem and was proved in [19, Corollary 4.4].

THEOREM 3.8. *The quasi-variety of PŁ′-algebras is generated by* $[0,1]_S$.

THEOREM 3.9. *Let* $\mathcal{C}$ *be either PŁ or PŁ′. Then every* $\mathcal{C}$*-algebra is a subdirect product of linearly ordered* $\mathcal{C}$*-algebras.*

Proof. The claim for PŁ-algebras is proved in [17, Theorem 5.1]. The claim for PŁ′-algebras is a trivial consequence of Theorem 2.9 and the fact that PŁ′-algebras are exactly the subreducts of ŁΠ-algebras (cf. [19, Theorem 4.2]). □

Now we will study the relation between nontrivial zero-divisors and infinitesimal elements of PŁ-algebras. We recall the definition of an infinitesimal element and continue with the lemma showing the distributivity of $\odot$ w.r.t. $\oplus$.

DEFINITION 3.10. *An element a in a PŁ-algebra is said to be infinitesimal iff* $a > \mathbf{0}$ *and* $na \leq \neg a$ *for each* $n \in \mathbb{N}$*, where* $na = a \oplus \cdots \oplus a$.

LEMMA 3.11. *In each PŁ-algebra the following inequality holds:*

$$b \odot (x \oplus y) \leq (b \odot x) \oplus (b \odot y)\,.$$

Proof. The inequality is equivalent to $(b \odot (x \oplus y)) \ominus ((b \odot x) \oplus (b \odot y)) = \mathbf{0}$. Now $(b \odot (x \oplus y)) \ominus ((b \odot x) \oplus (b \odot y)) = (b \odot (x \oplus y)) \otimes \neg(b \odot x) \otimes \neg(b \odot y) = [(b \odot (x \oplus y)) \ominus (b \odot x)] \ominus (b \odot y) = b \odot [((x \oplus y) \ominus x) \ominus y] = b \odot [(x \oplus y) \otimes \neg x \otimes \neg y] = b \odot [(x \oplus y) \ominus (x \oplus y)] = \mathbf{0}$. □ □

PROPOSITION 3.12. *Let* $\mathbf{L}$ *be a linearly ordered PŁ-algebra, and* $a \in L$, $a > \mathbf{0}$*. If a is a zero-divisor then a is an infinitesimal.*

Proof. Let us suppose that a is a zero-divisor which is not infinitesimal. Then there exists $n \in \mathbb{N}$ such that $na = \mathbf{1}$. By Lemma 3.11 $a \odot na \leq (a \odot a) \oplus \cdots \oplus (a \odot a) = \mathbf{0}$ because a is a zero-divisor. Thus $a = a \odot \mathbf{1} = a \odot na = \mathbf{0}$, a contradiction. □

Further we will show interesting examples which have important consequences in the sequel.

EXAMPLE 3.13. *Let us take the following set:*

$$L_{1,\infty} = \{0, 1, 2, \ldots, \infty, \infty - 1, \infty - 2, \ldots\}\,,$$

where we identify ∞ *with* $\infty - 0$*. The operations are defined as follows:*

$$\begin{array}{ll}
n \in \mathbb{N}: & \neg n = \infty - n, \\
 & \neg(\infty - n) = n. \\
k, n \in \mathbb{N}: & k \oplus n = k + n, \\
 & (\infty - k) \oplus (\infty - n) = \infty, \\
 & k \oplus (\infty - n) = \begin{cases} (\infty - n + k) & \text{if } k \leq n, \\ \infty & \text{otherwise.} \end{cases} \\
k, n \in \mathbb{N}: & k \odot n = 0, \\
 & k \odot (\infty - n) = k, \\
 & (\infty - k) \odot (\infty - n) = (\infty - k - n).
\end{array}$$

The structure $(L_{1,\infty}, \oplus, \odot, \neg, 0, \infty)$ *is a PŁ-algebra. Observe also that this algebra possesses nontrivial zero-divisors, thus it is not a PŁ′-algebra. Notice that the MV-reduct of this algebra is the well-known Chang algebra, thus the elements of* $L_{1,\infty}$ *are ordered as follows:* $0 < 1 < 2 < \cdots < \infty - 2 < \infty - 1 < \infty$.

We show how to generate this PŁ-algebra with nontrivial zeros-divisors from the standard PŁ-algebra. It is the well-known fact that each algebra in a variety generated by $[0,1]_S$ *can be obtained as* $A \in HSP([0,1]_S)$*, where P means the direct* *product, S a subalgebra, and H a homomorphic image. So we will construct the example in the following three steps.*

(1) *Step P: Take the algebra of all functions $L = [0,1]^{[0,1]}$.*
(2) *Step S: Restrict to the subalgebra $S \subseteq L$ of all continuous piecewise polynomial functions with integer coefficients such that either $f(0) = 0$ or $f(0) = 1$.*
(3) *Step H: Factorise by the equivalence $\sim$, where $f \sim g$ iff $f(0) = g(0)$ and $f'(0) = g'(0)$. By $f'(0)$, we denote the right-derivative of f in 0.*

EXAMPLE 3.14. *Now we will show an example of a PŁ-algebra which cannot be generated from the standard PŁ-algebra. Firstly, we construct an o-monoid and then we construct the algebra of polynomials over this monoid. In this way, we obtain a ring; its interval algebra is the desired PŁ-algebra (cf. Theorem 3.7).*

The following example of an o-monoid can be found in [8]*. For any $a, b, c, d \in \mathbb{N}$, $\langle a, b, c\rangle$ will denote the sub-o-monoid of $\mathbb{N}$ generated by a, b, c, and $\langle a, b, c\rangle/d$ will denote the o-monoid obtained by identifying with infinity all elements of $\langle a, b, c\rangle$ that are greater than or equal to d.*

Let $S = \{32^*\} \cup \langle 9, 12, 16\rangle/30$ denote the o-monoid obtained from $\langle 9, 12, 16\rangle/30$ by adding one additional element, denoted by 32^*. This element satisfies $16 + 16 = 32^*$, $32^* + z = \infty$, and the whole monoid is to be ordered as follows:

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty\,.$$

All the relations that do not involve 32^* are as in $\mathbb{N}$, so we have only to check that $x \le y$ implies $x + z \le y + z$ if x, y, or z is equal to 32^*, but it is easy to see.

Let R be the o-ring of integers. Then the monoid ring $R[S]$ is the set of all finite formal sums $r_1X^{s_1} + \cdots + r_nX^{s_n}$, where X is an indeterminate, $r_i \in R$ and $s_i \in S$. Multiplication is defined by $X^sX^t = X^{s+t}$ and by distributivity (so $X^0 = 1$). An element $r_1X^{s_1} + \cdots + r_nX^{s_n}$ is said to be in normal form if $r_i \neq 0$ and $s_1 < \cdots < s_n$.

We identify X^∞ with 0 and denote the resulting quotient of $R[S]$ by $R[S]_h$. An element $r_1X^{s_1} + \cdots + r_nX^{s_n}$ in normal form is positive iff $r_1 > 0$. Thus for all $a, b \in S$, $a < b$ implies $X^a > X^b$.

It can be checked that $R[S]_h$ is an o-ring. Finally, we get the desired PŁ-algebra $\mathbf{L}$ *as the interval algebra of $R[S]_h$. In* $\mathbf{L}$ *the following identity, which is valid in $[0,1]_S$, does not hold:*

$$(x_1 \odot z_1 \ominus y_1 \odot z_2) \wedge (x_2 \odot z_2 \ominus y_2 \odot z_1) \wedge (y_1 \odot y_2 \ominus x_1 \odot x_2) = \mathbf{0} \tag{1}$$

Indeed, let us evaluate the variables as follows:

$$\begin{array}{lll} x_1 = X^{16} & y_1 = X^{18} & z_1 = X^{16} \\ x_2 = X^{12} & y_2 = X^{9} & z_2 = X^{12} \end{array}$$

Then the terms in brackets in (1) attain the following values:

$$\begin{aligned} x_1 \odot z_1 \ominus y_1 \odot z_2 &= X^{16} \odot X^{16} \ominus X^{18} \odot X^{12} = X^{32^*} > 0\,, \\ x_2 \odot z_2 \ominus y_2 \odot z_1 &= X^{12} \odot X^{12} \ominus X^{9} \odot X^{16} = X^{24} \ominus X^{25} > 0\,, \\ y_1 \odot y_2 \ominus x_1 \odot x_2 &= X^{18} \odot X^{9} \ominus X^{16} \odot X^{12} = X^{27} \ominus X^{28} > 0\,. \end{aligned}$$

To see that (1) is valid in the standard PŁ-algebra $[0,1]_S$, just observe that whenever one of the variables x_i, y_i, z_i is 0, then the equality trivially holds. Further, if $x_1z_1 > y_1z_2$ and $x_2z_2 > y_2z_1$, then $\prod x_i \prod z_j > \prod y_i \prod z_j$ and this implies $\prod x_i > \prod y_i$.

The following theorem is a consequence of the previous examples. Let $\mathbf{P}Ł'$, $\mathbf{P}Ł$, and $[\mathbf{0},\mathbf{1}]_{\mathbf{S}}$ respectively denote the quasi-variety of PŁ′-algebra, the variety of PŁ-algebras, and the variety generated by $[0,1]_S$ resp.

THEOREM 3.15. *The following holds:*

(1) $\mathbf{P}Ł'$ *is not a variety.*
(2) $\mathbf{P}Ł' \subsetneqq [\mathbf{0},\mathbf{1}]_{\mathbf{S}} \subsetneqq \mathbf{P}Ł$.

Proof. (1) The proof can be found in [19, Theorem 3.1]. However, we give a very simple alternative proof: Example 3.13 shows that $\mathbf{P}Ł'$ is not closed under HSP, so it is not a variety.

(2) The first inequality is a consequence of Theorem 3.8, the second is obvious. The strictness of the first inequality is demonstrated by Example 3.13 and the strictness of the second one is demonstrated by Example 3.14.

□

REMARK 3.16. *The fact that the variety of PŁ-algebras is not generated by* $[0,1]_S$ *is already mentioned in Montagna's paper* [17, Problem 1]*, but there is no proof, only a reference to Isbell's paper* [15]*. In that paper, Isbell proved that the equational theory of formally real f-rings (lattice-ordered rings satisfying all lattice-ring identities that are true in a totally-ordered field) does not have a finite base, or even a base with a finite number of variables. Thus it seems to us that the connection between the second part of Theorem 3.15 and Isbell's paper is not so straightforward. For this reason, we gave an alternative proof of the second part of Theorem 3.15 which is easier for the reader to follow.*

3.3. **Completeness.** In this section, we are going to deal with the strong completeness theorem for PŁ and PŁ′ logic. We start with the proof of soundness and then we introduce Lindenbaum PŁ-algebra and PŁ′-algebra.

THEOREM 3.17. *PŁ logic is sound w.r.t. PŁ-algebras, i.e. if φ is a theorem of PŁ then φ is an* $\mathbf{L}$*-tautology for each PŁ-algebra* $\mathbf{L}$*.*

Furthermore, PŁ′ logic is sound w.r.t. PŁ′-algebras.

Proof. Let $\mathbf{L}$ be a PŁ-algebra. Since $\mathbf{L}^*$ is an MV-algebra, we know that the axioms of Łukasiewicz logic hold in $\mathbf{L}$ and modus ponens is a sound deduction rule. Axioms (P1)–(P3) and (P5) are obviously $\mathbf{L}$-tautologies (cf. conditions (1)–(4) in the definition of PŁ-algebra).

We check $\mathbf{L}$-tautologicity of (P4). By Proposition 2.3,4, we know that $a \odot b \to a = \mathbf{1}$ iff $a \odot b \leq a = a \odot \mathbf{1}$. Now using [17, Lemma 2.9(ii)], the proof of the first statement is done.

To prove the second statement, just observe that the rule (ZD) is obviously sound in each PŁ′-algebra. □ □

This theorem has two important corollaries. The first states the connection of our logics and Łukasiewicz logic and the second is the promised proof that the deduction theorem does not hold in PŁ′.

COROLLARY 3.18. *PŁ and PŁ′ are conservative extensions of Łukasiewicz logic.*

Proof. We show the proof for PŁ. The proof for PŁ′ is analogous. Let φ be a formula of Łukasiewicz logic which is a theorem of PŁ. Due to 3.17, φ is an $\mathbf{L}$-tautology for each PŁ-algebra $\mathbf{L}$. Thus it is a tautology in the standard PŁ-algebra. Since φ is a formula of Łukasiewicz logic, it is also a tautology in the standard MV-algebra. Using the standard completeness theorem of Łukasiewicz logic we conclude that φ is a theorem of Łukasiewicz logic. □ □

COROLLARY 3.19. *PŁ′ does not satisfy the deduction theorem in the same form as PŁ.*

Proof. Since obviously $\{\neg(v \odot v)\} \vdash \neg v$, the deduction theorem would give us that for some n the formula $(\neg(v \odot v))^n \to \neg v$ is a theorem of PŁ′. Hence $(\neg(v \odot v))^n \to \neg v$ is $[0,1]_S$-tautology (by the latter theorem), i.e., there is n such that $(\neg(x \odot x))^n \leq \neg x$ for each $x \in [0,1]$. Notice that the derivatives of $(\neg(x \odot x))^n$ and $\neg x$ at the point 0 are equal to 0 and -1, respectively. Thus for each n, there is x such that $(\neg(x \odot x))^n > \neg x$, a contradiction. □ □

DEFINITION 3.20. *Let $\mathcal{C}$ be either PŁ or PŁ′ and T a theory over $\mathcal{C}$. For each formula φ, let $[\varphi]_T$ denote the set $\{\psi \mid T \vdash \varphi \equiv \psi\}$. Let L_T be the set of all classes $[\varphi]_T$. We define the operations as: $[\varphi]_T * [\psi]_T = [\varphi * \psi]_T$, where $*$ on the left is a defined operation and on the right side is the corresponding connective in PŁ. The resulting structure $\mathbf{L}_T = (L_T, \oplus, \neg, \odot, \mathbf{0}, \mathbf{1})$ is called the* Lindenbaum PŁ-algebra of T-equivalent formulas.

The above definition of the operations is correct due to known properties of Lindenbaum MV-algebra and theorem (TP4).

LEMMA 3.21. *Let $\mathcal{C}$ be either PŁ or PŁ′ and T a theory over $\mathcal{C}$. Then $\mathbf{L}_T$ is a $\mathcal{C}$-algebra. Furthermore, $\mathbf{L}_T$ is a linear $\mathcal{C}$-algebra iff T is complete.*

The proof is a straightforward generalization of the proof of an analogous lemma for BL-algebras (see [11]). Now we are ready to prove the strong completeness theorem.

THEOREM 3.22 (Strong Completeness). *Let $\mathcal{C}$ be either PŁ or PŁ′, T a theory over $\mathcal{C}$, and φ a formula. Then the following are equivalent:*

(1) $T \vdash \varphi$,
(2) $e(\varphi) = \mathbf{1}_{\mathbf{L}}$ *for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T,*
(3) $e(\varphi) = \mathbf{1}_{\mathbf{L}}$ *for each $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T.*

Proof. The implication (1⇒2) follows from soundness (Theorem 3.17).

We prove the implication (2⇒3) indirectly. Assume there is a $\mathcal{C}$-algebra $\mathbf{L}$ and an $\mathbf{L}$-model e of theory T such that $e(\varphi) < \mathbf{1}_{\mathbf{L}}$. Since $\mathbf{L}$ is the subdirect product of the family $(A_i)_{i \in I}$ of linearly ordered $\mathcal{C}$-algebras (cf. Theorem 3.9), there must be an index i and a projection $\pi : \mathbf{L} \to A_i$ such that $\pi \circ e$ is a model of T and $\pi \circ e(\varphi) < 1_{A_i}$.

Finally, we prove the implication (3⇒1). Assume that $T \not\vdash \varphi$ and $\mathbf{L}_T$ is the Lindenbaum algebra of T. Define the $\mathbf{L}_T$-evaluation e by $e(v) = [v]_T$. Since $e(\varphi) = [\varphi]_T$, e is an $\mathbf{L}_T$-model of T. Since $e(\varphi) < \mathbf{1}_{\mathbf{L}_T}$, the proof is done. □ □

COROLLARY 3.23 (Completeness). *Let $\mathcal{C}$ be either PŁ or PŁ′ and φ a formula. Then the following are equivalent:*

(1) $\mathcal{C} \vdash \varphi$,
(2) *φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-algebra $\mathbf{L}$,*
(3) *φ is an $\mathbf{L}$-tautology for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$.*

The question whether our logics possess the standard completeness is answered by the following two theorems. The first is a consequence of Theorem 3.15. The proof of the second is based on Theorem 3.8.

THEOREM 3.24. *PŁ-logic does not fulfil the standard completeness property.*

THEOREM 3.25 (Finite Strong Standard Completeness). *Let T be a finite theory over PŁ′ and φ be a formula. Then*

(1) *$T \vdash \varphi$ iff $e(\varphi) = 1$ for each $[0,1]_S$-model e of theory T,*
(2) *PŁ$'$ $\vdash \varphi$ iff φ is $[0,1]_S$-tautology.*

Proof. (1) One direction is obvious. To prove the second, recall that if $T \not\vdash \varphi$ then there is a linearly ordered PŁ$'$-algebra $\mathbf{L}$ and an $\mathbf{L}$-model e of theory T such that $e(\varphi) < \mathbf{1_L}$. Let us denote by t_ψ the term corresponding to the formula ψ. Then the inequality $e(\varphi) < \mathbf{1_L}$ is equivalent to the fact that the quasi-identity $\bigwedge_{\psi \in T}(t_\psi = \mathbf{1}) \Rightarrow (t_\varphi = \mathbf{1})$ is not valid in $\mathbf{L}$ (the symbols $\bigwedge$ and $\Rightarrow$ stand for the classical logical connectives). Due to Theorem 3.8, the same quasi-identity does not hold in $[0,1]_S$ and the rest of the claim easily follows.

(2) Trivial. □ □

COROLLARY 3.26. *The logic PŁ$'$ is strictly stronger than the logic PŁ.*

At the end of this section we present a corollary of the strong completeness Theorem 3.22, namely the analogy of Theorem 3.4 for the PŁ$'$ logic, we promised.

THEOREM 3.27. *Let T be a theory over PŁ or PŁ$'$ and φ a formula such that $T \not\vdash \varphi$. Then there is a complete supertheory T' such that $T' \not\vdash \varphi$.*

Proof. We show the proof for the PŁ$'$ logic. Since $T \not\vdash \varphi$, there is a linearly ordered PŁ$'$-algebra $\mathbf{L}$ and the $\mathbf{L}$-model e such that $e(\varphi) < \mathbf{1_L}$. Take $T' = \{\psi \mid e(\psi) = \mathbf{1_L}\}$. Observe that $T \subseteq T'$ (because e is the $\mathbf{L}$-model of T) and $T' \not\vdash \varphi$ (if $T' \vdash \varphi$ then $e(\varphi) = \mathbf{1}$ (due to the completeness theorem)—a contradiction). Finally, T' is complete (each pair (φ, ψ) satisfies either $e(\varphi) \leq e(\psi)$ or $e(\psi) \leq e(\varphi)$, thus either $\varphi \to \psi \in T'$ or $\psi \to \varphi \in T'$). □ □

4. PŁ$_\triangle$ logic

In this section, we extend the language of PŁ logic by a unary connective $\triangle$ and introduce PŁ$_\triangle$ logic. The connective was introduced in Gödel logic by Baaz (see [1]) and generalized to BL by Hájek (see [11]).

DEFINITION 4.1. *Let $\mathcal{C}$ be either PŁ or PŁ$'$. The $\mathcal{C}_\triangle$ logic results from $\mathcal{C}$ by adding axioms (Ł$\triangle$1) – (Ł$\triangle$5) and the deduction rule of necessitation.*

LEMMA 4.2. *The formulas $\triangle\varphi \otimes \psi \equiv \triangle\varphi \wedge \psi$ and $\triangle\varphi \odot \psi \equiv \triangle\varphi \wedge \psi$ are theorems of the PŁ logic.*

Proof. The first formula is the known theorem of $L_\triangle$. For the proof of the second one, use the first formula and theorem (TP5). □

THEOREM 4.3. *The PŁ$'_\triangle$ logic can be equivalently defined as an extension of PŁ$_\triangle$ by the following axiom:*

(P$\triangle$) $\quad \triangle\neg(\varphi \odot \varphi) \to \neg\varphi\,.$

Proof. Firstly, we show that the deduction rule (ZD) can be derived in PŁ$_\triangle$ extended by axiom $(P\triangle)$. Let $\neg(\varphi \odot \varphi)$ be a theorem. Then $\triangle\neg(\varphi \odot \varphi)$ is a theorem as well and so $\neg\varphi$ is provable (by modus ponens and axiom $(P\triangle)$).

Converse direction: It is sufficient to prove $\neg(\neg(P\triangle) \odot \neg(P\triangle))$ (let us denote this formula by F). Then proof is done by the use of (ZD). Observe that $\neg(P\triangle) \equiv \triangle\neg(\varphi\odot\varphi)\otimes\varphi$. Thus by Lemma 4.2 we get $\neg F \equiv (\triangle\neg(\varphi\odot\varphi)\wedge\varphi)\odot(\triangle\neg(\varphi\odot\varphi)\wedge\varphi)$. After repeated use of theorem (TP6) we get $\neg F \equiv (\triangle\neg(\varphi \odot \varphi) \odot \triangle\neg(\varphi \odot \varphi)) \wedge (\triangle\neg(\varphi \odot \varphi) \odot \varphi) \wedge (\varphi \odot \triangle\neg(\varphi \odot \varphi)) \wedge (\varphi \odot \varphi)$.

By use of Lemma 4.2 we may write $\neg F \equiv \triangle\neg(\varphi \odot \varphi) \wedge \varphi \wedge (\varphi \odot \varphi)$. Finally, by the obvious fact that $\varphi \wedge (\varphi \odot \varphi) \equiv \varphi \odot \varphi$ is a theorem (use (H9), (H5)) and using Lemma 4.2, we get $F \equiv \neg(\triangle\neg(\varphi \odot \varphi) \otimes (\varphi \odot \varphi)) \equiv \triangle\neg(\varphi \odot \varphi) \to \neg(\varphi \odot \varphi)$. Since $\triangle\neg(\varphi \odot \varphi) \to \neg(\varphi \odot \varphi)$ is an instance of axiom (Ł$\triangle$3), the formula F is a theorem. □ □

Due to the previous result, we can prove the deduction theorem for PŁ$_\triangle$ and PŁ$'_\triangle$ in the same way as for Ł$_\triangle$. See [11, Theorem 2.4.14] for details.

THEOREM 4.4 (Deduction theorem). *Let T be a theory over PŁ$_\triangle$ or PŁ$_\triangle$ and φ, ψ be formulas. Then $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash \triangle\varphi \to \psi$.*

Now let us introduce the algebras corresponding to PŁ$_\triangle$ and PŁ$'_\triangle$.

DEFINITION 4.5. *A PŁ$_\triangle$-algebra is a structure $\mathbf{L} = (L, \oplus, \neg, \odot, \triangle, \mathbf{0}, \mathbf{1})$, where the reduct $\mathbf{L}^* = (L, \oplus, \neg, \triangle, \mathbf{0}, \mathbf{1})$ is an MV$_\triangle$-algebra and the reduct $\widehat{\mathbf{L}} = (L, \oplus, \neg, \odot, \mathbf{0}, \mathbf{1})$ is a PŁ-algebra. A PŁ$'_\triangle$-algebra is a PŁ$_\triangle$-algebra where the following identity holds:*

(1') $\triangle\neg(\varphi \odot \varphi) = \triangle\neg\varphi$.

It can be shown that our PŁ$'_\triangle$-algebras and PMV$_\triangle$-algebras defined in Montagna's paper [18] coincide. Montagna proved that there is a categorical equivalence between categories of PMV$_\triangle$-algebras and δ-f-rings (f-rings extended by an operation corresponding to $\triangle$).

An alternative way to introduce PŁ$'_\triangle$-algebras is to define PŁ$'_\triangle$-algebras as PŁ$_\triangle$-algebras satisfying quasi-identity (5) from Definition 3.5. The proof that both definitions coincide is easy.

REMARK 4.6. *If $(L, \oplus, \neg, \odot, \mathbf{0}, \mathbf{1})$ is a linearly ordered PŁ-algebra, then we may introduce the operation $\triangle$ by setting $\triangle(x) = \mathbf{1}$ iff $x = \mathbf{1}$ and $\triangle(x) = \mathbf{0}$ otherwise (see [11]). Then the structure $(L, \oplus, \neg, \odot, \triangle, \mathbf{0}, \mathbf{1})$ is a linearly ordered PŁ$_\triangle$-algebra.*

EXAMPLE 4.7. *According to Remark 4.6, we may extend the definition of the standard PŁ-algebra $[0, 1]_S$ by $\triangle$. Then we obtain the standard PŁ$_\triangle$-algebra which we denote $[\mathbf{0}, \mathbf{1}]_{\mathbf{S}\triangle} = ([0, 1], \oplus, \neg, \odot, \triangle, 0, 1)$.*

THEOREM 4.8. *Every PŁ$_\triangle$-algebra is a subdirect product of linearly ordered PŁ$_\triangle$-algebras.*

Proof. Just observe that the following formula holds in every PŁ$_\triangle$-algebra:

$$\triangle(x \equiv y) \wedge \triangle(a \equiv b) \leq (x \odot a) \equiv (y \odot b) .$$

Indeed, by (TP4) we get $(x \equiv y) \otimes (a \equiv b) \to (x \odot a) \equiv (y \odot b) = \mathbf{1}$. Then by ($\triangle$5) and Proposition 2.5 we obtain $\triangle(x \equiv y) \wedge \triangle(a \equiv b) \leq \triangle((x \odot a) \equiv (y \odot b))$. Finally we use ($\triangle$3).

Thus we have satisfied the conditions of [7, Lemma 3] and we can follow the proof of [7, Theorem 4]. □ □

As the corollary, we obtain the subdirect representation theorem for PŁ$'_\triangle$-algebras which was already proven by Montagna in [18, Proposition 2.4 (i)].

COROLLARY 4.9. *Every PŁ$'_\triangle$-algebra is a subdirect product of linearly ordered PŁ$'_\triangle$-algebras.*

THEOREM 4.10 (Strong Completeness). *Let $\mathcal{C}$ be either PŁ$_\triangle$ or PŁ$'_\triangle$, T be a theory over $\mathcal{C}$ and φ be a formula. Then the following are equivalent:*

(1) *$T \vdash \varphi$,*
(2) *$e(\varphi) = \mathbf{1_L}$ for each $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T,*
(3) *$e(\varphi) = \mathbf{1_L}$ for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T.*

Proof. The implication (1⇒2) is soundness which is obvious. The implication (2⇒3) is trivial. Finally, we prove the implication (3⇒1). Assume $T \not\vdash \varphi$. Then we found a complete supertheory $T' \not\vdash \varphi$ (this can be done by the straightforward modification of Theorem 3.4—due to the deduction theorem). Let us take the Lindenbaum $\mathcal{C}$-algebra $\mathbf{L}_{T'}$ of T' (its construction is analogous to the construction of the Lindenbaum PŁ-algebra). Then $\mathbf{L}_{T'}$ is linearly ordered and $e(v) = [v]_{T'}$ is obviously an $\mathbf{L}_{T'}$-model of T'. Since $e(\varphi) < \mathbf{1}_{\mathbf{L}_{T'}}$, the proof is done. □ □

THEOREM 4.11. *The $P\text{Ł}'_\triangle$ logic is strictly stronger than $P\text{Ł}_\triangle$ logic.*

Proof. This as an obvious consequence of Corollary 3.26. However, the connective $\triangle$ allows us to find a simpler proof. According to Remark 4.6 and Example 3.13, we have a $P\text{Ł}_\triangle$-algebra $\mathbf{L}$ with an element $c > 0$ such that $c \odot c = 0$. Then $\neg c = (\infty - c) < \infty$, thus $\triangle\neg c = 0$. Since $\neg(c \odot c) = \infty$ and so $\triangle(\neg(c \odot c)) = \infty$, we know that the axiom (P$\triangle$) is not an $\mathbf{L}$-tautology. Thus (P$\triangle$) is not a theorem of $P\text{Ł}_\triangle$. □ □

THEOREM 4.12 (Finite Strong Standard Completeness). *Let T be a finite theory over $P\text{Ł}'_\triangle$ and φ be a formula.*

1. *$P\text{Ł}'_\triangle \vdash \varphi$ iff φ is the $[0,1]_{S\triangle}$-tautology.*
2. *$T \vdash \varphi$ iff $e(\varphi) = 1$ for each $[0,1]_{S\triangle}$-model e of T.*

Proof. (1): Montagna and Panti proved that the standard $P\text{Ł}'_\triangle$-algebra $[0,1]_{S\triangle}$ generates the whole variety of $P\text{Ł}'_\triangle$-algebras (see [20, Theorem 2.4]). Thus the statement follows easily.

(2): Again we prove the non-trivial implication only. Let $T = \{\varphi_1, \dots, \varphi_n\}$ and let e be an $[0,1]_{S\triangle}$-evaluation. Firstly, if e is a $[0,1]_{S\triangle}$-model of T, then $e(\varphi) = 1$. Thus $e(\triangle\varphi_1 \otimes \cdots \otimes \triangle\varphi_n \to \varphi) = 1$. Secondly, if e is not a $[0,1]_{S\triangle}$-model, then $e(\triangle\varphi_1 \otimes \cdots \otimes \triangle\varphi_n \to \varphi) = 1$ as well. Thus $\triangle\varphi_1 \otimes \cdots \otimes \triangle\varphi_n \to \varphi$ is a $[0,1]_{S\triangle}$-tautology and by (1): $P\text{Ł}'_\triangle \vdash \triangle\varphi_1 \otimes \cdots \otimes \triangle\varphi_n \to \varphi$. Thus $T \vdash \triangle\varphi_1 \otimes \cdots \otimes \triangle\varphi_n \to \varphi$ and, because $T \vdash \triangle\varphi_1 \otimes \cdots \otimes \triangle\varphi_n$, the proof is complete. □ □

Now we summarize known facts about connections between our logics and Łukasiewicz and ŁΠ logic.

THEOREM 4.13.

(1) *PŁ, $P\text{Ł}'$, $P\text{Ł}_\triangle$, and $P\text{Ł}'_\triangle$ logics are conservative extensions of Łukasiewicz logic.*
(2) *$P\text{Ł}_\triangle$ and $P\text{Ł}'_\triangle$ logics are conservative extensions of Łukasiewicz logic with $\triangle$.*
(3) *$P\text{Ł}_\triangle$ logic is a conservative extension of PŁ logic and $P\text{Ł}'_\triangle$ logic is a conservative extension of $P\text{Ł}'$ logic.*
(4) *$P\text{Ł}'_\triangle$ logic is not a conservative extension of PŁ logic.*
(5) ŁΠ *logic is a conservative extension of $P\text{Ł}'$ and $P\text{Ł}'_\triangle$ logics.*
(6) ŁΠ *logic is not a conservative extension of PŁ and $P\text{Ł}_\triangle$ logics.*

Proof. (1): For PŁ it is already proven in Theorem 3.18, the proof for the remaining logics is analogous.

(2) Analogous to (1).

(3) It follows from the fact that we can extend each linearly ordered PŁ-algebra by $\triangle$ and from the completeness theorems for both logics.

(4) Since $P\text{Ł}'_\triangle$ is a conservative extension of $P\text{Ł}'$ and $P\text{Ł}'$ is strictly stronger than PŁ, the proof easily follows.

(5) A consequence of Theorems 2.8, 3.25, and 4.12.

(6) A consequence of (5) and the fact that $P\text{Ł}_\triangle$ is strictly weaker than $P\text{Ł}'_\triangle$ (and that PŁ is strictly weaker than $P\text{Ł}'$). □ □

5. Pavelka style extension of PŁ

In this section we add rational constants into our language together with the book-keeping axioms. Rational Pavelka logic (RPL) was introduced in the series of papers by Pavelka [22] and simplified to its modern form in [11]. RPL has interesting properties and has been widely studied.

A lot details about this logic can be found in the recent book by Novák, Perfilieva and Močkoř [21]. In this section we will fix the algebra of truth values to the standard algebra of the respective logic.

DEFINITION 5.1. *Let $\mathcal{L}$ be one of the logics PŁ, PŁ$'$, PŁ$_\triangle$, PŁ$'_\triangle$. The language of the logic RP$\mathcal{L}$ arises from the language of $\mathcal{L}$ by adding a truth constant $\overline{r}$ for each $r \in \mathbb{Q} \cap [0,1]$. The notion of evaluation extends by the condition $e(\overline{r}) = r$. The axioms of RP$\mathcal{L}$ logic are the axioms of $\mathcal{L}$ plus the following book-keeping axioms for each rational $r, s \in [0,1]$:*

$$\begin{aligned}
(\overline{r} \oplus \overline{s}) &\equiv \overline{\min(1, r+s)}\,,\\
(\overline{r} \odot \overline{s}) &\equiv \overline{r \cdot s}\,,\\
\neg\overline{r} &\equiv \overline{1-r}\,,\\
\triangle\overline{r} &\equiv \overline{\triangle r} \qquad (\textit{in the case of RPPŁ}_\triangle \textit{ and RPPŁ}'_\triangle).
\end{aligned}$$

The deduction rules of RP$\mathcal{L}$ are the same as the deduction rules of $\mathcal{L}$. The logics RPPŁ$_\triangle$ and RPPŁ$'_\triangle$ have one additional infinitary deduction rule (IR):

from $\overline{r} \to \varphi$ for all $r < 1$, derive φ.

Since we added the infinitary deduction rule IR, we have to change the notion of proof. Let T be a theory, then the set $C_{RP\mathcal{L}}(T)$ of all provable formulas in T is the smallest set of formulas, which contains T, axioms of RP$\mathcal{L}$ and is closed under all deduction rules. For simplicity, we shall write $T \vdash \varphi$ to denote $\varphi \in C_{RP\mathcal{L}}(T)$.

Now we have to prove that our infinitary deduction rule doesn't violate the deduction theorem. We cannot use the standard way of proving this theorem since our notion of proof is now different.

THEOREM 5.2 (Deduction theorem). *Let φ and ψ be formulas.*

(1) *Let T be a theory over RPPŁ. Then $T \cup \{\varphi\} \vdash \psi$ iff there is n such that $T \vdash \varphi^n \to \psi$.*
(2) *Let T be a theory over RPPŁ$_\triangle$ or RPPŁ$'_\triangle$. Then $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash \triangle\varphi \to \psi$.*

Proof. Statement 1 is proved as usual (see [11, Theorem 2.2.18]).

Statement 2 is more complex because of the infinitary deduction rule (IR). We prove this for RPPŁ$_\triangle$ (the proof for RPPŁ$'_\triangle$ is analogous). Let $K = \{\psi \mid \triangle\varphi \to \psi \in C_{RP\mathcal{L}}(T)\}$. We shall show that $K = C_{RP\mathcal{L}}(T \cup \{\varphi\})$. One direction—$K \subseteq C_{RP\mathcal{L}}(T \cup \{\varphi\})$—is obvious. We prove the reverse direction by showing that K is a set containing $T \cup \{\varphi\}$, axioms of RP$\mathcal{L}$ and closed under all deduction rules. The first two conditions obviously hold. We prove that K is closed under modus ponens. Assume $\psi, \psi \to \chi \in K$. Then $T \vdash \triangle\varphi \to \psi$ and $T \vdash \triangle\varphi \to (\psi \to \chi)$. Thus $T \vdash \triangle\varphi \otimes \triangle\varphi \to \psi \otimes (\psi \to \chi)$ which leads to $T \vdash \triangle\varphi \to \chi$. Hence $\chi \in K$.

Now we prove that K is closed under the necessitation. Assume $\psi \in K$. Then $T \vdash \triangle\varphi \to \psi$. Thus $T \vdash \triangle(\triangle\varphi \to \psi)$ (because $C_{RP\mathcal{L}}(T)$ is closed under the necessitation). Then $T \vdash \triangle\triangle\varphi \to \triangle\psi$ (by axiom (Ł$\triangle$5)) which leads to $T \vdash \triangle\varphi \to \triangle\psi$ (by axiom (Ł$\triangle$3)). Hence $\triangle\psi \in K$.

Finally, we prove that K is closed under (IR). Assume $\overline{r} \to \psi \in K$ for each $r < 1$. Then $T \vdash \triangle\varphi \to (\overline{r} \to \psi)$ which leads to $T \vdash \overline{r} \to (\triangle\varphi \to \psi)$ for each $r < 1$. Since $C_{RP\mathcal{L}}(T)$ is closed under (IR), we get $T \vdash \triangle\varphi \to \psi$. Hence $\psi \in K$.

□

DEFINITION 5.3. *Let $\mathcal{L}$ be one of the PŁ, PŁ$'$, PŁ$_\triangle$, PŁ$'_\triangle$, and T a theory over RP$\mathcal{L}$ and φ be a formula.*

(1) *The* truth degree *of φ over T is $||\varphi||_T = \inf\{e(\varphi) \mid e$ is a model of $T\}$ (by the word "model" we mean the $[0,1]_S$-model in the case of RPPŁ, RPPŁ$'$, and the $[0,1]_{S\triangle}$-model in the case of RPPŁ$_\triangle$, RPPŁ$'_\triangle$).*

(2) *The* provability degree *of φ over T is $|\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \varphi\}$.*

LEMMA 5.4. *Let T be a theory over either RPPŁ$_\triangle$ or RPPŁ$'_\triangle$ and φ a formula. Then $|\varphi|_T = 1$ iff $T \vdash \varphi$.*

Proof. Since $|\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \varphi\} = 1$, $T \vdash \overline{r} \to \varphi$ for all $r < 1$. Thus by the deduction rule (IR) we obtain $T \vdash \varphi$. The reverse implication is trivial. □ □

LEMMA 5.5. *Let $\mathcal{L}$ be one of the PŁ, PŁ$_\triangle$, PŁ$'_\triangle$. Let T be a consistent complete theory over RP$\mathcal{L}$. Then*

(1) *For each φ, $|\varphi|_T = \sup\{r|T \vdash \overline{r} \to \varphi\} = \inf\{r|T \vdash \varphi \to \overline{r}\}$.*
(2) *The provability degree commutes with all connectives, i.e.,*
$$|\neg\varphi|_T = 1 - |\varphi|_T,\ |\varphi \to \psi|_T = |\varphi|_T \to |\psi|_T,\ |\varphi \odot \psi|_T = |\varphi|_T \odot |\psi|_T.$$
Moreover, in the case of RPPŁ$_\triangle$ and RPPŁ$'_\triangle$, the following holds:
$$|\triangle\varphi|_T = \triangle|\varphi|_T.$$
Thus the evaluation $e(v) = |v|_T$ is a model of T.

Proof. (1) See [11, Lemma 3.3.8(1)].
(2) For the proofs for negation and implication see [11, Lemma 3.3.8(2)]. The proof for $\odot$: $|\varphi|_T \odot |\psi|_T = \sup\{r|T \vdash \overline{r} \to \varphi\} \odot \sup\{s|T \vdash \overline{s} \to \psi\} = \sup\{r \odot s|T \vdash \overline{r} \to \varphi, T \vdash \overline{s} \to \psi\} \le \sup\{r \odot s|T \vdash \overline{r} \odot \overline{s} \to \varphi \odot \psi\}$ (since we know that if $T \vdash \overline{r} \to \varphi$ and $T \vdash \overline{s} \to \psi$ then $T \vdash \overline{r} \odot \overline{s} \to \varphi \odot \psi$, cf. theorem (TP3)). Conversely, assume that $|\varphi|_T \odot |\psi|_T < t < t' < |\varphi \odot \psi|_T$. Then there are r, s such that $t = r \odot s$, $r > |\varphi|_T$, and $s > |\psi|_T$. Due to (1), $T \vdash \varphi \to \overline{r}$, $T \vdash \psi \to \overline{s}$. Using (TP3) we get $T \vdash \varphi \odot \psi \to \overline{t}$. We also know that $T \vdash \overline{t'} \to \varphi \odot \psi$. Thus $T \vdash \overline{t'} \to \overline{t}$. Since $\overline{t' \to t} < 1$, we get that T is inconsistent, a contradiction. If $T \vdash \overline{t' \to t}$, then $T \vdash \overline{t' \to t} \otimes \ldots \otimes \overline{t' \to t}$ for any n-fold conjunction. As $\otimes$ is an Archimedean t-norm, $T \vdash \overline{0}$.

Finally, we have to show the proof for $\triangle$. Firstly, assume $|\varphi|_T < 1$. Then $\triangle|\varphi|_T = 0$ and $T \vdash \varphi \to \overline{r}$ for some $r < 1$ (cf. (1)). Further, $T \vdash \triangle(\varphi \to \overline{r})$ and $T \vdash \triangle\varphi \to \overline{0}$ (by necessitation, axiom ($\triangle$5), and the book-keeping axiom for $\triangle$). Thus $|\triangle\varphi|_T = 0$ as well.

Secondly, assume that $|\varphi|_T = 1$. Then $\triangle|\varphi|_T = 1$ and, by Lemma 5.4, $T \vdash \varphi$. Thus $T \vdash \triangle\varphi$ and $|\triangle\varphi|_T = 1$ as well. □ □

THEOREM 5.6. *Let $\mathcal{L}$ be one of the PŁ, PŁ$'$, PŁ$_\triangle$, PŁ$'_\triangle$ and T a theory over RP$\mathcal{L}$ and φ be a formula. Then $|\varphi|_T = ||\varphi||_T$.*

Proof. For RPPŁ, RPPŁ$_\triangle$, and RPPŁ$'_\triangle$, see the proof of completeness of RPL [11, 3.3.9].

For RPPŁ$'$, we have the following chain of inequalities: $||\varphi||_T = |\varphi|_T \le |\varphi|'_T \le ||\varphi||'_T$, where $|\varphi|_T$ and $|\varphi|'_T$ is the provability degree in RPPŁ, RPPŁ$'$, respectively (analogously for the truth degrees). Since $||\varphi||_T = ||\varphi||'_T$, $|\varphi|'_T = ||\varphi||'_T$ □ □

REMARK 5.7. *The proof of Pavelka style completeness for RPPŁ could be obtained as a corollary of* [21, Corollary 4.6]*, but this is not possible for RPPŁ$_\triangle$ and RPPŁ$'_\triangle$.*

We can also obtain Pavelka style completeness for RPPŁ from [11, Section 3.3]*, where the author defines the logic RPL($\odot$). It is Łukasiewicz logic plus book-keeping axioms of RPPŁ and the axioms:*

$$(\varphi \to \psi) \quad \to \quad (\varphi \odot \chi \to \psi \odot \chi)\,,$$
$$(\varphi \to \psi) \quad \to \quad (\chi \odot \varphi \to \chi \odot \psi)\,.$$

This logic enjoys the same Pavelka's style completeness as RPPL (see [11, Theorem 3.3.19]*).*

Thus we have three different logics, namely RPPŁ, RPPŁ′, RPL($\odot$). All of them enjoy the same Pavelka's style completeness. However, the question whether these logics (as sets of theorems) coincide seems to be open for us.

Now we prove the strong standard completeness for RPPŁ$_\triangle$ and RPPŁ$'_\triangle$ and show that these logics coincide.

THEOREM 5.8. *Let T be a theory over RPPŁ$_\triangle$ or RPPŁ$'_\triangle$ and φ be a formula. Then $T \vdash \varphi$ iff $e(\varphi) = 1$ for all standard models e.*

Proof. The proof follows from Theorem 5.6 and Lemma 5.4. □ □

COROLLARY 5.9.

(1) *The RPPŁ$_\triangle$ and RPPŁ$'_\triangle$ logics coincide.*
(2) *RPPŁ′ is a conservative extension of PŁ′ and RPPŁ$_\triangle$ is a conservative extension of PŁ$'_\triangle$.*
(3) *RPPŁ$_\triangle$ is a conservative extension of RPL.*
(4) *RŁΠ is a conservative extension of RPPŁ$_\triangle$.*

Proof. (1) This is an obvious consequence of the latter theorem and the fact that standard algebras for these logics are the same.
(2) Let φ be a formula of the language of PŁ′ such that RPPŁ′ $\vdash \varphi$. Then φ is a $[0,1]_S$-tautology. By Theorem 3.25 we get that PŁ′ $\vdash \varphi$ (for RPPŁ$'_\triangle$ the proof is analogous).
(3) As RPL also enjoys standard completeness (cf. [11, Theorem 3.3.14]), we may proceed in the same way as in 2.
(4) It is the consequence of [7, Corollary 5] and the latter theorem. □
□

6. The predicate logics

This section deals with a predicate version of the PŁ, PŁ$_\triangle$ and PŁ$'_\triangle$ logics. The following definitions are analogous to the definitions of the corresponding concepts in [11].

DEFINITION 6.1. *A* predicate language *is a pair* $((\mathbf{P}, \mathbf{A}), \mathbf{C})$*, where* $\mathbf{P}$ *is a non-empty set of predicates, each predicate P together with a positive natural number $n = \mathbf{A}(P)$ called the arity of P, and $\mathbf{C}$ is a potentially empty set of the object constants. The* logical symbols *are object variables x, y, $\ldots$, logical connectives $\oplus, \neg, \odot$ (and $\triangle$ in case of PŁ$_\triangle$ or PŁ$'_\triangle$ logic), truth constant $\overline{0}$ and the quantifier $\forall$ (the quantifier $\exists$ is defined as $\neg\forall\neg$).*

DEFINITION 6.2. Terms *of the predicate language $I = ((\mathbf{P}, \mathbf{A}), \mathbf{C})$ are the object constants and variables.* Atomic formulas *have the following form: $P(t_1, t_2, \ldots, t_n)$, where P is a predicate, n its arity, and $t_1, t_2, \ldots, t_n$ are terms. The truth constants are atomic formulas as well. Let φ and ψ be formulas, x an object variable. Then $\varphi\oplus\psi$, $\neg\varphi$, $\varphi\odot\psi$, $(\forall x)\varphi$ (and $\triangle\varphi$ in case of PŁ$_\triangle$ or PŁ$'_\triangle$ logic) are formulas. Each formula is constructed from the atomic formulas by iterating these rules.*

DEFINITION 6.3. *Let I be a predicate language, $\mathcal{C}$ either PŁ, PŁ$_\triangle$ or PŁ$'_\triangle$,* $\mathbf{L}$ *a linearly ordered $\mathcal{C}$-algebra. An* $\mathbf{L}$-structure $\mathbf{M}$ *for I has the following form $\mathbf{M} = (M, (r_P)_{P\in\mathbf{P}}, (m_c)_{c\in\mathbf{C}})$, where M is a non-empty domain, r_P is n-ary fuzzy relation $M^n \to L$ for each n-ary predicate P from $\mathbf{P}$, and $m_c \in M$ for each object constant c from $\mathbf{C}$.*

DEFINITION 6.4. *Let I be a predicate language, $\mathcal{C}$ either PL, $PL_\triangle$ or $PL'_\triangle$, $\mathbf{L}$ a linearly ordered $\mathcal{C}$-algebra and $\mathbf{M}$ an $\mathbf{L}$-structure for I. An $\mathbf{M}$-*evaluation *of the object variables is a mapping v from the set of the object variables into the domain M. Let v and v' be two $\mathbf{M}$-evaluations. Then $v \equiv_x v'$ means that $v(y) = v'(y)$ for each object variable y different from x.*

DEFINITION 6.5. *Let I be a predicate language, $\mathcal{C}$ either PL, $PL_\triangle$ or $PL'_\triangle$, $\mathbf{L}$ a linearly ordered $\mathcal{C}$-algebra, $\mathbf{M}$ an $\mathbf{L}$-structure for I, v an $\mathbf{M}$-evaluation. The* value of the term *is defined as follows: $||x||^{\mathbf{L}}_{\mathbf{M},v} = v(x)$ and $||c||^{\mathbf{L}}_{\mathbf{M},v} = m_c$. A* truth value of the formula φ *in $\mathbf{M}$ for an evaluation v is defined as follows:*

$$\begin{aligned}
||P(t_1,\, t_2,\, \ldots,\, t_n)||^{\mathbf{L}}_{\mathbf{M},v} &= r_P(||t_1||^{\mathbf{L}}_{\mathbf{M},v},\, ||t_2||^{\mathbf{L}}_{\mathbf{M},v},\, \ldots,\, ||t_n||^{\mathbf{L}}_{\mathbf{M},v})\,,\\
||\varphi \oplus \psi||^{\mathbf{L}}_{\mathbf{M},v} &= ||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \oplus ||\psi||^{\mathbf{L}}_{\mathbf{M},v}\,,\\
||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v} &= \neg||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\,,\\
||\varphi \odot \psi||^{\mathbf{L}}_{\mathbf{M},v} &= ||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \odot ||\psi||^{\mathbf{L}}_{\mathbf{M},v}\,,\\
||\bar{0}||^{\mathbf{L}}_{\mathbf{M},v} &= \mathbf{0}\,,\\
||\triangle\varphi||^{\mathbf{L}}_{\mathbf{M},v} &= \triangle||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\,, \text{ in the case of } PL_\triangle \text{ or } PL'_\triangle\,,\\
||(\forall x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} &= \inf\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \mid v' \equiv_x v\}\,.
\end{aligned}$$

If the infimum does not exist, we take its value as undefined.

DEFINITION 6.6. *Let I be a predicate language, $\mathcal{C}$ either PL, $PL_\triangle$ or $PL'_\triangle$, $\mathbf{L}$ a linearly ordered $\mathcal{C}$-algebra, $\mathbf{M}$ an $\mathbf{L}$-structure for I and φ a formula of I.* A truth value of the formula φ *in $\mathbf{M}$ is defined as follows:*

$$||\varphi||^{\mathbf{L}}_{\mathbf{M}} = \inf\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \mid v \text{ is an } \mathbf{M}\text{-evaluation}\}.$$

We say that $\mathbf{M}$ is a safe *$\mathbf{L}$-structure, if $||\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ is defined for each φ and v, and that φ is an $\mathbf{L}$-*tautology *if $||\varphi||^{\mathbf{L}}_{\mathbf{M}} = \mathbf{1}$ for each safe $\mathbf{L}$-structure $\mathbf{M}$.*

LEMMA 6.7. *Let I be a predicate language, $\mathcal{C}$ either PL, $PL_\triangle$ or $PL'_\triangle$, $\mathbf{L}$ a linearly ordered $\mathcal{C}$-algebra, $\mathbf{M}$ a safe $\mathbf{L}$-structure for I, and v an $\mathbf{M}$-evaluation. Then*

$$||(\exists x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \sup\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \mid v' \equiv_x v\}.$$

Proof. We start with the proof that $\inf \neg b_v = \neg \sup b_v$. First, we show that $\neg \sup b_v$ is a lower bound: $b_v \leq \sup b_v$ for each v iff $\neg b_v \geq \neg \sup b_v$ for each v.

Second, we show that $\neg \sup b_v$ is the greatest lower bound. Let us suppose $z \leq \neg b_v$ for each v. Then $\neg z \geq b_v$ for each v. Hence $\neg z \geq \sup b_v$ and obviously $z \leq \neg \sup b_v$ for each v.

The proof can be completed by the following set of equations: $||(\exists x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||\neg((\forall x)\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \neg \inf\{\neg||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}|v' \equiv_x v\} = \neg\neg \sup\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}|v' \equiv_x v\} = \sup\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}|v' \equiv_x v\}$. □ □

Here we finally define the axiomatic system of our predicate logic. For the following definition, we suppose that the reader is familiar with the notion of free and bounded occurrence of an object variable in a formula and the notion of substitutable term into the formula φ.

DEFINITION 6.8. *Let $\mathcal{C}$ be either PL, $PL_\triangle$ or $PL'_\triangle$, and I be a predicate language. The logic $\mathcal{C}\forall$ isgiven by the following axioms and the deduction rules:*

- *(i)* *the formulas resulting from the axioms of $\mathcal{C}$ by the substitution of the propositional variables by the formulas of I,*
- *(ii)* ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, *where t is substitutable for x in φ,*
- *(iii)* ($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where x is not free in χ,*
- *(iv)* *deduction rules are modus ponens, generalization (from $\varphi(x)$ infer $(\forall x)\varphi(x)$), and necessitation of $\triangle$ in case of $PL_\triangle$ or $PL'_\triangle$ logic.*

LEMMA 6.9. *Let $\mathcal{C}$ be either PŁ, PŁ$_\triangle$ or PŁ$'_\triangle$. Then the logical axioms of quantifiers are* **L***-tautologies for each linearly ordered $\mathcal{C}$-algebra* **L** *and the deduction rules are sound.*

Proof. The axioms ($\forall$1) and ($\forall$2) are **L**-tautologies for each MV-algebra. Each $\mathcal{C}$-algebra is an expansion of an MV-algebra, and in both axioms only Łukasiewicz connectives are used (as structural connectives between formulas φ and χ; the internal structure of these formulas is unimportant), so the claim holds. Soundness of deduction rules is obvious. □ □

Now we are ready to prove the completeness theorem of our predicate logic. The proof of completeness of our predicate logics is analogous to the proof of completeness of the basic predicate logic (cf. [11, Theorem 5.2.9]).

We need to recall the definition of theory and of complete and consistent theory from propositional logic. These definitions remain as they stand, with only one exception—we restrict ourselves to the theories of closed formulas.

We also need to give a definition of model.

DEFINITION 6.10. *Let I be a predicate language, $\mathcal{C}$ either PŁ, PŁ$_\triangle$ or PŁ$'_\triangle$,* **L** *a linearly ordered $\mathcal{C}$-algebra,* **M** *an* **L***-structure for I, and T a theory over $\mathcal{C}\forall$. Then we say that* **M** *is an* **L***-model of T if $||\varphi||^{\mathbf{L}}_{\mathbf{M}} = \mathbf{1}$ for each φ from T.*

THEOREM 6.11 (Strong Completeness). *Let I be a predicate language, $\mathcal{C}$ either PŁ, PŁ$_\triangle$ or PŁ$'_\triangle$, T a theory over $\mathcal{C}\forall$, φ a closed formula. Then $T \vdash \varphi$ iff $||\varphi||^{\mathbf{L}}_{\mathbf{M}} = \mathbf{1}$ for each linearly ordered $\mathcal{C}$-algebra* **L** *and each safe* **L***-model* **M** *of T.*

Proof. Inspect the corresponding proof for the basic predicate logic in [11, Section 5.2]. □ □

THEOREM 6.12. *It holds:*

(1) *PŁ$_\triangle\forall$ logic is a conservative extension of PŁ$\forall$ logic.*
(2) *PŁ$'_\triangle\forall$ logic is strictly stronger than PŁ$_\triangle\forall$ logic.*
(3) ŁΠ$\forall$ *logic is a conservative extension of PŁ$'_\triangle\forall$ logic.*[1]
(4) ŁΠ$\forall$ *logic is not a conservative extension of PŁ$_\triangle\forall$ logic.*

Proof. (1) It follows from the fact that we can extend each linearly ordered PŁ-algebra by $\triangle$ and from the completeness theorems for both logics.

(2) An obvious consequence of Theorem 4.11.

(3) As it was shown by Montagna (cf. [19, Theorem 4.2]), the PŁ$'$-algebras are exactly subreducts of ŁΠ-algebras, thus obviously PŁ$'_\triangle$-algebras are also subreducts of ŁΠ-algebras. In other words, we can extend each linearly ordered PŁ$'_\triangle$-algebra to an ŁΠ-algebra. The completeness theorems for both logics complete the proof (cf. Theorem 6.11 and [3, Corollary 3.1.17]).

(4) A trivial consequence of (2) and (3). □ □

The questions whether PŁ$\forall$ is a conservative extension of the Łukasiewicz predicate logic, and whether ŁΠ$\forall$ is a conservative extension of PŁ$\forall$ logic seems to be open. The standard completeness of PŁ$\forall$, PŁ$_\triangle\forall$, and PŁ$'_\triangle\forall$ logics is a related problem. But this question can be answered. Here we assume that the reader is familiar with the basic concepts of undecidability and arithmetical hierarchy ([11, Section 6.1] is satisfactory for our needs). In the following we assume that our predicate language is at most countable.

THEOREM 6.13. *The set of $[0,1]_S$-tautologies of PŁ$\forall$ is Π_2-complete.*

[1]For the definition and details about ŁΠ$\forall$, see [3].

Proof. The Π_2-hardness is an obvious corollary of [11, Theorem 6.3.4], where it is proven that the set of $[0,1]_S$-tautologies of Ł$\forall$ is Π_2-complete.

The fact that the set of $[0,1]_S$-tautologies of PŁ$\forall$ is in Π_2 is a corollary of upcoming Theorem 6.19 (of the Pavelka style completeness of the extension of PŁ$\forall$ by rational constants). Thus we know that φ is the $[0,1]_S$-tautology iff $(\forall r \in \mathbb{Q} \cap [0,1))(\exists \text{ proof } \omega)(\omega \text{ is the proof of } \overline{r} \to \varphi)$. □ □

THEOREM 6.14. *The set of $[0,1]_{S\triangle}$-tautologies of PŁ$_\triangle\forall$ is not arithmetical.*

Proof. The proof is an obvious modification of the analogous proof for product predicate logic (see [12, Corollary 2]). □ □

COROLLARY 6.15. *Let $\mathcal{C}$ be either PŁ, PŁ$_\triangle$ or PŁ$'_\triangle$. Then the $\mathcal{C}\forall$ logic has not the standard completeness property.*

Another corollary is that all our logics (understood as sets of $[0,1]_S$ or $[0,1]_{S\triangle}$-tautologies) are undecidable. We present a partial answer to the problem of decidability of our logics understood as sets of theorems.

THEOREM 6.16. *Let $\mathcal{C}$ be either PŁ$_\triangle$ or PŁ$'_\triangle$. Then the $\mathcal{C}\forall$ logic (as set of theorems) is undecidable.*

Proof. The formula φ' is created from the formula φ by replacing each atomic formula $P(t_1, \dots, t_n)$ with the formula $\triangle P(t_1, \dots, t_n)$. Notice that formula ψ of a classical predicate language is provable iff formula ψ' is provable in $\mathcal{C}_\triangle\forall$ (this is obvious from the definition of $\triangle$ and completeness of the $\mathcal{C}_\triangle\forall$). Thus if the $\mathcal{C}_\triangle\forall$ logic would be decidable, it would be a contradiction to the undecidability of the classical predicate logic. □ □

6.1. **Rational Pavelka style of predicate logic.** In this section we build Rational Pavelka's style extension of the predicate logics defined in the previous subsection. Our approach will be analogous to the one that of Section 5. As we proved there, adding of the infinitary rule (IR) turns the rational extensions of PŁ$_\triangle$ and PŁ$'_\triangle$ into the same logics. Thus in this section we will develop only RPPŁ$\forall$ and RPPŁ$_\triangle\forall$ logics. Again we restrict ourselves to the standard algebras of the truth values ($[0,1]_S$ and $[0,1]_{S\triangle}$).

DEFINITION 6.17. *Let $\mathcal{C}$ be either PŁ or PŁ$_\triangle$.*

- *We extend the set of logical symbols by truth constant $\overline{r}$ for each $r \in \mathbb{Q} \cap [0,1]$ (see Definition 6.1).*
- *We extend the definition of a formula by a clause that $\overline{r}$ is a formula (see Definition 6.2).*
- *We extend the definition of truth value by a condition $||\overline{r}||^{\mathbf{L}}_{\mathbf{M},v} = r$ (see Definition 6.5).*
- *The logic RP$\mathcal{C}\forall$ results from $\mathcal{C}\forall$ by adding the book-keeping axioms from Definition 5.1 (see Definition 6.8).*
- *The deduction rules of RPPŁ$\forall$ are modus ponens and generalization, the deduction rules of RPPŁ$_\triangle\forall$ are modus ponens, generalization, necessitation, and (IR) (see Definitions 5.1, 6.8).*

DEFINITION 6.18. *Let $\mathcal{C}$ be either RPPŁ or RPPŁ$_\triangle$. Let T be a theory over $\mathcal{C}\forall$ and φ be a formula.*

(1) *The* truth degree *of φ over T is* $||\varphi||_T = \inf\{||\varphi||_{\mathbf{M}} \mid \mathbf{M} \text{ is a model of} T\}$.

(2) *The* provability degree *of φ over T is* $|\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \varphi\}$.

THEOREM 6.19 (Pavelka's style completeness). *Let $\mathcal{C}$ be either RPPŁ or RPPŁ$_\triangle$. Let T be a theory over $\mathcal{C}\forall$ and φ be a formula. Then*

$$||\varphi||_T = |\varphi|_T.$$

Proof. Inspect the corresponding proof of RPŁ$\forall$ in [11, Section 5.4]. The crucial point is [11, Lemma 5.4.7]. The necessary modification of the proof of this lemma is done by means of Lemma 5.5. □ □

LEMMA 6.20. *Let T be a theory over either RPPŁ$_\triangle\forall$ and φ be a formula. Then $|\varphi|_T = 1$ iff $T \vdash \varphi$.*

Proof. An analogy of the proof of Lemma 5.4. □ □

THEOREM 6.21 (strong standard completeness). *Let T be a theory over RPPŁ$_\triangle\forall$ and φ be a formula. Then $T \vdash \varphi$ iff $||\varphi||_{\mathbf{M}} = 1$ for all standard models $\mathbf{M}$.*

Proof. The proof can be done by a straightforward application of Theorem 6.19 and Lemma 6.20. □ □

COROLLARY 6.22. *RŁΠ$\forall$ is a conservative extension of RPPŁ$_\triangle\forall$.*[2]

Proof. It is the consequence of [3, Theorem 3.2.6] and the previous theorem. □ □

At the end of this section we prove a consequence of the standard completeness of RPPŁ$_\triangle\forall$. We make a small modification of the language of RPPŁ$_\triangle\forall$ and show that the resulting logic TT coincides with the famous logic of Takeuti and Titani.

The logic TT results from RPPŁ$_\triangle\forall$ by omitting truth constants $\overline{r}$ such that r can not be expressed in the form $\frac{k}{2^n}$. In other words, we may say that the logic TT has only one additional truth constant, $\overline{\mathbf{\frac{1}{2}}}$, and the other truth constants are defined by using the connectives of PŁ$_\triangle\forall$.

Now we can easily prove the analogy of Theorem 6.21 for TT. Just go through the proofs leading to this theorem and notice that the set of truth constants in TT is dense (as in the case of RPPŁ$_\triangle\forall$). Thus all the proofs will be sound for TT as well. This gives us the following corollary:

COROLLARY 6.23. *RPPŁ$_\triangle\forall$ is a conservative extension of TT.*

Takeuti and Titani's logic was introduced by Takeuti and Titani in their work [23]. It is a predicate fuzzy logic based on Gentzen's system of intuitionistic predicate logic. The connectives used by this logic are just the connectives of RPPŁ$_\triangle\forall$ logic. This logic has two additional deduction rules (named $\mathcal{R}1$, $\mathcal{R}2$) and 46 axioms (namely **F1** – **F46**). We will not present the axiomatic system and we only recall that this logic is sound and complete w.r.t. the standard PL$_\triangle$-algebra (cf. [23, Theorem 1.4.3]). All this leads us to the following conclusion:

THEOREM 6.24. *Takeuti and Titani logic coincides with the logic TT. Furthermore, RPPŁ$_\triangle\forall$ logic is a conservative extension of Takeuti and Titani logic.*

This theorem allows to translate the results from the Takeuti and Titani's logic into our much more simpler (in syntactical sense) logical system of the TT or RPPŁ$_\triangle\forall$ logic. An interesting corollary of this theorem and the previous corollary is a very simple proof of one part of [4, Theorem 10]:

THEOREM 6.25. *Takeuti and Titani's logic is contained in RŁΠ$\forall$ logic.*

◇

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[2]For the definition and details about RŁΠ$\forall$ see [3].

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