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Algebraic Properties of Fuzzy Logics

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Contents

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Chapter 1

Introduction

As the title of this thesis suggests, it should be a contribution to the field of fuzzy logic. In order to make it clear, we firstly explain what we understand under the term “fuzzy logic”. The word “fuzzy” in a relation with mathematics was likely used for the first time by L. A. Zadeh in his paper [38] on fuzzy sets. He came with an idea to introduce a new kind of a set (so-called fuzzy set) to which its elements belong with a certain degree. In the classical setting an element either belongs to a set or not. If A is a classical set then the formula $x \in A$ is either absolutely true or absolutely false. In the case of a fuzzy set A, an element x can attain more than two degrees of its membership. Thus the formula $x \in A$ may be only partially satisfied. In order to define fuzzy sets in a proper way, we need a logical calculus which is able to cope with partially true statements. Such logical calculus is called fuzzy logic. Similarly as in the classical setting fuzzy logic can be propositional or predicate. We will focus here only on propositional fuzzy logics.

This thesis is devoted to the research direction started by Hájek. He introduced one of the most successful fuzzy logics, so-called *Hájek's Basic Logic* (BL for short). In his monograph [17] one can find a lot of interesting results about this calculus (like completeness theorem) and also connections to other already known calculi (like Łukasiewicz logic or Gödel logic). The logic BL has an algebraic type of semantics, so-called *BL-algebras.* They play an analogous role for BL as Boolean algebras for the classical logic. BL-algebras form in fact a subvariety of residuated lattices[1]. A motivation example of a BL-algebra is the real unit interval endowed with a continuous t-norm[2] interpreting a conjunction and the corresponding residuum interpreting an implication. Such BL-algebra is often

[1]A residuated lattice is a lattice ordered monoid whose monoid operation possesses a so-called residuum or residual (for details see [32]).

[2]A t-norm is a commutative, associative, binary operation on the real unit interval which is monotone in each argument and 1 is a neutral element.

called a *standard* BL-algebra. Hájek proved that BL is sound and complete w.r.t. the class of BL-algebras, i.e., a formula is provable in BL if and only if it is a tautology in all BL-algebras. Later it was shown in [10] that BL is also complete w.r.t. the standard BL-algebras. This kind of result is usually referred to as *Standard Completeness Theorem.*

In a standard BL-algebra the t-norm is continuous. However, for the existence of the corresponding residuum only left-continuity is sufficient. Thus Esteva and Godo introduced a weaker logic so-called *Monoidal T-norm Based Logic* (MTL for short) capturing this fact. An algebraic counterpart of MTL is the class of MTL-algebras. Similarly to the case of BL it was showed that MTL is complete w.r.t. the class of MTL-algebras. Jenei and Montagna even proved in [28] that MTL enjoys Standard Completeness Theorem.

The logic BL has three basic schematic extensions; Łukasiewicz logic, Gödel logic, and the product logic. It is a natural question what happens if we add the axiomatic schemata corresponding to the basic extensions of BL to MTL. Hájek showed in [19] that MTL plus the schema characteristic of Gödel logic already collapses to Gödel logic. If we add to MTL the schema characteristic of Łukasiewicz logic, we obtain a strictly weaker logic than Łukasiewicz logic, so-called *Involutive Monoidal T-norm Based Logic* (IMTL for short). Finally, the extension of MTL by the schemata characteristic of the product logic is called *Product Monoidal T-norm Based Logic* (ΠMTL for short) and is strictly weaker than the product logic. It is just ΠMTL which this thesis is focused on.

1.1 Goals of this thesis

We would like to present here mainly two results concerning ΠMTL. The first one is a solution of an open problem whether ΠMTL satisfies Standard Completeness Theorem. This question was posed by Esteva, Gispert, Godo, and Montagna in [13]. In Chapter 3 we provide a positive answer to this question. Thus we show in fact that the variety of ΠMTL-algebras is generated by standard ΠMTL-algebras, i.e., ΠMTL-algebras in the real unit interval $[0,1]$.

The second part of the thesis (Chapter 4) gives a characterization of the structure of the standard ΠMTL-algebras since they are the generators of the whole variety. Let $\mathbf{L}$ be a standard ΠMTL-algebra. Firstly, we show that the ℓ-monoid reduct of $\mathbf{L}$ can be extended to a totally ordered Abelian group by making fractions since the monoid operation of $\mathbf{L}$ is cancellative. Then we use Hahn's Embedding Theorem (see e.g. [14, 15]) and prove that the ℓ-monoid reduct of $\mathbf{L}$ can be embedded into a full Hahn group $\mathbf{V}$. Finally, we describe those elements in $\mathbf{V}$ whose preimages belong to $\mathbf{L}$. More precisely, we present a method how to construct from a full Hahn group a standard subdirectly irreducible ΠMTL-algebra and then we prove that each standard subdirectly irreducible ΠMTL-algebra can be obtained in this way. Thus we get a characterization of the standard subdirectly irreducible ΠMTL-algebras up to an isomorphism.

1.2 Notation

Throughout the text we use the following notation. The set of naturals, integers, rationals, and reals are denoted respectively by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$. The set of non-positive reals is denoted by $\mathbb{R}^- = \{r \in \mathbb{R} \mid r \leq 0\}$. The set of non-positive integers is denoted by $\mathbb{Z}^- = \{z \in \mathbb{Z} \mid z \leq 0\}$. Ordinals are denoted by Greek letters. Let α be an ordinal. Then α^* stands for the reversely ordered α, e.g. ω^* is the set of natural numbers ordered as follows: $0 > 1 > 2 \ldots$. If $f : A \to B$ is a function and $C \subseteq A$ then the restriction of f to C is denoted by $f \restriction C$.

1.2.1 Partially ordered sets

Let $(P, \leq)$ be a partially ordered set (*poset*). The poset P is said to be inversely well ordered (i.w.o.) if each subset of P contains a maximum. Then clearly each element in P has a predecessor. Further, the poset $(P, \preccurlyeq)$, where $x \preccurlyeq y$ iff $y \leq x$ for $x, y \in P$, is called the *dual* of P and is denoted by P^∂. In any poset P intervals will be respectively denoted by

$$\begin{aligned}
&[x,y] = \{z \in P \mid x \leq z \leq y\}, &&(x,y] = \{z \in P \mid x < z \leq y\},\\
&[x,y) = \{z \in P \mid x \leq z < y\}, &&(x,y) = \{z \in P \mid x < z < y\},\\
&[x,\to) = \{z \in P \mid x \leq z\}, &&(x,\to) = \{z \in P \mid x < z\},\\
&(\leftarrow,x] = \{z \in P \mid z \leq x\}, &&(\leftarrow,x) = \{z \in P \mid z < x\}.
\end{aligned}$$

We also use without mentioning the alternative signature for a lattice using the lattice order $\leq$ instead of $\wedge, \vee$.

1.2.2 Universal algebra

Algebras are denoted by boldface capitals (e.g. $\mathbf{L}$). The corresponding universes are denoted by the same capital (e.g. L). Let $\mathbf{L}$ be an algebra and $G \subseteq L$. Then the subalgebra generated by G will be denoted by $\mathbf{Sg}(G)$. The congruence lattice of $\mathbf{L}$ will be denoted by $\operatorname{Con} \mathbf{L}$ and the minimum and the maximum congruence will be denoted by Δ, ∇ respectively. Let $\theta \in \operatorname{Con} \mathbf{L}$ be a congruence. The factor algebra corresponding to θ is denoted by $\mathbf{L}/\theta$. An equivalence class (i.e., an element of $\mathbf{L}/\theta$) containing an element $a \in L$ is denoted by $[a]_\theta$. If S is a subset of L then the restriction of θ to S^2 is denoted by $\theta\restriction_S$.

1.2.3 Lattice ordered algebraic structures

A commutative *lattice ordered monoid* (ℓ-monoid) $\mathbf{M} = (M, *, \leq, \mathbf{1})$ is a structure where $(M, *, 1)$ is a commutative monoid, $(M, \leq)$ is a lattice, and the operation $*$ is order-preserving, i.e., for all $a, b, c \in M$, $a \leq b$ implies $a * c \leq b * c$. Since in this text we deal only with commutative ℓ-monoids, the adjective "commutative" is often omitted. An ℓ-monoid is said to be *integral* if the neutral element $\mathbf{1}$ is also the top element of the lattice reduct. In the case when an ℓ-monoid $\mathbf{M}$ is totally ordered (i.e., the lattice reduct forms a chain) we call it an o-monoid. An ℓ-monoid $\mathbf{M}$ is said to be *cancellative* if for all $x, y, z \in M$, $x * z = y * z$ implies $x = y$.

Let I be a well ordered index set and for any $i \in I$, let $\mathbf{M}_i = (M_i, *_i, \leq_i, \mathbf{1}_i)$ be an o-monoid. Then a *lexicographic product* $\mathbf{M} = \prod_{i \in I} \mathbf{M}_i$ is the following o-monoid. The monoid reduct of $\mathbf{M}$ is just the direct product of the monoid reducts of $\mathbf{M}_i$. Let $f, g \in M$. The order on $\mathbf{M}$ is defined as follows: $f < g$ iff $f(i) < g(i)$, where i is the least index such that $f(i) \neq g(i)$.

An Abelian *lattice ordered group* (ℓ-group) $\mathbf{G} = (G, *, ^{-1}, \leq, \mathbf{1})$ is a structure where $(G, *, ^{-1}, \mathbf{1})$ is an Abelian group, $(G, \leq)$ is a lattice, and $*$ is order-preserving, i.e., for all $a, b, c \in M$, $a \leq b$ implies $a * c \leq b * c$. Again if $(G, \leq)$ forms a chain then $\mathbf{G}$ is referred to as an o-group.

Chapter 2

State of the art

Nowadays fuzzy logic is a widely developed area of mathematical logic. Thus it is not possible to present here all important results belonging to this area. Nevertheless, we will try to collect in this chapter most of the definitions and results related to our work in order to make this thesis reasonably self-contained.

2.1 Residuated lattices

Since in many cases the set of truth values of a fuzzy logic forms a residuated lattice, we describe residuated lattices in this section. Such lattices were firstly introduced by Ward and Dilworth in [37] as a generalization of ideal lattices of rings. More precisely, it can be shown that the collection of all two-sided ideals of a ring forms a residuated lattice. General algebraic facts about residuated lattices can be found in papers [2, 4, 20, 32].

A *commutative residuated lattice* $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is an algebraic structure, where $(L, *, \mathbf{1})$ is a commutative monoid, $(L, \wedge, \vee)$ is a lattice, and $(*, \rightarrow)$ form a residuated pair, i.e.,

$$x * y \leq z \text{ iff } x \leq y \rightarrow z\,. \tag{2.1}$$

The operation $\rightarrow$ is called a *residuum.* When we refer to a commutative residuated lattice, we will omit the word commutative since we will deal here only with the commutative case. The symbol a^n stands for $a * \cdots * a$ (n times). In the absence of parenthesis, $*$ is performed first, followed by $\rightarrow$, and finally $\vee$ and $\wedge$.

It follows from the definition that $*$ is order-preserving, i.e., $a \leq b$ implies $a * c \leq b * c$. Indeed, $b * c \leq b * c$ iff $b \leq c \rightarrow b * c$ by (2.1). Since $a \leq b$, we get $a \leq c \rightarrow b * c$. Thus $a * c \leq b * c$ by (2.1). The residuum is decreasing in the first argument and increasing in the second one. Let $a \leq b$. By (2.1) we have $b \rightarrow c \leq$

$b \to c$ iff $b * (b \to c) \leq c$. Since $*$ is order-preserving, $a * (b \to c) \leq b * (b \to c) \leq c$. Thus $b \to c \leq a \to c$ by (2.1). For the second argument we have $c \to a \leq c \to a$ iff $c * (c \to a) \leq a$. Thus $c * (c \to a) \leq b$ and we get $c \to a \leq c \to b$ by (2.1).

Further, the inequality $a * x \leq b$ has a greatest solution for x (namely $a \to b$). Thus

$$a \to b = \max\{x \in L \mid a * x \leq b\}\,.$$

In particular, the residuum is uniquely determined by $*$ and $\leq$. The existence of the residuum has the following consequence which will be useful for us later.

Proposition 2.1.1 *Let* $\mathbf{L}$ *be a residuated lattice.*

1. *The operation* $*$ *preserves all existing joins in each argument, i.e., if* $\bigvee X$ *and* $\bigvee Y$ *exist for* $X, Y \subseteq L$ *then* $\bigvee_{x\in X, y\in Y} x * y$ *exists and*

$$\left(\bigvee X\right) * \left(\bigvee Y\right) = \bigvee_{x\in X, y\in Y} x * y\,.$$

2. *The residuum preserves all existing meets in the second argument and convert existing joins to meets in the first argument, i.e., if* $\bigvee X$ *and* $\bigwedge Y$ *exist for* $X, Y \subseteq L$ *then for any* $z \in L$, $\bigwedge_{x\in X}(x \to z)$ *and* $\bigwedge_{y\in Y}(z \to y)$ *exist and*

$$\left(\bigvee X\right) \to z = \bigwedge_{x\in X}(x \to z) \text{ and } z \to \left(\bigwedge Y\right) = \bigwedge_{y\in Y}(z \to y)\,.$$

Here we collect several identities valid in any residuated lattice used throughout the text.

Proposition 2.1.2 *The following identities hold in any residuated lattice* $\mathbf{L}$*.*

1. $x * (x \to y) \leq y$,
2. $x \to y \leq z * x \to z * y$,
3. $(x \to y) * (y \to z) \leq x \to z$,
4. $x * y \to z = x \to (y \to z)$,
5. $(x \vee y) * z = (x * z) \vee (y * z)$,
6. $\mathbf{1} \to x = x$.

A residuated lattice $\mathbf{L}$ is said to be *integral* if $\mathbf{1}$ is the top element of L. In this case we have that $x \leq y$ implies $x \to y = \mathbf{1}$. Indeed, $\mathbf{1} * x \leq y$ iff $\mathbf{1} \leq x \to y$. If a residuated lattice possesses a bottom element $\mathbf{0}$, then we have $\mathbf{0} * x = \mathbf{0}$. Further, a totally ordered residuated lattice is referred to as a *residuated chain*. A residuated lattice $\mathbf{L}$ is said to be *cancellative* if for any $x, y, z \in L$, $x * z = y * z$ implies $x = y$.

The class of commutative residuated lattices forms a finitely based variety. An equational basis is given, for example, by a basis for lattice identities, monoid identities, $x * ((x \to z) \wedge y) \leq z$, and $y \leq x \to (x * y \vee z)$ (see [32]). Moreover, the variety is an *ideal variety*, i.e., congruences are determined by their $\mathbf{1}$-congruence classes, and these are further characterized as convex subalgebras. A subalgebra $\mathbf{S}$ is *convex* if $[x, y] \subseteq S$ for all $x, y \in S$. The following two theorems were presented in [20].

Theorem 2.1.3 *The lattice* $\mathrm{CS}(\mathbf{L})$ *of convex subalgebras of a commutative residuated lattice* $\mathbf{L}$ *is isomorphic to its congruence lattice* $\mathrm{Con}\,\mathbf{L}$*. Let* $\mathbf{H} \in \mathrm{CS}(\mathbf{L})$ *and* $\theta \in \mathrm{Con}\,\mathbf{L}$*. The isomorphism is given by the mutually inverse maps*

$$\mathbf{H} \mapsto \theta_H = \{\langle a, b\rangle \mid (b \to a) \wedge \mathbf{1} \in H \text{ and } (a \to b) \wedge \mathbf{1} \in H\} \text{ and } \theta \mapsto [\mathbf{1}]_\theta\,.$$

In fact the congruences are already determined by the negative part of $[\mathbf{1}]_\theta$. Let $\mathbf{L}$ be a residuated lattice. Then L^- denotes the negative part of $\mathbf{L}$, i.e.,

$$L^- = \{x \in L \mid x \leq \mathbf{1}\}\,.$$

Theorem 2.1.4 *Let* $\mathbf{S}$ *be a convex submonoid of* $\mathbf{L}$ *such that* $S \subseteq L^-$*. Then defining the set* H_S *by*

$$H_S = \{a \in L \mid s \leq a \leq s \to \mathbf{1} \text{ for some } s \in S\}\,,$$

$\mathbf{H}_S$ *is a convex subalgebra of* $\mathbf{L}$ *and* $S = H_S^-$*. Conversely, if* $\mathbf{H}$ *is any convex subalgebra of* $\mathbf{L}$ *then, setting* $S_H = H^-$*,* $\mathbf{S}_H$ *is a convex submonoid of* $\mathbf{L}$ *and* H *can be recovered from* S_H *as described above. Moreover, the mutually inverse maps* $\mathbf{H} \mapsto \mathbf{S}_H$ *and* $\mathbf{S} \mapsto \mathbf{H}_S$ *establish a lattice isomorphism between the lattice* $\mathrm{CS}(\mathbf{L})$ *of convex subalgebras of* $\mathbf{L}$ *and the lattice* $\mathrm{CM}(\mathbf{L})$ *of convex submonoids of* $\mathbf{L}$ *whose underlying sets are subsets of* L^-*.*

2.1.1 Abelian totally ordered groups

One of the typical examples of a residuated lattice is an Abelian lattice ordered group (ℓ-group). In this case $a \to b = a^{-1} * b$. Such objects were extensively studied in mathematics (for details see [14, 15]). Since we will deal with a subclass of Abelian totally ordered ℓ-groups (o-groups), we also recall several facts about them useful in the sequel. It follows from Theorem 2.1.3 that the congruences on an o-group $\mathbf{G}$ are completely determined by convex subgroups, i.e., Con $\mathbf{G}$ is isomorphic to the chain of all convex subgroups. Let V be a convex subgroup of $\mathbf{G}$. Then the factor o-group will be denoted by $\mathbf{G}/V$. The equivalence class of $x \in G$ w.r.t. the congruence given by V will be denoted by $[x]_V$.

A convex subgroup generated by an element g is said to be *principal* and will be denoted by V^g. The principal convex subgroups are characterized in the following lemma (see [15, Lemma 3.1.5]).

Lemma 2.1.5 *If* $\mathbf{G}$ *is an o-group and* $g \in G$*, then*

$$V^g = \{f \in G \mid |f| \leq |g|^n \text{ for some } n \in \mathbb{N}\}\,,$$

where $|g| = g \vee g^{-1}$*.*

Since a union of any system of convex subgroups is again a convex subgroup, each V^g has a predecessor, namely the largest convex subgroup not containing g, i.e., the union of all such subgroups. This predecessor will be denoted by V_g. Following the terminology from [15], V_g, the largest convex subgroup not containing g, will be called a *value* of g.

2.2 Hájek's Basic Logic

Probably one of the most successful fuzzy logic is Hájek's Basic Logic (BL for short). Many details about BL can be found in Hájek's monograph [17]. Here we will deal only with the propositional version of BL. The original motivation was to introduce a truth-functional many-valued logic where the set of truth values is the real unit interval $[0, 1]$ with the usual order and a conjunction is interpreted by a continuous t-norm $*$. A *t-norm* $* : [0, 1]^2 \to [0, 1]$ is a commutative, associative operation which is non-decreasing in each argument and $a * 1 = a$ for all $a \in [0, 1]$. Since the only deduction rule of BL is modus ponens, the interpretation of an implication $\to$ is chosen so that it is a maximal function from $[0, 1]^2$ to $[0, 1]$ such that modus ponens is still a correct deduction rule. The maximality of $\to$ follows from the fact that we want to have the deduction rule powerful. It can be shown that $[0, 1]_* = ([0, 1], *, \to, \leq, 1)$ forms an integral residuated chain. Thus $a \to b = \max\{z \in [0, 1] \mid a * z \leq b\}$. Since we want to have in our logic an absolute falsity, the bottom element 0 is added into the signature of $[0, 1]_*$, i.e., $[0, 1]_* = ([0, 1], *, \to, \leq, 0, 1)$. Moreover, the continuity of $*$ implies a divisibility of $[0, 1]_*$, i.e., if $b \leq a$ then there exists $z \in [0, 1]$ (namely $a \to b$) such that $a * z = b$. Hence the meet is definable in $[0, 1]_*$ as $a \wedge b = a * (a \to b)$. Also the join can be defined by $a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$ since $[0,1]_*$ is integral and totally ordered. It may be helpful for a reader to keep this motivation in mind when reading the formal definition of BL.

2.2.1 Syntax and semantics of BL

Let $[0,1]_* = ([0,1], *, \to, \leq, 0, 1)$ be the same as in the previous paragraph. The language of BL contains as usual a countable set of propositional variables, a conjunction &, an implication $\Rightarrow$, and the constant $\bar{0}$. Further connectives are defined as follows:

$$
\begin{array}{cll}
\varphi \wedge \psi & \text{is} & \varphi \& (\varphi \Rightarrow \psi), \\
\varphi \vee \psi & \text{is} & ((\varphi \Rightarrow \psi) \Rightarrow \psi) \wedge ((\psi \Rightarrow \varphi) \Rightarrow \varphi), \\
\neg \varphi & \text{is} & \varphi \Rightarrow \bar{0}, \\
\varphi \equiv \psi & \text{is} & (\varphi \Rightarrow \psi) \& (\psi \Rightarrow \varphi), \\
\bar{1} & \text{is} & \neg \bar{0}.
\end{array}
$$

An evaluation e of propositional variables is a mapping assigning to each propositional variable p its truth value $e(p)$. This can be uniquely extended to the evaluation of all formulas as follows:

$$
\begin{aligned}
e(\bar{0}) &= 0\,, \\
e(\varphi \& \psi) &= e(\varphi) * e(\psi)\,, \\
e(\varphi \Rightarrow \psi) &= e(\varphi) \to e(\psi)\,.
\end{aligned}
$$

The following formulas are the *axioms* of BL capturing the above-mentioned properties of the integral residuated chain $[0,1]_*$ from the motivation paragraph:

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))$
(A2) $\varphi \& \psi \Rightarrow \varphi$
(A3) $\varphi \& \psi \Rightarrow \psi \& \varphi$
(A4) $\varphi \& (\varphi \Rightarrow \psi) \Rightarrow \psi \& (\psi \Rightarrow \varphi)$
(A5a) $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\varphi \& \psi \Rightarrow \chi)$
(A5b) $(\varphi \& \psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))$
(A6) $((\varphi \Rightarrow \psi) \Rightarrow \chi) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \chi) \Rightarrow \chi)$
(A7) $\bar{0} \Rightarrow \varphi$

The *deduction rule* of BL is modus ponens, i.e. from φ and $\varphi \Rightarrow \psi$ derive ψ. The notion of a proof in BL is defined as in the classical logic.

The axioms (A1), (A2), (A3), and (A5) capture the facts that $[0,1]_*$ is an integral commutative residuated lattice. The axiom (A4) expresses the fact that $[0,1]_*$ is divisible. The axiom (A6) is a variant of the proof by cases expressing somehow that $[0,1]_*$ is totally ordered. The fact that $[0,1]_*$ is a bounded lattice is captured in (A7).

It can be easily checked that all axioms of BL are valid in $[0,1]_*$, i.e., for any evaluation all axioms are evaluated by 1. The formulas to which any evaluation assigns 1 are called 1-tautologies. Furthermore, since modus ponens preserves 1-tautologies, all formulas provable in BL are 1-tautologies.

Now, we are going to give a definition of a BL-algebra. BL-algebras play the same role in BL like Boolean algebras in the classical logic. They form a subvariety of integral residuated lattices.

Definition 2.2.1 A structure $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a BL-*algebra* if the following conditions are satisfied for all $x, y \in L$:

1. $(L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is an integral residuated lattice,
2. $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice,
3. $x \wedge y = x * (x \rightarrow y)$,
4. $(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}$.

A totally ordered BL-algebra is referred to as BL-*chain*. The last identity in the definition $(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}$ is called *prelinearity*.

Observe that in any integral residuated chain the prelinearity equation is trivially satisfied. Further, it is clear that the integral residuated lattice $[0,1]_*$ from the motivation paragraph is a BL-chain. The BL-chain $[0,1]_* = ([0,1], *, \rightarrow, \leq, 0, 1)$ where $*$ is a continuous t-norm and $\rightarrow$ is the corresponding residuum will be called *standard* BL-chain.

Let $\mathbf{L}$ be a BL-algebra. In the same way as it was done at the beginning of this section, an $\mathbf{L}$-evaluation is a mapping from the set of all formulas to L assigning to each propositional variable p an element of L and satisfying $e(\overline{0}) = \mathbf{0}$, $e(\varphi \& \psi) = e(\varphi) * e(\psi)$, $e(\varphi \Rightarrow \psi) = e(\varphi) \rightarrow e(\psi)$. A formula φ is an $\mathbf{L}$-*tautology* if $e(\varphi) = \mathbf{1}$ for each $\mathbf{L}$-evaluation e. The logic BL is sound with respect to $\mathbf{L}$-tautologies, i.e., whenever φ is provable in BL then φ is an $\mathbf{L}$-tautology for each BL-algebra $\mathbf{L}$.

Thanks to the prelinearity it was shown (see [17, Lemma 2.3.16]) that the variety of BL-algebras is generated by BL-chains.

Theorem 2.2.2 (Subdirect Representation) *Each BL-algebra is isomorphic to a subdirect product of BL-chains.*

2.2.2 Structure of continuous t-norms

Among all continuous t-norms there are three t-norms which play an important role in the characterization of continuous t-norms.

Gödel t-norm:	$x * y = \min\{x, y\}$,
Product t-norm:	$x * y = xy$ (the usual product of reals),
Łukasiewicz t-norm:	$x * y = \max\{0, x + y - 1\}$.

The following result was firstly presented in [36]. Let $*$ be a continuous t-norm. An element $x \in [0, 1]$ is said to be an *idempotent* if $x * x = x$. It can be shown that the set E of all idempotents of $*$ is closed subset of $[0, 1]$. Thus $\mathcal{I}_o = [0, 1] - E$ is an at most countable union of non-overlapping open intervals. Let $\mathcal{I}$ be the set of closed intervals such that $[a, b] \in \mathcal{I}$ iff $(a, b) \in \mathcal{I}_o$. Further, let $I \in \mathcal{I}$ and $*_I$ be the restriction of $*$ on I^2. We say that $*_I$ is isomorphic to a t-norm $*'$ if there is a bijection $f : I \to [0, 1]$ preserving the operation $*_I$, i.e., for any $a, b \in I$ we have $f(a *_I b) = f(a) *' f(b)$.

Theorem 2.2.3 *Let $*$ be any continuous t-norm and the set $\mathcal{I}$ be as above.*

1. *For each $I \in \mathcal{I}$, $*_I$ is isomorphic either to the product t-norm or Łukasiewicz t-norm.*
2. *If for $x, y \in [0, 1]$ there is no $I \in \mathcal{I}$ such that $x, y \in I$, then $x * y = \min\{x, y\}$.*

2.2.3 Some schematic extensions of BL

A logical calculus $\mathcal{C}$ is a *schematic extension* of BL if it results from BL by adding some axiom schemata. Let $\mathcal{C}$ be a schematic extension of BL. Then BL-algebra $\mathbf{L}$ is a $\mathcal{C}$-algebra if all axioms of $\mathcal{C}$ are $\mathbf{L}$-tautologies. For each of the three basic t-norms there are corresponding schematic extensions of BL.

Definition 2.2.4 *Gödel logic* G is a schematic extension of BL by the axiom schema $\varphi \Rightarrow (\varphi \& \varphi)$.

The algebras of truth values for Gödel logic are called Gödel algebras. Such algebras were firstly considered by Kurt Gödel in the proof that the intuitionistic propositional calculus is not complete w.r.t. any finitely-valued semantics (cf. [16]).

Definition 2.2.5 The *product logic* Π is a schematic extension of BL by the following axiom schemata:

(Π1)	$\neg\neg\chi \Rightarrow ((\varphi \& \chi \Rightarrow \psi \& \chi) \Rightarrow (\varphi \Rightarrow \psi))$,
(Π2)	$\varphi \wedge \neg\varphi \Rightarrow \overline{0}$.

The corresponding algebras of truth values are called *product algebras*. The product logic was firstly axiomatized and studied by Esteva, Godo, Hájek in [12].

Definition 2.2.6 *Łukasiewicz logic* Ł is a schematic extension of BL by the axiom schema $\neg\neg\varphi \Rightarrow \varphi$.

The algebras of truth values are known under the names *MV-algebras* or *Wajsberg algebras*. Łukasiewicz logic was introduced by Jan Łukasiewicz in [34]. A deep investigation of MV-algebras can be found in [7].

2.2.4 Structure of BL-chains

Theorem 2.2.3 shows that each continuous t-norm can be composed up to an isomorphism from the three basic t-norms. Similar representation can be proved also for BL-chains. The results from this part of the text are taken from [18, 10].

An element u in a BL-algebra is called *idempotent* if $u * u = u$.

Lemma 2.2.7 *Let* $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ *be a BL-chain and* $u \leq v$ *be two idempotents of* $\mathbf{L}$. *Then* $[u, v]_{\mathbf{L}} = ([u, v], *', \rightarrow', \wedge, \vee, u, v)$ *is a BL-chain where* $*'$ *is the restriction of* $*$ *to* $[u, v]$ *and for all* $x, y \in [u, v]$,

$$x \rightarrow' y = \begin{cases} x \rightarrow y\,, & \text{if } x > y\,, \\ v\,, & \text{otherwise}\,. \end{cases}$$

Note that if $u = v$ then $[u, u]_{\mathbf{L}}$ is the trivial one-element BL-algebra.

A pair (X, Y) of subsets of L is called a *cut* in a BL-algebra $\mathbf{L}$ if the following conditions are satisfied:

1. $X \cup Y = L$,
2. $x \leq y$, for all $x \in X$ and $y \in Y$,
3. Y is closed under $*$,
4. $x * y = x$, for all $x \in X$ and $y \in Y$.

A BL-chain is called *saturated* if for each cut (X, Y) there is an idempotent u such that $x \leq u \leq y$ for all $x \in X$ and $y \in Y$.

Theorem 2.2.8 ([18, Theorem 3]) *Each BL-chain* $\mathbf{L}$ *can be isomorphically embedded into a saturated BL-chain* $\overline{\mathbf{L}}$. *Moreover,* $\mathbf{L}$ *is dense*[1] *in* $\overline{\mathbf{L}}$.

The structure of a saturated BL-chain can be described in terms of so-called ordinal sum which is a way how to connect several BL-chains in order to obtain a new one. The construction for BL-chains comes from [18, Definition 4].

[1] In the sense that for any two elements $u < u'$ in $\overline{L} - L$ there is an $x \in L$ such that $u < x < u'$.

Definition 2.2.9 (Ordinal sum) Let $(I, \leq)$ be a chain with a least element 0 and a greatest element 1. For each $\alpha \in I$, let α^+ be the successor of α, if it exists, otherwise $\alpha^+ = \alpha$. Let $\{\mathbf{L}_\alpha \mid \alpha \in I\}$ be an indexed family of BL-chains such that $\mathbf{L}_\alpha$ has the least element α, the greatest α^+, and non-extremal elements do not belong to $\mathbf{L}_\beta$ for $\beta \neq \alpha$. The *ordinal sum* $\bigoplus_{\alpha \in I} \mathbf{L}_\alpha$ is defined as follows:

1. the universe is $\bigcup_{\alpha \in I} L_\alpha$,
2. for $x \in L_\alpha$, $y \in L_\beta$, we put $x \leq y$ iff $\alpha < \beta$ or $[\alpha = \beta$ and $x \leq_\alpha y]$,
3. $x * y = x *_\alpha y$ for $x, y \in L_\alpha$,
4. $x * y = \min\{x, y\}$ for $x \in L_\alpha$, $y \in L_\beta$, and $\alpha \neq \beta$,
5. $x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ x \to_\alpha y, & \text{if } x > y \text{ and } x, y \in L_\alpha, \\ y, & \text{if } x \in L_\alpha,\ y \in L_\beta,\ \alpha \neq \beta, \text{ and } x > y. \end{cases}$

Lemma 2.2.10 *The ordinal sum* $\mathbf{L} = \bigoplus_{\alpha \in I} L_\alpha$ *is a BL-chain. Moreover, for each* $\alpha \in I$, $\mathbf{L}_\alpha = [\alpha, \alpha^+]_{\mathbf{L}}$.

Now, we are going to present a representation theorem for saturated BL-chains as it was given in [10]. There is also another very important representation for BL-chains in terms of hoops (for details, see [1]) but it goes behind the scope of this text.

Let $\mathbf{L}$ be a saturated BL-chain and $E \subseteq L$ be the set of all its idempotents. The following was shown in [10, Lemma 3.5]: 1. for any $c \in E$ there is a maximal closed interval $[a, b] \subseteq E$ such that $c \in [a, b]$; 2. for any $c \notin E$ there exists a closed interval $[a, b]$ such that $c \in [a, b]$ and $[a, b] \cap E = \{a, b\}$. Let $\mathcal{I}(E) = \{[a, b] \mid a, b \in E,\ a < b, (a, b) \cap E = \emptyset\}$, $\mathcal{G}(E)$ be the set of maximal proper (non-singletons) intervals of idempotents, and $E_{is} = E - (\mathcal{I}(E) \cup \mathcal{G}(E))$.

Theorem 2.2.11 ([10, Theorem 3.6]) *Let* $\mathbf{L}$ *be a saturated BL-chain. Then*

1. *For each* $[a, b] \in \mathcal{I}(E)$*, the BL-algebra* $[a, b]_{\mathbf{L}}$ *is either an MV-algebra or a product algebra.*

2. *If* $x, y \in L$ *are such that there is no interval* $I \in \mathcal{I}(E)$ *with* $x, y \in I$*, then* $x * y = \min\{x, y\}$*. In particular, for each* $[a, b] \in \mathcal{G}(E)$*,* $[a, b]_{\mathbf{L}}$ *is a Gödel algebra.*

3. *Let* $(I, \leq)$ *be the totally ordered set defined by*

$$I = \{a \in L \mid a \in E_{is} \text{ or } \exists b \in L \text{ such that } [a, b] \in \mathcal{I}(E) \cup \mathcal{G}(E)\},$$

and $\leq$ is induced by the order of $\mathbf{L}$. For each $a \in I$, let $\mathbf{L}_a$ be either $[a,a]_{\mathbf{L}}$ if $a \in E_{is}$ or $[a,b]_{\mathbf{L}}$ if the corresponding interval $[a,b] \in \mathcal{I}(E) \cup \mathcal{G}(E)$. Then

$$\mathbf{L} = \bigoplus_{a \in I} \mathbf{L}_a \,.$$

2.2.5 Completeness theorems

In [17, Theorem 2.3.22] Hájek proved that any schematic extension $\mathcal{C}$ of BL is complete w.r.t. the class of all $\mathcal{C}$-algebras. Further, thanks to Theorem 2.2.2 he showed that BL is complete w.r.t. the class of all totally ordered $\mathcal{C}$-algebras ($\mathcal{C}$-chains).

Theorem 2.2.12 (Completeness) *Let $\mathcal{C}$ be a schematic extension of BL and φ be a formula. Then the following are equivalent:*

1. $\mathcal{C} \vdash \varphi$.
2. *φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-algebra $\mathbf{L}$.*
3. *φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-chain $\mathbf{L}$.*

In particular, the completeness theorem can be applied to Gödel logic G, Łukasiewicz logic Ł, and the product logic Π.

Let T be a theory over $\mathcal{C}$. An $\mathbf{L}$-*model* e of T is an evaluation e such that $e(\psi) = \mathbf{1}$ for all $\psi \in T$. The following theorem comes from [17, Theorem 2.4.3].

Theorem 2.2.13 (Strong completeness) *Let T be a theory over $\mathcal{C}$ and φ be a formula. Then the following are equivalent:*

1. $T \vdash \varphi$.
2. *$e(\varphi) = \mathbf{1}$ for each $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T.*
3. *$e(\varphi) = \mathbf{1}$ for each $\mathcal{C}$-chain $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T.*

It was mentioned at the beginning of Section 2.2 that BL should be a logic of continuous t-norms, i.e., the intended algebras of truth values are standard BL-chains $[0,1]_*$. It was recently proved (see [10, Theorem 5.2]) that BL is really a logic of the standard BL-chains.

Theorem 2.2.14 (Standard completeness) *Let φ be a formula. Then BL $\vdash \varphi$ iff φ is a* 1*-tautology in all standard BL-chains.*

The analogous results hold also for Gödel logic G, Łukasiewicz logic Ł, and the product logic Π. Let $[0,1]_G$, $[0,1]_L$, and $[0,1]_\Pi$ denote the standard BL-chains where the monoid operation is Gödel, Łukasiewicz, and the product t-norm respectively.

Theorem 2.2.15 *Let $\mathcal{L}$ be either* G, Ł, *or* Π, *and φ be a formula. Then $\mathcal{L} \vdash \varphi$ iff φ is a* 1*-tautology in* $[0,1]_{\mathcal{L}}$.

The fact that the latter theorem is valid for Gödel logic was proved by Dummett in [9], for Łukasiewicz logic it was proved by Chang in [6], and finally for the product logic it was proved by Hájek in [17].

2.3 Monoidal t-norm based logic

In the case of BL the motivation algebra of truth values was a standard BL-chain $[0,1]_* = ([0,1], *, \rightarrow, \leq, 0, 1)$ where $*$ was a continuous t-norm and $\rightarrow$ was the corresponding residuum. However, for the existence of the residuum it is sufficient for a t-norm to be only left-continuous[2]. Thus Esteva and Godo came with an idea to axiomatize a logic of a left-continuous t-norms. In their paper [11] they introduced so-called monoidal t-norm based logic (MTL for short) and proved that MTL is complete w.r.t. the class of all MTL-chains (bounded integral residuated chains). The fact that MTL is really a logic of left-continuous t-norms and their residua was proved later by Jenei and Montagna in [28].

2.3.1 Syntax and semantics of MTL

Let $*$ be a left-continuous t-norm. Similarly as in the case of BL, the algebra $[0,1]_* = ([0,1], *, \rightarrow, \leq, 0, 1)$ forms again a bounded integral residuated chain. However, $[0,1]_*$ need not be divisible. Consequently, the meet $a \wedge b$ cannot be in general expressed as $a * (a \rightarrow b)$ and we have only one inequality $a * (a \rightarrow b) \leq a \wedge b$. Thus the language of MTL contains one more connective than the language of BL. More precisely, it consists of a countable set of propositional variables, a conjunction &, an implication $\Rightarrow$, the truth constant $\overline{0}$, and the minimum conjunction $\wedge$. Derived connectives are defined as follows:

[2] A t-norm $*$ is said to be *left-continuous* if whenever $\langle x_n \rangle_{n \in \mathbb{N}}$ and $\langle y_n \rangle_{n \in \mathbb{N}}$ are increasing sequences of reals in $[0,1]$ such that $\sup\{x_n \mid n \in \mathbb{N}\} = x$ and $\sup\{y_n \mid n \in \mathbb{N}\} = y$, then $\sup\{x_n * y_n \mid n \in \mathbb{N}\} = x * y$.

$$
\begin{array}{rll}
\varphi \vee \psi & \text{is} & ((\varphi \Rightarrow \psi) \Rightarrow \psi) \wedge ((\psi \Rightarrow \varphi) \Rightarrow \varphi)\,, \\
\neg\varphi & \text{is} & \varphi \Rightarrow \bar{0}\,, \\
\varphi \equiv \psi & \text{is} & (\varphi \Rightarrow \psi)\&(\psi \Rightarrow \varphi)\,, \\
\bar{1} & \text{is} & \neg\bar{0}\,.
\end{array}
$$

In [11], the authors introduced a Hilbert style calculus for MTL with an axiomatization similar to BL. They introduced new axioms for the minimum conjunction $\wedge$ (i.e., the axioms (B4) and (B5)) and changed the axiom (A4) to a weaker form (B6). The following are the axioms of MTL:

$$
\begin{array}{ll}
\text{(B1)} & (\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))\,, \\
\text{(B2)} & \varphi\&\psi \Rightarrow \varphi\,, \\
\text{(B3)} & \varphi\&\psi \Rightarrow \psi\&\varphi\,, \\
\text{(B4)} & (\varphi \wedge \psi) \Rightarrow \varphi\,, \\
\text{(B5)} & (\varphi \wedge \psi) \Rightarrow (\psi \wedge \varphi)\,, \\
\text{(B6)} & (\varphi\&(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \wedge \psi)\,, \\
\text{(B7a)} & (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\varphi\&\psi \Rightarrow \chi)\,, \\
\text{(B7b)} & (\varphi\&\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))\,, \\
\text{(B8)} & ((\varphi \Rightarrow \psi) \Rightarrow \chi) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \chi) \Rightarrow \chi)\,, \\
\text{(B9)} & \bar{0} \Rightarrow \varphi\,.
\end{array}
$$

The deduction rule of MTL is modus ponens.

Algebras of truth values for MTL are so-called MTL-algebras and they form a subvariety of integral residuated lattices.

Definition 2.3.1 An MTL-*algebra* is a structure $(L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ where the following conditions are satisfied:

1. $(L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is an integral residuated lattice,

2. $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice,

3. $(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}$ for all $x, y \in L$.

A totally ordered MTL-algebra is called an MTL-*chain*. An MTL-chain whose underlying set is the real interval $[0, 1]$ is referred to as a *standard* MTL-chain.

Let $\mathbf{L}$ be an MTL-algebra. The notions of $\mathbf{L}$-evaluation and $\mathbf{L}$-tautology are defined in the analogous way as it was done in the section on BL. Then it can be easily checked that MTL is sound with respect to $\mathbf{L}$-tautologies, i.e., if φ is a provable formula in MTL then φ is an $\mathbf{L}$-tautology in each MTL-algebra $\mathbf{L}$.

Similarly as in the case of BL, thanks to the prelinearity it was shown (see [11]) that the variety of MTL-algebras is generated by MTL-chains.

Theorem 2.3.2 (Subdirect Representation) *Each MTL-algebra is isomorphic to a subdirect product of MTL-chains.*

2.3.2 Structure of left-continuous t-norms

Although the structure of continuous t-norms is completely characterized, there are not many results on the structure of left-continuous t-norms. So far mainly construction methods were published (see e.g. [29, 31]). We present here the most related results to this text.

Definition 2.3.3 Let $(S, \leq)$ be a complete ordered set. Let D be a dense subset of S^2, and let $f(x, y)$ be a non-decreasing function from D^2 into S. Then function $\hat{f}$ from S^2 into S defined for every $x, y \in S$ by $\hat{f}(x, y) = \sup\{f(d, e) \mid d, e \in D,\ d \leq x,\ e \leq y\}$ is said to be the *completion* of f.

A left-continuous t-norm $*$ is said to be *cancellative* if for any $x, y, z \in [0, 1]$, $z \neq 0$, $x * z = y * z$ implies $x = y$. In [30] the authors showed that each left-continuous t-norm can be obtained as the completion of a continuous function on a dense subset of $[0, 1]^2$. Moreover, they proved that every cancellative left-continuous t-norm is the completion of a continuous t-norm on $\mathbb{Q} \cap [0, 1]$. Further, they proved that the set of discontinuity points of a left-continuous t-norm is a first-category set and its measure is zero.

Construction method 2.3.4 Let $\mathbf{M} = (M, *, \leq, \mathbf{1})$ be a countable, commutative, totally ordered, integral monoid. The following method, how to construct a left-continuous t-norm from $\mathbf{M}$, was presented by Jenei and Montagna in [29]. Without any loss of generality we can suppose that $\mathbf{M}$ is bounded. If not then we can extend M by a new bottom element $\mathbf{0}$ and define $\mathbf{0} * x = \mathbf{0}$ for all $x \in M$. Let us define a new universe as follows:

$$M' = \{\langle a, q\rangle \mid a \in M,\ q \in \mathbb{Q} \cap (0, 1]\} \cup \{\langle \mathbf{0}, 1\rangle\},$$

the order $\leq'$ is lexicographic and the monoid operation is defined as follows:

$$\langle a, q\rangle *' \langle b, r\rangle = \begin{cases} \langle a * b, 1\rangle, & \text{if } a * b < \min\{a, b\}, \\ \min\{\langle a, q\rangle, \langle b, r\rangle\}, & \text{otherwise.} \end{cases}$$

Then $\mathbf{M}' = (M', *', \leq', \langle \mathbf{1}, 1\rangle)$ is a bounded, countable, commutative, totally ordered, integral monoid. The universe M' is dense, i.e., for any $x, y \in M'$ such that $x < y$ there is $z \in M'$ such that $x < z < y$. The monoid $\mathbf{M}$ can be embedded into $\mathbf{M}'$ by an embedding $a \mapsto \langle a, 1\rangle$. As $\mathbf{M}'$ is countable and dense, there exists an order-isomorphism ϕ from M' to $\mathbb{Q} \cap [0,1]$. Thus $\mathbf{M}'$ is isomorphic to $(\mathbb{Q} \cap [0,1], \circ, \leq, 1)$, where $x \circ y = \phi(\phi^{-1}(x) *' \phi^{-1}(y))$. Then the completion of $\circ$ is a left-continuous t-norm.

Finally, we present here two construction methods of left-continuous t-norms again from [29] which will be useful for us later.

Construction method 2.3.5 Let $\mathbf{Z}_i = (\mathbb{Z}^-, \oplus_i, \leq, 0)$, $i = 1, \ldots, n$ be a finite family of integral o-monoids with the usual order of $\mathbb{Z}^-$, $\mathbf{R} = (\mathbb{R}^-, \oplus_r, \leq, 0)$ be a commutative ℓ-monoid with the usual order of $\mathbb{R}^-$, without zero divisors (i.e., for all $x, y < 0$, $x \oplus_r y < 0$), and $\oplus_r$ be a left-continuous operation[3]. Then the lexicographic product $\mathbf{M} = (\prod_{i=1}^{n} \mathbf{Z}_i) \times \mathbf{R}$ is an integral o-monoid $(M, \oplus, \leq, \overline{0})$, where $\overline{0} = \langle 0, \ldots, 0 \rangle$. Let $\bar{a} = \langle a_1, \ldots, a_{n+1} \rangle \in M$ and let us define $\bar{b} = \langle b_1, \ldots, b_{n+1} \rangle$, where $b_k = -1 + \sum_{i=1}^{k} a_i$. Then M can be mapped onto $(0,1]$ by a mapping ϕ as follows:

$$\phi(\bar{a}) = \sum_{i=1}^{n+1} 2^{b_i} \,.$$

Clearly, ϕ is a order-isomorphism between M and $(0,1]$. Hence $\mathbf{M}$ is isomorphic to an integral commutative ℓ-monoid $((0,1], *, \leq, 1)$, where $a*b = \phi(\phi^{-1}(a) \oplus \phi^{-1}(b))$. If we extend $*$ to $[0,1]$ by setting $0 * x = 0$, we obtain a left-continuous t-norm. We will use a shorter notation $\langle \bar{a}, r \rangle$ for an element of M where $\bar{a} \in \prod_{i=1}^{n} \mathbf{Z}_i$ and $r \in \mathbb{R}^-$. The left-continuity of $*$ follows from the fact that if $\lim_{k\to\infty} \langle \bar{a}_k, r_k \rangle = \langle \bar{a}, r \rangle$ then $\lim_{k\to\infty} r_k = r$ and $\bar{a}_k = \bar{a}$ for almost all $k \in \mathbb{N}$ since $\prod_{i=1}^{n} \mathbf{Z}_i$ is i.w.o.

The second method is similar to the previous one and can be obtained by a limit procedure.

Construction method 2.3.6 Let $\mathbf{Z}_i = (\mathbb{Z}^-, \oplus_i, \leq, 0)$, $i \in \mathbb{N} - \{0\}$ be an indexed family of integral o-monoids with the usual order of $\mathbb{Z}^-$. Then again the lexicographic product $\mathbf{M} = \prod_{i \in \mathbb{N}} \mathbf{Z}_i$ forms an integral o-monoid $(M, \oplus, \leq, \overline{0})$. To each sequence $\langle a_i \rangle_{i=1}^{\infty} \in M$ let us again assign a cumulative sum $\langle b_i \rangle_{i=1}^{\infty}$ such that $b_k = -1 + \sum_{i=1}^{k} a_i$. Obviously, $\langle b_i \rangle_{i=1}^{\infty}$ is a non-increasing sequence of integers. Then we can map M onto $(0,1]$ by a mapping ϕ as follows:

$$\phi(\langle a_i \rangle_{i=1}^{\infty}) = \sum_{i=1}^{\infty} 2^{b_i} \,.$$

The mapping ϕ is an order-isomorphism between M and $(0,1]$. Thus we can get a t-norm $*$ from $\oplus$ in the same way as before. Moreover, it can be shown that $*$ is left-continuous.

[3] In the same sense as the left-continuity of a t-norm.

2.3.3 Some schematic extensions of MTL

In Section 2.2.3 we presented three schematic extensions of BL; Gödel, Łukasiewicz, and the product logic. It is a natural question what happens if we extend MTL by the same axiom schemata.

Definition 2.3.7 *Involutive monoidal t-norm based logic* IMTL is a schematic extension of MTL by the axiom schema $\neg\neg\varphi \Rightarrow \varphi$.

This logic was already investigated in the original paper on MTL [11], where the authors proved that IMTL is a strictly weaker logic than Łukasiewicz logic.

The schematic extension of MTL by $\varphi \Rightarrow (\varphi \& \varphi)$ was studied by Hájek in [19], where it was shown that such extension already collapses into Gödel logic. In the same paper Hájek also introduced the third possible schematic extension of MTL.

Definition 2.3.8 *Product monoidal t-norm based logic* ΠMTL is a schematic extension of MTL by the following axiom schemata:

(Π1) $\neg\neg\chi \Rightarrow ((\varphi \& \chi \Rightarrow \psi \& \chi) \Rightarrow (\varphi \Rightarrow \psi))$,
(Π2) $\varphi \wedge \neg\varphi \Rightarrow \overline{0}$.

It was shown (see [19, Theorem 2]) that ΠMTL is a strictly weaker logic than the product logic. The algebras of truth values for ΠMTL are so-called ΠMTL-algebras. Since this thesis is focused on ΠMTL, we will investigate ΠMTL-algebras in more detail at the beginning of Chapter 3 and in Chapter 4.

2.3.4 Completeness theorems

In [11, Theorem 2] Esteva and Godo proved that any schematic extension $\mathcal{C}$ of MTL is complete w.r.t. the class of all $\mathcal{C}$-algebras and thanks to Theorem 2.3.2 to the class of all $\mathcal{C}$-chains as well.

Theorem 2.3.9 (Completeness) *Let $\mathcal{C}$ be a schematic extension of* MTL *and φ be a formula. Then the following are equivalent:*

1. $\mathcal{C} \vdash \varphi$.
2. *φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-algebra $\mathbf{L}$.*
3. *φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-chain $\mathbf{L}$.*

In particular, the completeness theorem can be applied to IMTL and ΠMTL. Further, we have also completeness for theories.

Theorem 2.3.10 (Strong completeness) *Let T be a theory over $\mathcal{C}$ and φ be a formula. Then the following are equivalent:*

1. $T \vdash \varphi$.
2. $e(\varphi) = \mathbf{1}$ *for each $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T.*
3. $e(\varphi) = \mathbf{1}$ *for each $\mathcal{C}$-chain $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T.*

Recently, Jenei and Montagna proved that MTL is really logic of a left-continuous t-norms and their residua (for details, see [28]). The analogous result also holds for IMTL as it was shown by Esteva, Gispert, Godo, and Montagna in [13].

Theorem 2.3.11 (Standard completeness) *Let $\mathcal{L}$ be either* MTL *or* IMTL, *and φ be a formula. Then $\mathcal{L} \vdash \varphi$ iff φ is a* 1*-tautology in all standard $\mathcal{L}$-chains.*

The author of [13] also tried to prove that ΠMTL is complete w.r.t. standard ΠMTL-chains. They showed that ΠMTL is complete w.r.t. ΠMTL-chains whose underlying set is $\mathbb{Q} \cap [0, 1]$. However, they did not succeeded in extending of this result to the whole real unit interval and left this question as an open problem (see [13, Page 12]). A positive answer was given by us in [22] and a detailed proof of the following theorem is presented in Chapter 3.

Theorem 2.3.12 *Let φ be a formula. Then* $\Pi\mathrm{MTL} \vdash \varphi$ *iff φ is a* 1*-tautology in all standard* ΠMTL*-chains.*

Chapter 3

Standard Completeness Theorem for ΠMTL

The main aim of this chapter is to present a solution to the open problem mentioned at the end of the previous chapter and prove Theorem 2.3.12. Thus we will show here that ΠMTL is complete with respect to the class of standard ΠMTL-chains. Moreover, we will show that it is sufficient to consider only ΠMTL-chains with finitely many Archimedean classes.

Note that one direction of this statement (if ΠMTL $\vdash \varphi$ then φ is a 1-tautology in all standard ΠMTL-chains) already follows from Theorem 2.3.9. The second is difficult and we will prove it in Section 3.3. However, we have to firstly collect several results on ΠMTL-algebras which we will need in the sequel. The results in this chapter come from our paper [22].

3.1 ΠMTL-algebras and filters

Definition 3.1.1 A ΠMTL-algebra $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is an MTL-algebra satisfying the following identities:

1. $\neg\neg z \rightarrow [(x * z \rightarrow y * z) \rightarrow (x \rightarrow y)] = \mathbf{1}$,
2. $x \wedge \neg x = \mathbf{0}$,

where $\neg x = x \rightarrow \mathbf{0}$. A totally ordered ΠMTL-algebra is called a ΠMTL-*chain*.

In order to study the structure of ΠMTL-algebras, we have to work with congruences. By Theorem 2.1.3 the congruence lattice of a residuated lattice is isomorphic to the collection of all convex subalgebras. In our case we follow the terminology of [11] and use the notion of a filter instead of the convex subalgebra since the bottom element $\mathbf{0}$ is in the signature of a ΠMTL-algebra and a filter need not be a subalgebra.

Definition 3.1.2 Let $\mathbf{L} = (L, *, \rightarrow, \leq, \mathbf{0}, \mathbf{1})$ be a ΠMTL-algebra. A *filter* F in $\mathbf{L}$ is a subset of L satisfying:

1. if $x, y \in F$, then $x * y \in F$,
2. if $x \in F$, $x \leq y$, then $y \in F$.

Throughout the paper the collection of the filters of a ΠMTL-algebra $\mathbf{L}$ will be denoted by $\mathcal{F}$. By applying Theorem 2.1.3 we get that Con $\mathbf{L}$ is isomorphic to $\mathcal{F}$. Moreover, it gives us the following lemma.

Lemma 3.1.3 *For any filter F in a* ΠMTL-*algebra* $\mathbf{L}$*, let us define the following equivalence relation in L:*

$$x \sim_F y \text{ iff } x \rightarrow y \in F \text{ and } y \rightarrow x \in F.$$

Then $\sim_F$ is a congruence and the quotient $\mathbf{L}/F$ is a ΠMTL-*algebra.*

We will denote the equivalence class containing an element $x \in L$ with respect to a filter F by $[x]_F = \{a \in L \mid a \sim_F x\}$. Clearly, $[\mathbf{1}]_F = F$. Observe also that if $\mathbf{L}$ is a ΠMTL-chain then only one of the implications in the definition of $\sim_F$ is important because for all x, y either $x \leq y$ or $y \leq x$, thus either $x \rightarrow y = \mathbf{1}$ or $y \rightarrow x = \mathbf{1}$.

As it will be seen later on, the filters in ΠMTL-chains are also related to so-called Archimedean classes (see [14]).

Definition 3.1.4 Let $\mathbf{L}$ be a ΠMTL-chain, a, b be elements of L, and $\sim$ be an equivalence on L defined as follows:

$$a \sim b \text{ iff there exists an } n \in \mathbb{N} \text{ such that } a^n \leq b \leq a \text{ or } b^n \leq a \leq b.$$

Then for any $a \in L$ the equivalence class $[a]_\sim$ is called an *Archimedean class*.

Archimedean classes correspond to the subsets of L where the elements behave like in an Archimedean ℓ-monoid, i.e., for any pair of elements $x, y \in [a]_\sim$, such that $x \leq y$, there is an $n \in \mathbb{N}$ such that $y^n \leq x$.

3.2 ΠMTL-chains

In this section we list several basic statements about general ΠMTL-chains which will be useful in the sequel. In [19, Lemma 4] Hájek proved the following result.

Lemma 3.2.1 *An* MTL*-chain* $\mathbf{L}$ *is a* ΠMTL*-chain if and only if for any* $x, y, z \in L$*,* $z \neq \mathbf{0}$*, we have* $x * z = y * z$ *implies* $x = y$*.*

Observe that by Lemma 3.2.1 we obtain for $a, b, c \in L$, $c \neq \mathbf{0}$, that $a < b$ implies $a * c < b * c$, in particular $a^2 < a$ and $a * b < a$ for $b < \mathbf{1}$. Moreover, we get $a * c \to b * c = a \to b$ for $c \neq \mathbf{0}$. Indeed, $a \to b \leq a * c \to b * c$ holds in any residuated lattice (see Proposition 2.1.2(2)). Since $a * c * (a * c \to b * c) \leq b * c$, we get $a * (a * c \to b * c) \leq b$ by Lemma 3.2.1. Thus $a * c \to b * c \leq a \to b$.

Moreover, due to Lemma 3.2.1 it can be shown that there is a connection between cancellative residuated chains and ΠMTL-chains.

Lemma 3.2.2 *Let* $\mathbf{L} = (L, *, \to, \leq, \mathbf{0}, \mathbf{1})$ *be a* ΠMTL*-chain and* $L_0 = L - \{\mathbf{0}\}$*. Then the subreduct* $\mathbf{L}_0 = (L_0, *, \to, \leq, \mathbf{1})$ *is an integral cancellative residuated chain.*

PROOF: By Lemma 3.2.1 we get $x * y > \mathbf{0}$ for any $x, y \in L_0$. Since $x \to y \geq \mathbf{1} \to y = y$, we get $x \to y \in L_0$ for any $x, y \in L_0$. Thus L_0 is really subuniverse of the reduct $(L, *, \to, \leq, \mathbf{1})$. Moreover, $x * z = y * z$ implies $x = y$ for any $x, y, z \in L_0$ by Lemma 3.2.1. Since $\mathbf{1}$ is the top element, $\mathbf{L}_0$ is an integral cancellative residuated chain. □

Also the other direction is possible. If we have an integral cancellative residuated chain $\mathbf{L} = (L, *, \to, \leq, \mathbf{1})$, we can extend it to a ΠMTL-chain by adding a bottom element $\mathbf{0}$. Let $L' = L \cup \{\mathbf{0}\}$ be the new universe. The order $\leq'$ is an extension of $\leq$ in such a way that $\mathbf{0} < x$ for all $x \in L$. The operations are defined as follows:

$$x *' y = \begin{cases} x * y & x, y \in L\,, \\ \mathbf{0} & x = \mathbf{0} \text{ or } y = \mathbf{0}\,, \end{cases}$$

$$x \to' y = \begin{cases} x \to y & x, y \in L\,, \\ \mathbf{1} & x = \mathbf{0}\,, \\ \mathbf{0} & y = \mathbf{0} \text{ and } x > \mathbf{0}\,. \end{cases}$$

Lemma 3.2.3 *Let* $\mathbf{L} = (L, *, \to, \leq, \mathbf{1})$ *be an integral cancellative residuated chain. Then the structure* $\mathbf{L}' = (L', *', \to', \leq', \mathbf{0}, \mathbf{1})$ *is a* ΠMTL*-chain.*

PROOF: It can be easily checked that $\mathbf{L}'$ is an integral bounded residuated chain. Since $\mathbf{L}$ is cancellative, we get by Lemma 3.2.1 that $\mathbf{L}'$ is a ΠMTL-chain. □

As it was shown in Lemmata 3.2.2 and 3.2.3, there is a tight connection between integral cancellative residuated chains and ΠMTL-chains. Thus the results presented in the subsequent sections are also applicable to the integral cancellative residuated chains if we omit everywhere the bottom element $\mathbf{0}$.

Lemma 3.2.4 *Let* $\mathbf{L}$ *be a* ΠMTL*-chain. Then any union of filters of* $\mathbf{L}$ *is again a filter.*

PROOF: Let $\{F_i \mid i \in I\}$ be an indexed family of filters of $\mathbf{L}$. We will show that $F = \bigcup_{i \in I} F_i$ is a filter. Suppose that $a, b \in F$. Then a belongs to F_i for some $i \in I$ and $b \in F_j$ for some $j \in I$. Since the set of all filters of $\mathbf{L}$ is linearly ordered, either $F_i \subseteq F_j$ or $F_j \subseteq F_i$. Without any loss of generality suppose that $F_i \subseteq F_j$. As $a * b$ belongs to F_j, $a * b \in F$.

Now, suppose that $a \in F$ and $b \geq a$. Since $a \in F_i$ for some $i \in I$, we get $b \in F_i$. Thus b belongs to F. □

The next trivial result characterizes the *principal* filters, i.e., the filters generated by a single element. A principal filter F generated by b is denoted by F^b. The set of all principal filters of a ΠMTL-chain will be denoted by $\mathcal{P}$.

Lemma 3.2.5 *Let* $\mathbf{L}$ *be a* ΠMTL*-chain and* $b \in L$. *Then the principal filter generated by* b *is of the form:*

$$F^b = \{z \in L \mid (\exists n \in \mathbb{N})(b^n \leq z)\}\,.$$

Observe, that each filter F is a union of principal filters since $F = \bigcup_{b \in F} F^b$. Moreover, if Con $\mathbf{L}$ is finite (i.e., $\mathbf{L}$ has only finite number of filters) then all filters are principal. Indeed, as the collection of all filters forms a chain, we get $F = \bigcup_{b \in F} F^b = F^c$ for some $c \in F$.

Let F^b be a principal filter. Then by Lemma 3.2.4 the union of all filters not containing b is a filter. Clearly, it is the largest filter not containing b. Thus we obtain the following lemma.

Lemma 3.2.6 *Each principal filter generated by* b *has a predecessor.*

Following the notation from *o*-groups we will denote the predecessor of F^b by F_b.

Further, we list several easy results about Archimedean classes.

Lemma 3.2.7 *Let* **L** *is a* ΠMTL*-chain and* $b \in L$*. Then the Archimedean classes of* **L**, $[a]_\sim$, $a \in L$*, have the following properties:*

1. $[a]_\sim$ *is closed under* $*$.
2. $[a]_\sim$ *is a left-open interval for* $a \neq \mathbf{0}, \mathbf{1}$.
3. $[a * b]_\sim = [a \wedge b]_\sim$.

PROOF:

1. Suppose that $x, y \in [a]_\sim$ and $x \leq y$. Then $x^2 \leq x * y \leq x$, thus $x * y$ belongs to $[a]_\sim$.
2. The left-openness follows from the fact that $x \in [a]_\sim$ implies $x^2 \in [a]_\sim$ and $x^2 < x$ from cancellativity. Finally, we have to show that there is no gap in $[a]_\sim$. Suppose that $x, y \in [a]_\sim$, $z \in L$, and $x < z < y$. Then there is an n such that $y^n \leq x < z < y$. Thus $z \in [a]_\sim$.
3. Without any loss of generality suppose that $a \leq b$. Then $a^2 \leq a * b \leq a$. Thus $a * b \in [a]_\sim = [a \wedge b]_\sim$.

□

Note that if **L** is a non-trivial ΠMTL-chain then there are always at least two Archimedean classes, $\{\mathbf{0}\}$ and $\{\mathbf{1}\}$, and $L/\sim$ is totally ordered because of Lemma 3.2.7(2), i.e., $[a]_\sim < [b]_\sim$ iff $a \notin [b]_\sim$ and $a < b$.

As we mentioned in the previous section, the Archimedean classes are related to the filters. This connection is described by the next proposition.

Proposition 3.2.8 *Let* $(\mathcal{C}, \leq)$ *be the chain of all Archimedean classes of a* ΠMTL*-chain* **L**. *Then the dual chain* $\mathcal{C}^\partial$ *of* $\mathcal{C}$ *is isomorphic to the chain of all pricipal filters* $\mathcal{P}$. *Let* $C \in \mathcal{C}$. *The order-isomorphism* $\phi : \mathcal{C} \to \mathcal{P}$ *is defined as follows:*

$$\phi(C) = F^b\,, \text{ for some } b \in C\,.$$

PROOF: Firstly, we have to show that the definition of ϕ is independent of the choice of b. We prove that $F^b = F^c$ for $b, c \in C$. Let $x \in F^b$. By Lemma 3.2.5 we have $n \in \mathbb{N}$ such that $b^n \leq x$. As b, c belong to the same Archimedean class, there exists $m \in \mathbb{N}$ such that $c^m \leq b^n$. Thus $c^m \leq x$ and $x \in F^c$. Consequently, $F^b \subseteq F^c$. The case $F^c \subseteq F^b$ is completely analogous.

Secondly, we prove the ϕ is order-preserving and injective. Let $C_1, C_2 \in \mathcal{C}$ such that $C_1 < C_2$. Further, let $b \in C_1$ and $c \in C_2$. Since $b < c$, we get $F^b \supseteq F^c$.

Moreover, we show that $F^b \neq F^c$. Suppose that $F^b = F^c$. Then $b \in F^c$ and it follows that there is $n \in \mathbb{N}$ such that $c^n \leq b$. Thus b and c must belong to the same Archimedean class contradicting the fact that $C_1 < C_2$.

Finally, we show that ϕ is onto. Let $F^b \in \mathcal{P}$. By Lemma 3.2.6, F^b has a predecessor F_b. Now, we show that $C = F^b - F_b$ is an Archimedean class such that $\phi(C) = F^b$. Clearly, b belongs to C because b generates F^b. Thus it is sufficient to show that C is an Archimedean class. Let $x, y \in C$ such that $x \leq y$. Assume that $x \leq y^n$ for all $n \in \mathbb{N}$. Then F^y does not contain x but $F^y \supsetneq F_b$ contradicting the fact that F_b is a predecessor of F^b. □

From the last paragraph of the proof of Proposition 3.2.8, it follows that the inverse of isomorphism between $\mathcal{C}$ and $\mathcal{P}$ is $\phi^{-1}(F^b) = F^b - F_b$ where F_b is the predecessor of F^b.

As a corollary of the previous proposition, we obtain the following statement.

Corollary 3.2.9 *A* ΠMTL-*chain* **L** *has a finite number of Archimedean classes if and only if* Con **L** *is finite.*

PROOF: Let $\mathcal{F}$ be the collection of all filters in **L**. If **L** has finite number of Archimedean classes then the chain of all principal filters $\mathcal{P}$ is finite by Proposition 3.2.8. Since each filter $F = \bigcup_{b \in F} F^b$, we get $\mathcal{F} = \mathcal{P}$. Thus Con **L** is finite. On the other hand, if Con **L** is finite then all filters are principal and Proposition 3.2.8 finishes the proof. □

We finish this section with several examples of ΠMTL-chains. One of the simplest examples is the standard product algebra $[0,1]_\Pi$. It has three Archimedean classes: the singletons $\{0\}$, $\{1\}$, and the open interval $(0,1)$. Now we present an example of a ΠMTL-chain with four Archimedean classes. Such ΠMTL-chain was firstly considered by Hájek in [19] and it is based on the construction method 2.3.5.

Example 3.2.10 Let $L = \mathbb{Z}^- \times \mathbb{R}^-$. The order on L will be lexicographic, i.e., $\langle k, r \rangle \leq \langle m, s \rangle$ iff $k < m$ or $k = m$ and $r \leq s$. The operations are defined as follows:

$$\begin{aligned} \langle k, r \rangle * \langle m, s \rangle &= \langle k+m, r+s \rangle \,, \\ \langle k, r \rangle \to \langle m, s \rangle &= \begin{cases} \langle 0, 0 \rangle & \text{if } \langle k, r \rangle \leq \langle m, s \rangle \,, \\ \langle m-k, \min\{0, s-r\} \rangle & \text{if } \langle k, r \rangle > \langle m, s \rangle \,. \end{cases} \end{aligned}$$

It can be shown that $\mathbf{L} = (L, *, \to, \leq, \langle 0, 0 \rangle)$ is an integral cancellative residuated chain. By Lemma 3.2.3, **L** can be extended to a ΠMTL-chain by adding a bottom

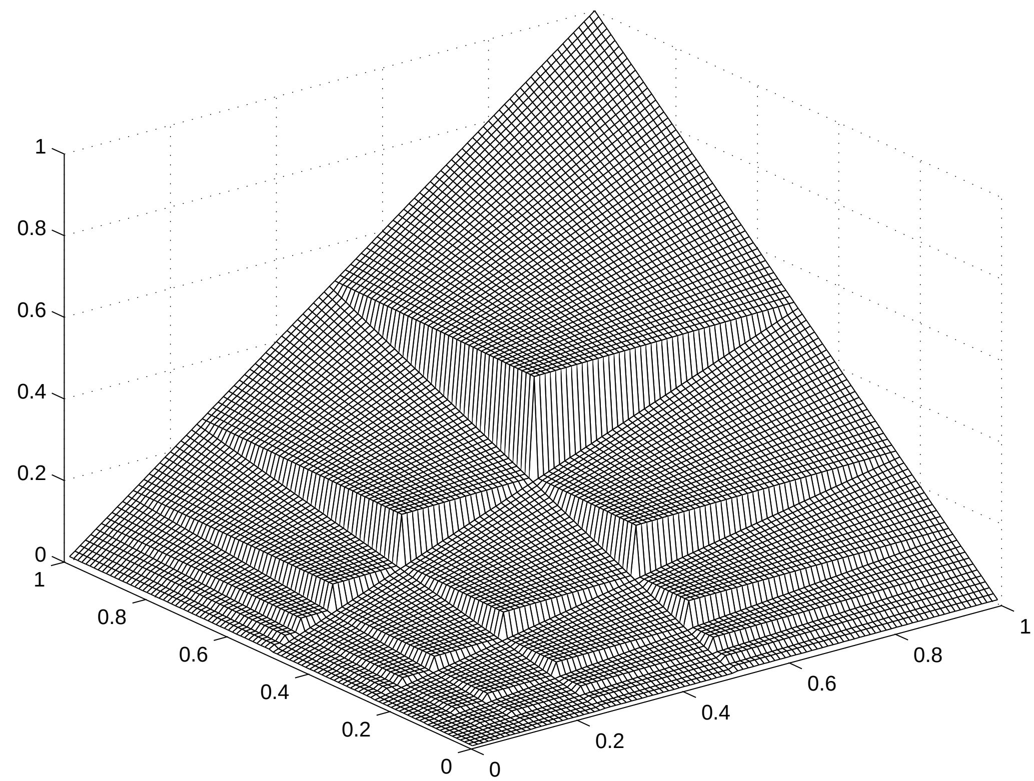

Figure 3.1: A left-continuous cancellative t-norm corresponding to the monoid operation of the ΠMTL-chain from Example 3.2.10.

element. Moreover, the extension of **L** possesses four Archimedean classes: again the singletons with top and bottom element, $\{\langle 0, x\rangle \in L \mid x < 0\}$ since $\langle -1, 0\rangle < \langle 0, x\rangle^n$ for all $n \in \mathbb{N}$, and $\{\langle y, x\rangle \in L \mid y < 0\}$.

If we map the ΠMTL-chain from the previous example onto the real interval $[0, 1]$ as it is described in the construction method 2.3.5, we get a left-continuous t-norm which is depicted in Figure 3.1.

It is clear that if we take a cancellative left-continuous t-norm and the corresponding residuum, we obtain a ΠMTL-chain. Other examples of left-continuous cancellative t-norms are presented in Figure 3.2.

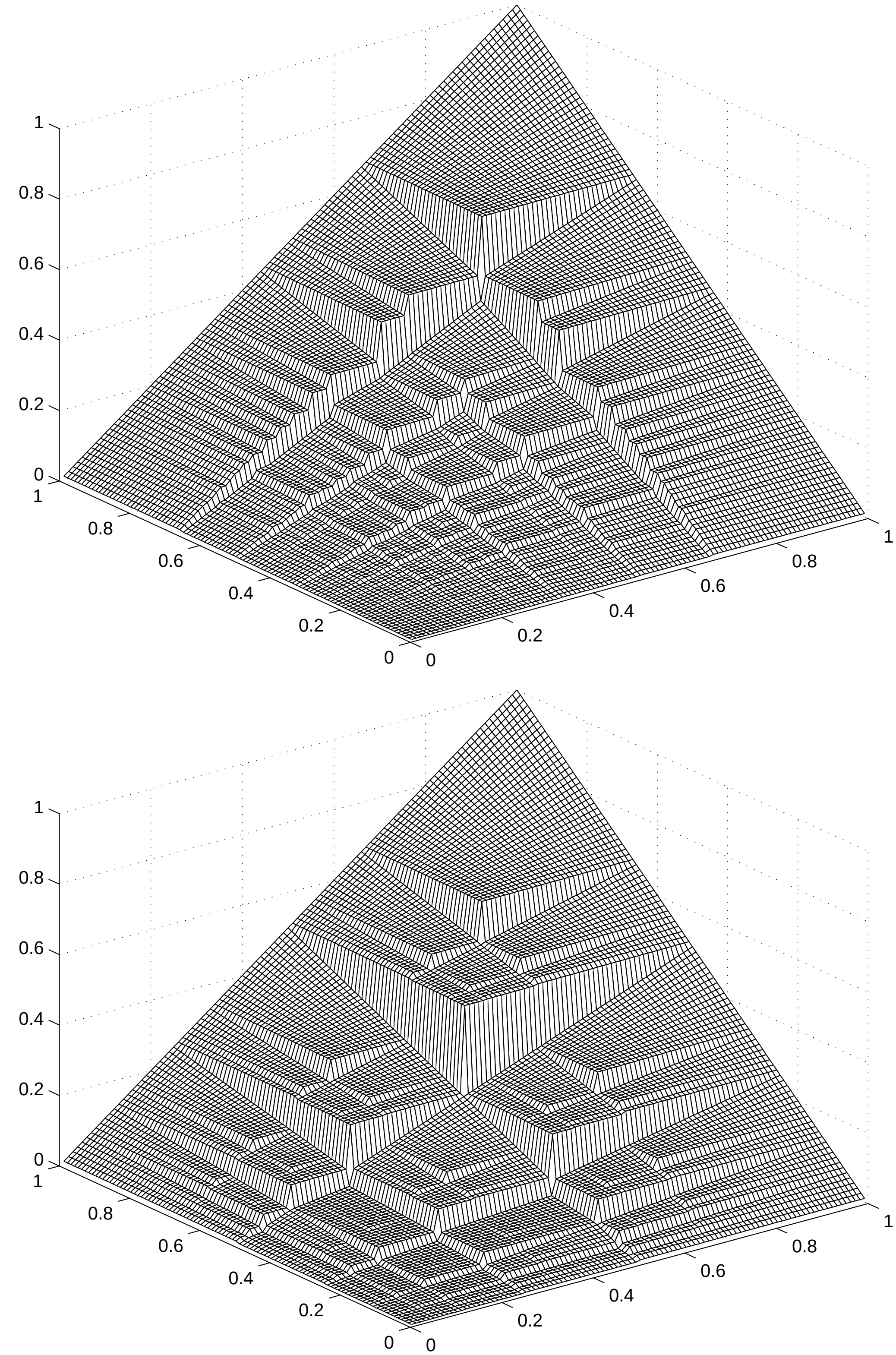

Figure 3.2: Examples of cancellative left-continuous t-norms.

3.3 Proof of Standard Completeness Theorem

Now we are going to prove Standard Completeness Theorem for ΠMTL. The proof has several steps. We will start with a formula φ which is not valid in a ΠMTL-chain **L**. Then we construct a new ΠMTL-chain **S** such that φ is not valid in **S**, too, and **S** has a more transparent structure. The next step is to extend **S** to a continuum. Finally, we will show that this extension of **S** is order-isomorphic to $[0,1]$.

We know from Theorem 2.3.9 that whenever ΠMTL $\not\vdash \varphi$ then there exists a ΠMTL-chain $\mathbf{L} = (L, *_L, \rightarrow_L, \leq, \mathbf{0}, \mathbf{1})$ and an **L**-evaluation $e_{\mathbf{L}}$ such that $e_{\mathbf{L}}(\varphi) < \mathbf{1}$. Let us denote the set of all subformulas of φ by B. Since B is finite, we can assume that $B = \{\psi_1, \ldots, \psi_n\}$. Then let us define the following set:

$$G = \{a_i \in L \mid a_i = e_{\mathbf{L}}(\psi_i),\ \psi_i \in B,\ 1 \leq i \leq n\}\,. \tag{3.1}$$

Let **S** be a submonoid of **L** generated by G, i.e. $\mathbf{S} = (S, *, \leq, \mathbf{0}, \mathbf{1})$, where

$$S = \{a_1^{k_1} *_L \cdots *_L a_n^{k_n} \mid a_i \in G,\ k_i \in \mathbb{N},\ 1 \leq i \leq n\} \cup \{\mathbf{0}, \mathbf{1}\}\,,$$

and $*$ denotes the restriction of $*_L$ to S.

Lemma 3.3.1 *Let $\mathbf{C} = (C, *, \leq, \mathbf{1})$ be an integral o-monoid and **K** be a finitely generated submonoid of **C**. Then K is i.w.o.*

PROOF: The proof of this lemma is based on Dickson's lemma stating that each subset of $(\mathbb{N}, \leq)^n$ has only finitely many minimal elements (the proof of Dickson's lemma in a little bit different form can be found in [8]). Let $g_1, \ldots, g_n$ be the generators of **K** and M be a subset of K. To each element $g_1^{k_1} * \cdots * g_n^{k_n} \in M$ we can assign an n-tuple $(k_1, \ldots, k_n) \in \mathbb{N}^n$. Thus there is a subset $H \subseteq \mathbb{N}^n$ such that $(k_1, \ldots, k_n) \in H$ implies $g_1^{k_1} * \cdots * g_n^{k_n} \in M$. Moreover, if $(k_1, \ldots, k_n) \leq (t_1, \ldots, t_n)$, we obtain $g_1^{k_1} * \cdots * g_n^{k_n} \geq g_1^{t_1} * \cdots * g_n^{t_n}$ since $*$ is order-preserving. Since H has only finitely many minimal elements, one of them must correspond to the maximum of M. □

Due to Lemma 3.3.1, **S** is i.w.o. and we can introduce a residuum on **S** as follows:

$$a \rightarrow b = \max\{z \in S \mid a * z \leq b\}\,.$$

Theorem 3.3.2 *The enriched submonoid* $\mathbf{S} = (S, *, \rightarrow, \leq, \mathbf{0}, \mathbf{1})$ *is a* ΠMTL*-chain and there exists an* **S***-evaluation* $e_{\mathbf{S}}$ *such that* $e_{\mathbf{S}}(\varphi) = e_{\mathbf{L}}(\varphi)$.

PROOF: Firstly, we know that $\mathbf{S}$ is an integral o-monoid and $S - \{\mathbf{0}\}$ forms a cancellative submonoid. Since $\mathbf{S}$ is a chain, the prelinearity axiom is obviously satisfied. Thus the only thing which we have to check is whether $(*, \rightarrow)$ form a residuated pair, i.e. $a * b \leq c$ iff $a \leq b \rightarrow c$. But this easily follows from the definition of $\rightarrow$. Hence $\mathbf{S}$ is a ΠMTL-chain.

Secondly, let us define an evaluation $e_{\mathbf{S}}(v) = e_{\mathbf{L}}(v)$ for each propositional variable v appearing in φ and $e_{\mathbf{S}}(v)$ arbitrary otherwise. Then we show by induction on the complexity of φ that $e_{\mathbf{S}}(\psi_i) = e_{\mathbf{L}}(\psi_i)$ for all subformulas ψ_i of φ, in particular $e_{\mathbf{S}}(\varphi) = e_{\mathbf{L}}(\varphi)$. The first step is obvious by the definition of $e_{\mathbf{S}}$. Now suppose that $\psi_k = \psi_i \& \psi_j$. Then $e_{\mathbf{S}}(\psi_k) = e_{\mathbf{S}}(\psi_i) * e_{\mathbf{S}}(\psi_j) = e_{\mathbf{L}}(\psi_i) * e_{\mathbf{L}}(\psi_j) = a_i *_L a_j = a_k = e_{\mathbf{L}}(\psi_k)$ (similarly for $\psi_k = \psi_i \wedge \psi_j$). Finally, suppose that $\psi_k = (\psi_i \Rightarrow \psi_j)$. Then $e_{\mathbf{S}}(\psi_k) = a_i \rightarrow a_j = \max\{z \in S \mid a_i * z \leq a_j\}$. Let $a_k = a_i \rightarrow_L a_j$ ($a_k \in S$ because $\psi_i \Rightarrow \psi_j$ is a subformula of φ). Then $a_i * a_k \leq a_j$. Thus $a_k \leq a_i \rightarrow a_j$. Now suppose that there is an element $z' \in S$ such that $z' > a_k$ and $a_i * z' \leq a_j$. Since $z' \in L$, we get $z' \leq a_i \rightarrow_L a_j = a_k$, a contradiction. Hence $a_i \rightarrow a_j = a_k = e_{\mathbf{L}}(\psi_k)$. □

Note that $\mathbf{S}$ need not be a subalgebra of $\mathbf{L}$ since $\mathbf{S}$ arises only from a submonoid of $\mathbf{L}$. However, the existence of the evaluation $e_{\mathbf{S}}$ such that $e_{\mathbf{S}}(\varphi) < \mathbf{1}$ is sufficient for us.

Since $\mathbf{S}$ is finitely generated using only $*$, there must be only finitely many Archimedean classes in $\mathbf{S}$ by Lemma 3.2.7(3).

Lemma 3.3.3 *There are only finitely many Archimedean classes in* $\mathbf{S}$*.*

Now we have the ΠMTL-chain $\mathbf{S}$ which countable and the evaluation $e_{\mathbf{S}}$ such that $e_{\mathbf{S}}(\varphi) < \mathbf{1}$. The next step is to build a new ΠMTL-chain $\mathbf{S}'$ order-isomorphic to $[0, 1]$ in which $\mathbf{S}$ can be embedded. The new universe is defined as follows:

$$S' = \{\langle s, r\rangle \mid s \in S - \{\mathbf{0}\},\ r \in (0, 1]\} \cup \{\langle \mathbf{0}, 1\rangle\}\,.$$

This construction is the same as in [28], except for the fact that we use reals as second components in the definition of S' instead of rationals.

The order $\leq'$ on S' is lexicographic, i.e., $\langle s_1, r_1\rangle \leq' \langle s_2, r_2\rangle$ iff $s_1 \leq s_2$ or $s_1 = s_2$ and $r_1 \leq r_2$. The operations are defined by the following formulas:

$$\begin{aligned} \langle a, x\rangle *' \langle b, y\rangle &= \langle a * b, xy\rangle\,, \quad (xy \text{ is the usual product of reals}) \\ \langle a, x\rangle \rightarrow' \langle b, y\rangle &= \begin{cases} \langle a \rightarrow b, 1\rangle & \text{if } a * (a \rightarrow b) < b\,, \\ \langle a \rightarrow b, \min\{1, y/x\}\rangle & \text{otherwise}\,. \end{cases} \end{aligned}$$

It is easy to check that $\mathbf{S'} = (S', *', \rightarrow', \leq', \langle \mathbf{0}, 1\rangle, \langle \mathbf{1}, 1\rangle)$ is a ΠMTL-chain.

Finally, the mapping $\Psi : S \to S'$ defined by $\Psi(x) = \langle x, 1\rangle$ is a ΠMTL-homomorphism since it satisfies the following equalities:

$$\Psi(x * y) = \langle x * y, 1\rangle = \langle x, 1\rangle *' \langle y, 1\rangle = \Psi(x) *' \Psi(y)\,,$$

and

$$\Psi(x \to y) = \langle x \to y, 1\rangle = \langle x, 1\rangle \to' \langle y, 1\rangle = \Psi(x) \to' \Psi(y)\,.$$

Moreover, Ψ obviously preserves the order, i.e., $x \leq y$ implies $\Psi(x) \leq' \Psi(y)$.

The remaining step is to find an order-isomorphism $\Phi : S' \to [0, 1]$. Since S is countable and has a minimum and a maximum, there exists an order-preserving mapping $\nu : S \to \mathbb{Q} \cap [0, 1]$ such that $\nu(\mathbf{0}) = 0$ and $\nu(\mathbf{1}) = 1$. Moreover, as S is i.w.o., for each $a \in S - \{\mathbf{0}\}$ there is a predecessor a^- of a. Now let us define a mapping Φ as follows:

$$\begin{aligned} \Phi(\mathbf{0}, 1) &= 0\,, \\ \Phi(a, x) &= \nu(a^-) + (\nu(a) - \nu(a^-))x\,. \end{aligned}$$

Since ν is order-preserving and the elements of S' are lexicographically ordered, the following result can be easily seen.

Theorem 3.3.4 *The mapping Φ is an order-isomorphism between S' and the real unit interval* $[0, 1]$.

Finally, we define the operations in $[0, 1]$ as usual:

$$a \odot b = \Phi(\Phi^{-1}(a) *' \Phi^{-1}(b))\,, \quad a \to_{\odot} b = \Phi(\Phi^{-1}(a) \to' \Phi^{-1}(b))\,.$$

Then $[\mathbf{0}, \mathbf{1}] = ([0, 1], \odot, \to_{\odot}, \leq, 0, 1)$ is a ΠMTL-chain and $[\mathbf{0}, \mathbf{1}] \not\models \varphi$, i.e.,

$$\Phi(\Psi(e_{\mathbf{S}}(\varphi))) < 1\,.$$

Thus the proof of Standard Completeness Theorem (Theorem 2.3.12) is done.

Theorem 3.3.5 (Standard Completeness Theorem) *A formula φ is provable in* ΠMTL *if and only if φ is a* 1*-tautology in all standard* ΠMTL*-chains.*

Since $\mathbf{S}$ has finitely many Archimedean classes by Lemma 3.3.3, we can a little bit strengthen the latter theorem. By the construction of $\mathbf{S}'$ the number of Archimedean classes increases by 1.

Lemma 3.3.6 *Let k be the number of Archimedean classes of* $\mathbf{S}$*. Then the number of Archimedean classes in* $\mathbf{S}'$ *is* $k + 1$*.*

PROOF: Let C be an non-trivial (i.e., $C \neq \{\mathbf{0}\}, \{\mathbf{1}\}$) Archimedean class in $\mathbf{S}$. Then there is a corresponding Archimedean class in $\mathbf{S}'$ described as follows:

$$C' = \{\langle a, x\rangle \in S' \mid a \in C\}\,.$$

Thus $\mathbf{S}'$ has at least k Archimedean classes. Finally, $C'_{k+1} = \{\langle \mathbf{1}, r\rangle \in S' \mid r < 1\}$ is clearly an Archimedean class not corresponding to any Archimedean class in $\mathbf{S}$. Hence the proof is done. □

Lemma 3.3.6 together with Corollary 3.2.9 gives us the following version of Standard Completeness Theorem.

Theorem 3.3.7 *Let φ be a* ΠMTL *formula. Then the following are equivalent:*

1. ΠMTL $\vdash \varphi$.
2. *φ is a* 1*-tautology in all standard* ΠMTL*-chains with finitely many Archimedean classes.*
3. *φ is a* 1*-tautology in all standard* ΠMTL*-chains with finite congruence lattice.*

3.4 Finite Strong Standard Completeness

In the previous section we proved that provable formulas are valid in $[0, 1]$ and vice versa. Nevertheless from the logical point of view, it is desirable to extend Theorem 3.3.5 also to theories.

Theorem 3.4.1 (Finite Strong Standard Completeness) *Let T be a finite theory over* ΠMTL *and φ be a formula. Then the following are equivalent:*

1. *$T \vdash \varphi$.*
2. *$e(\varphi) = \mathbf{1}$ for each standard* ΠMTL*-chain* $\mathbf{L}$ *with finitely many Archimedean classes and each* $\mathbf{L}$*-model e of T.*
3. *$e(\varphi) = \mathbf{1}$ for each standard* ΠMTL*-chain* $\mathbf{L}$ *with finite* Con $\mathbf{L}$ *and each* $\mathbf{L}$*-model e of T.*

PROOF: We will prove only the non-trivial direction. Suppose that $T \nvdash \varphi$. Then Theorem 2.3.10 gives us a ΠMTL-chain $\mathbf{L}$ and an $\mathbf{L}$-model e of T such that $e(\varphi) < \mathbf{1}$. We will proceed similarly as in Section 3.3. Let us define the following set:

$$M = \{\psi \mid \psi \text{ is a subformula of } \tau,\ \tau \in T \cup \{\varphi\}\}\,.$$

Then we construct a submonoid **S** of **L** generated by the set:

$$G = \{a \in L \mid a = e(\psi),\ \psi \in M\}.$$

Since **S** is finitely generated as in Section 3.3, **S** is i.w.o. by Lemma 3.3.1. We can define a residuum and show that **S** is a ΠMTL-chain. Further, if we define $e_{\mathbf{S}}(v) = e(v)$ for each propositional variable v, we obtain an **S**-evaluation such that $e_{\mathbf{S}}(\psi) = e(\psi)$ for all $\psi \in M$. This can be proved by a straightforward modification of the proof of Theorem 3.3.2. Moreover, since $e_{\mathbf{S}}(\tau) = e(\tau) = \mathbf{1}$ for all $\tau \in T$, $e_{\mathbf{S}}$ is an **S**-model of T.

Finally, **S** can be embedded into a ΠMTL-chain in $[0,1]$ with finitely many Archimedean classes in the same way as in the proof of Theorem 3.3.5. Thus there exists an embedding $\Phi : \mathbf{S} \to [\mathbf{0}, \mathbf{1}]$ such that $\Phi(e_{\mathbf{S}}(\varphi)) < 1$ and $\Phi(e_{\mathbf{S}}(\tau)) = 1$ for all $\tau \in T$. □

3.5 Generators of the variety of ΠMTL-algebras

Standard Completeness Theorem has also another important consequence. It shows that the variety of ΠMTL-algebras is generated by the class of all standard ΠMTL-chains.

Theorem 3.5.1 *The class of standard* ΠMTL*-chains generates the variety of* ΠMTL*-algebras.*

PROOF: Let $\mathcal{V}$ be the variety of all ΠMTL-algebras and $\mathcal{K}$ be the class of all standard ΠMTL-chains. By well-known Birkhoff's theorem $\mathcal{V}$ is an equational class, i.e., $\mathcal{V}$ satisfies some set of identities. In order to show that $\mathcal{K}$ generates $\mathcal{V}$, we have to prove that each identity valid in $\mathcal{K}$ is valid also in $\mathcal{V}$. Firstly, observe that an identity $\sigma = \tau$ is valid in some ΠMTL-algebra **L** iff $(\sigma \to \tau)*(\tau \to \sigma) = \mathbf{1}$ is valid in **L**. Let $\sigma = \tau$ be an identity valid in $\mathcal{K}$. Then the identity $(\sigma \to \tau)*(\tau \to \sigma) = \mathbf{1}$ is valid in $\mathcal{K}$ as well. By Theorem 3.3.5 we obtain that the formula $(\hat{\sigma} \Rightarrow \hat{\tau})\&(\hat{\tau} \Rightarrow \hat{\sigma})$ is provable in ΠMTL ($\hat{\sigma}, \hat{\tau}$ denote the corresponding translations of the terms σ, τ into language of ΠMTL). Thus by Theorem 2.3.9 we get that $(\hat{\sigma} \Rightarrow \hat{\tau})\&(\hat{\tau} \Rightarrow \hat{\sigma})$ is an **L**-tautology in any ΠMTL-algebra and $\sigma = \tau$ is valid in $\mathcal{V}$. □

We can again a little bit strengthen the latter theorem according to Theorem 3.3.7.

Theorem 3.5.2 *The class of standard* ΠMTL*-chains with finite congruence lattice generates the variety of* ΠMTL*-algebras.*

Furthermore, in the proof of Standard Completeness Theorem we constructed the ΠMTL-chain **S** and proved in fact that ΠMTL is complete w.r.t. those ΠMTL-chains whose ℓ-monoid reducts are finitely generated. The ℓ-monoid reduct of each such ΠMTL-chain is obviously a bounded integral o-monoid. Moreover, we showed that it is i.w.o. by Lemma 3.3.1. Now we are going to study its order type more closely.

Let $\mathbf{S} = (S, *, \rightarrow, \leq, \mathbf{0}, \mathbf{1})$ be a ΠMTL-chain whose ℓ-monoid reduct $\mathbf{S}' = (S, *, \leq, \mathbf{0}, \mathbf{1})$ is finitely generated. Then **S** has only finitely many Archimedean classes by Lemma 3.3.3. Further, Con **S** is finite by Corollary 3.2.9. Thus all filters of **S** are principal. Let

$$\mathcal{C} = \{C_0 = \{\mathbf{0}\}, C_1, \ldots, C_m, C_{m+1} = \{\mathbf{1}\}\}$$

be the chain of all Archimedean classes of **S**. By Proposition 3.2.8 the dual chain $\mathcal{C}^\partial$ is isomorphic to the chain of all filters of **S**:

$$\mathcal{F} = \{F_0 = S, F_1, \ldots, F_m, F_{m+1} = \{\mathbf{1}\}\}\,.$$

Observe that $F_i = \bigcup_{j \geq i} C_j$ and for $i < m+1$ we have $C_i = F_i - F_{i+1}$ by Proposition 3.2.8. In particular, $F_1 = \bigcup_{j \geq 1} C_j = S - \{\mathbf{0}\}$.

Let G be the finite set of generators of $\mathbf{S}'$ and $G_i = G \cap C_i$, $i = 1, \ldots, m$. Note that $\mathbf{Sg}(G_i) - \{\mathbf{0}, \mathbf{1}\} \subseteq C_i$.

Lemma 3.5.3 *Let $i \in \{1, \ldots, m\}$. Then the submonoid $\mathbf{Sg}(G_i)$ of $\mathbf{S}'$ is order-isomorphic to $(\omega + 1)^*$.*

PROOF: Clearly, $\mathbf{Sg}(G_i)$ is infinite since $\langle g^n \rangle_{n \in \mathbb{N}}$ is a strictly decreasing sequence for any $g \in G_i$ by cancellativity. Let $G_i = \{g_1, \ldots, g_r\}$, $x \in \mathbf{Sg}(G_i)$, and $x \neq \mathbf{0}$. Since all generators from G_i belong to the same Archimedean class, there must be $n \in \mathbb{N}$ for each $j = 1, \ldots, r$, such that we have $g_j^n \leq x$. Let $k \in \mathbb{N}$ be the minimal natural number such that $g_j^k \leq x$ for all $j = 1, \ldots, r$. Then $g_1^k * \cdots * g_r^k \leq x$. As the set

$$\{y \in \mathbf{Sg}(G_i) \mid y \geq g_1^k * \cdots * g_r^k\}$$

is finite, we get that there are only finitely many elements above x. Thus we have proved that the set of nonzero elements in $\mathbf{Sg}(G_i)$ is order-isomorphic to ω^*. If we add the element $\mathbf{0}$, we obtain that $\mathbf{Sg}(G_i)$ is order-isomorphic to $(\omega + 1)^*$. □

Corollary 3.5.4 *Let $H \subseteq \mathbf{Sg}(G_i)$ and H be infinite. Then for any $x \in \mathbf{Sg}(G_i)$, $x \neq \mathbf{0}$, there is an element $w \in H$ such that $w \leq x$.*

Due to commutativity and associativity, each element $x \in S$ can be expressed in the form:

$$x = p_1 * p_2 * \cdots * p_m\,, \quad p_i \in \mathbf{Sg}(G_i)\,.$$

Thus we can assign to each element x a vector $\bar{x} = (p_1, \ldots, p_m)$ and find a function h such that $x = h(\bar{x}) = p_1 * \cdots * p_m$. Note that $\mathbf{1} = h(\mathbf{1}, \ldots, \mathbf{1})$ and $\mathbf{0} = h(p_1, \ldots, p_m)$, if $p_i = \mathbf{0}$ for at least one $i \in \{1, \ldots, m\}$. The projection to the i-th coordinate is denoted by π_i. Observe that if $p_j > 0$ for $j = i, \ldots, m$, then the element represented by $(\mathbf{1}, \ldots, \mathbf{1}, p_i, \ldots, p_m)$ belongs to F_i, in addition, if $p_i < 1$ then $(\mathbf{1}, \ldots, \mathbf{1}, p_i, \ldots, p_m) \in C_i$. To each subset $M \subseteq S$ there exists a set of vectors $\bar{M}$ such that $h(\bar{M}) = M$ and $|M| \leq |\bar{M}|$.

Let us denote by $\mathbf{Sg}(G_1, \ldots, G_k)$ the universe of the submonoid of $\mathbf{S}$ generated by $\bigcup_{1 \leq j \leq k} G_j$. Since $\mathbf{S}$ is i.w.o., each equivalence class $[x]_{F_i}$ has a maximum. It is denoted by $m_x^{F_i}$.

Lemma 3.5.5 *Let* $x \in S - \{\mathbf{0}\}$ *and* $0 < i < m+1$. *Then* $m_x^{F_{i+1}} \in \mathbf{Sg}(G_1, \ldots, G_i)$, *i.e.,* $m_x^{F_{i+1}} = h(p_1, \ldots, p_i, \mathbf{1}, \ldots, \mathbf{1})$ *for some* $p_j \in \mathbf{Sg}(G_j)$, $j = 1, \ldots, i$.

PROOF: If $i = m$, then $\mathbf{Sg}(G_1, \ldots, G_m) = \mathbf{S}$ and obviously $m_x^{F_{m+1}} \in S$. Assume that $i < m$. If $x \in F_{i+1}$ then $m_x^{F_{i+1}} = \mathbf{1}$ and $\mathbf{1} \in \mathbf{Sg}(G_1, \ldots, G_i)$. Finally, suppose that $x \notin F_{i+1}$ and $m_x^{F_{i+1}} = p_1 * p_2 * \cdots * p_i * z$, $z < \mathbf{1}$, $z \in F_{i+1}$. Then $p_1 * p_2 * \cdots * p_i \to m_x^{F_{i+1}} = z$. Thus $p_1 * p_2 * \cdots * p_i \in [x]_{F_{i+1}}$, a contradiction with the condition that $m_x^{F_{i+1}}$ is maximal. □

Lemma 3.5.6 *Let* $x \in S - \{\mathbf{0}\}$ *and* $0 < i < m + 1$. *Then the set*

$$M = [x]_{F_i} \cap \mathbf{Sg}(G_1, \ldots, G_{i-1})$$

is finite.

PROOF: If $i = 1$ then $\mathbf{Sg}(G_1, \ldots, G_{i-1}) = \mathbf{Sg}(\emptyset) = \{\mathbf{0}, \mathbf{1}\}$. Since $[x]_{F_1} = [\mathbf{1}]_{F_1} = F_1$, the intersection $[x]_{F_i} \cap \mathbf{Sg}(G_1, \ldots, G_{i-1}) = \{\mathbf{1}\}$.

Let $i > 1$. Note that the elements from $\bar{M}$ are of the form $(p_1, \ldots, p_{i-1}, \mathbf{1}, \ldots, \mathbf{1})$ for some $p_j \in \mathbf{Sg}(G_j)$, $j = 1, \ldots, i-1$. Suppose that M is infinite. Then $\bar{M}$ is also infinite and there exists a minimal k, $k \leq i-1$, such that $\pi_k(\bar{M})$ is infinite and $\pi_j(\bar{M})$ is finite for all $j < k$. Thus there must be a subset $\bar{H} \subseteq \bar{M}$ such that $\pi_k(\bar{H})$ is infinite and $\pi_j(\bar{H}) = \{q_j\}$ for all $j < k$ and some $q_j \in \mathbf{Sg}(G_j)$.

Let us take an element $p_k \in \pi_k(\bar{H})$ and $p_k < \mathbf{1}$. Since $\pi_k(\bar{H}) \subseteq \mathbf{Sg}(G_k)$ is infinite, there must be an element $w \in \pi_k(\bar{H})$ such that $w \leq p_k^2$ by Corollary 3.5.4 and the fact that $p_k^2 \in \mathbf{Sg}(G_k)$.

Now let us take two vectors $\bar{a}, \bar{b} \in \bar{H}$ such that $\pi_k(\bar{a}) = p_k$ and $\pi_k(\bar{b}) = w$. Then $h(\bar{a}) \to h(\bar{b}) = a \to b \in F_i$ because $a, b \in [x]_{F_i}$. Let us denote $q = q_1 * \cdots * q_{k-1}$, $z_a = \pi_{k+1}(\bar{a}) * \cdots * \pi_{i-1}(\bar{a})$, and $z_b = \pi_{k+1}(\bar{b}) * \cdots * \pi_{i-1}(\bar{b})$. Then $a \to b = q * p_k * z_a \to q * w * z_b = p_k * z_a \to w * z_b \leq p_k * z_a \to p_k^2 * z_b = z_a \to p_k * z_b$. Thus $z_a \to p_k * z_b \in F_i$. Since $F_i \subseteq F_{k+1}$ and $z_a \in F_{k+1}$, we get $z_a * (z_a \to p_k * z_b) \in F_{k+1}$. Thus $p_k * z_b \in F_{k+1}$ because $z_a * (z_a \to p_k * z_b) \leq p_k * z_b$. Since $p_k * z_b \in C_k = F_k - F_{k+1}$, we get a contradiction. □

Lemma 3.5.7 *Let* $x \in S - \{\mathbf{0}\}$, $0 < i < m + 1$, $a, b \in [x]_{F_i}$, *and* $a \leq b$. *Then there exists an element* $w \in \mathbf{Sg}(G_i)$ *such that* $b * w \leq a$ *and* $b * w \in [x]_{F_i}$.

PROOF: Firstly, if $a = b$, then take $w = \mathbf{1}$. Secondly, if $a < b$, let $z = b \to a$. Then $z < \mathbf{1}$, $z \in F_i$, and $b * z \leq a$. We can write $z = h(\mathbf{1}, \ldots, \mathbf{1}, p_i, \ldots, p_m)$ for some $p_j \in \mathbf{Sg}(G_j)$, $j = i, \ldots, m$. There are two cases. In the first case, let $p_i < \mathbf{1}$. Then $p_i < p_{i+1} * \cdots * p_m$ because $p_i \in C_i$ and $p_{i+1} * \cdots * p_m \in C_{i+1}$ by Lemma 3.2.7(3). Let us take $w = p_i^2 < z$. Then $b * w \leq b * z \leq a$ and $w \in \mathbf{Sg}(G_i)$. Moreover, as $b \to b * w = w \in \mathbf{Sg}(G_i) \subseteq F_i$, $b * w$ belongs to $[x]_{F_i}$. In the second case, let $p_i = \mathbf{1}$. Then we can take any element $w \in \mathbf{Sg}(G_i)$, $\mathbf{0} < w < \mathbf{1}$. Since $w < z$ and $b * w \in [x]_{F_i}$, the proof is done. □

Using Lemmata 3.5.5, 3.5.6, and 3.5.7, we are going to prove the crucial structural lemma. This lemma describes the behaviour of the equivalence classes w.r.t. F_{i+1} which are subsets of one equivalence class w.r.t. F_i. Since S/F_{i+1} is a refinement of S/F_i, such a subset form the set $\{[y]_{F_{i+1}} \mid y \in [x]_{F_i}\}$.

Lemma 3.5.8 *Let* $x \in S - \{\mathbf{0}\}$ *and* $0 < i < m + 1$. *Then the set*

$$Y = \{[y]_{F_{i+1}} \mid y \in [x]_{F_i}\}$$

is order-isomorphic to ω^*.

PROOF: Observe that the set Y must be infinite. Since $\mathbf{Sg}(G_i) - \{\mathbf{0}, \mathbf{1}\} \subseteq C_i = F_i - F_{i+1}$, we get that the set $W = \{m_x^{F_i} * s \mid s \in \mathbf{Sg}(G_i) - \{\mathbf{0}\}\}$ is a subset of $[x]_{F_i}$ such that its elements are not equivalent w.r.t. F_{i+1}. Moreover W is order-isomorphic to ω^* by Lemma 3.5.3.

Let $M = [x]_{F_i} \cap \mathbf{Sg}(G_1, \ldots, G_{i-1})$. We will show that each element $z \in [x]_{F_i}$ can be expressed in the form $z = b * s$ for some $b \in M$ and some $s \in F_i$. Firstly, if $i = 1$, then $M = \{\mathbf{1}\}$. Since $z > \mathbf{0}$, we get $z \in F_1$ and $z = \mathbf{1} * z$. Secondly, assume that $i > 1$. Then $z = p_1 * \cdots * p_{i-1} * p_i * \cdots * p_m$ and we can write $z = p_1 * \cdots * p_{i-1} * s$ for $s = p_i * \cdots * p_m \in F_i$. Further, $p_1 * \cdots * p_{i-1} \in [x]_{F_i}$

because $p_1 * \cdots * p_{i-1} \to z = s$. Since $p_1 * \cdots * p_{i-1} \in \mathbf{Sg}(G_1, \ldots, G_{i-1})$, we get $p_1 * \cdots * p_{i-1} \in M$.

Thus for each maximum $m_y^{F_{i+1}} \in [x]_{F_i}$, we can write $m_y^{F_{i+1}} = b * s$ for some $b \in M$ and some $s \in F_i$. Since $m_y^{F_{i+1}} \in \mathbf{Sg}(G_1, \ldots, G_i)$ by Lemma 3.5.5, it follows that s must belong to $\mathbf{Sg}(G_i)$.

By Lemma 3.5.7 we can find for each $m_y^{F_{i+1}} \in [x]_{F_i}$ and for each $b \in M$, an element $w \in \mathbf{Sg}(G_i)$ such that $b * w \leq m_y^{F_{i+1}}$ and $b * w \in [x]_{F_i}$. Since $\mathbf{Sg}(G_i) - \{\mathbf{0}\}$ is order-isomorphic to ω^* by Lemma 3.5.3 and M is finite by Lemma 3.5.6, we get that the set

$$H = \{b * s \mid s \in \mathbf{Sg}(G_i),\ b \in M,\ b * s \geq m_y^{F_{i+1}}\}$$

is finite. Since $\{m_u^{F_{i+1}} \mid m_u^{F_{i+1}} \geq m_y^{F_{i+1}},\ u \in [x]_{F_i}\} \subseteq H$, the desired order-isomorphism # can be defined as follows:

$$\#[y]_{F_{i+1}} = |\{m_u^{F_{i+1}} \mid m_u^{F_{i+1}} \geq m_y^{F_{i+1}},\ u \in [x]_{F_i}\}| \,.$$

It is obvious that $\#[y]_{F_{i+1}} \leq |H|$. In other words, the natural number $\#[y]_{F_{i+1}}$ represents the position of $[y]_{F_{i+1}}$ within $[x]_{F_i}$. □

Now we define a mapping $\Phi : S - \{\mathbf{0}\} \to (\omega^m)^*$ as follows:

$$\Phi(x) = (\#[x]_{F_2}, \#[x]_{F_3}, \ldots, \#[x]_{F_{m+1}}) \,.$$

Lemma 3.5.9 *The mapping* Φ *is an order-isomorphism.*

PROOF: Firstly, we have to show that Φ is one-to-one and order-preserving. Consider two elements $x, y \in S - \{\mathbf{0}\}$ such that $x < y$. Then there exists a minimal $i \in \{2, \ldots, m+1\}$ such that $[x]_{F_i} < [y]_{F_i}$. Thus $\#[x]_{F_j} = \#[y]_{F_j}$ for all $j = 2, \ldots, i-1$, and $\#[x]_{F_i} < \#[y]_{F_i}$. Thus $\Phi(x) < \Phi(y)$.

Secondly, we have to show that the function Φ is onto. Consider an m-tuple $(n_2, n_3, \ldots, n_{m+1})$. By Lemma 3.5.8 we know that equivalence classes $[y]_{F_2}$ which are subsets of $[x]_{F_1} = [\mathbf{1}]_{F_1} = S - \{\mathbf{0}\}$ are order-isomorphic to ω^*. Thus we can find an equivalence class $[x_2]_{F_2}$ such that $\#[x_2]_{F_2} = n_2$. Then again by Lemma 3.5.8 we can find an equivalence class $[x_3]_{F_3} \subseteq [x_2]_{F_2}$ such that $\#[x_3]_{F_3} = n_3$. Repeating this procedure we finally find $[x_{m+1}]_{F_{m+1}}$ such that $\#[x_{m+1}]_{F_{m+1}} = n_{m+1}$. Since $F_{m+1} = \{\mathbf{1}\}$ is the trivial filter, we get $[x_{m+1}]_{F_{m+1}} = \{x_{m+1}\}$ and $\Phi(x_{m+1}) = (n_2, n_3, \ldots, n_{m+1})$ because $[x_2]_{F_2} \supseteq [x_3]_{F_3} \supseteq \cdots \supseteq [x_{m+1}]_{F_{m+1}} = \{x_{m+1}\}$. □

Lemma 3.5.9 gives us the final theorem on order type of **S**.

Theorem 3.5.10 *Let* $\mathbf{S} = (S, *, \to, \leq, \mathbf{0}, \mathbf{1})$ *be a* ΠMTL-*chain whose* ℓ-*monoid reduct* $\mathbf{S}' = (S, *, \leq, \mathbf{0}, \mathbf{1})$ *is finitely generated. Then* $\mathbf{S}$ *is order-isomorphic to* $(\omega^m + 1)^*$ *where* $m + 2$ *is the number of Archimedean classes of* $\mathbf{S}$.

Remark 3.5.11 Let $\mathbf{M} = (M, *, \leq, \mathbf{1})$ be any cancellative, integral, finitely generated o-monoid. In this section we have in fact proved that $\mathbf{M}$ is order-isomorphic to $(\omega^m)^*$ where $m+1$ is the number of Archimedean classes. Since $\mathbf{M}$ is i.w.o. by Lemma 3.3.1, $\mathbf{M}$ can be enriched by a residuum. Thus $\mathbf{M}$ is a cancellative, integral residuated chain. If we add to $\mathbf{M}$ a bottom element $\mathbf{0}$, we obtain a ΠMTL-chain by Lemma 3.2.3. Thus Theorem 3.5.10 can be applied to $\mathbf{M}$ as well.

Let $(\mathbb{Z}^-)^m = \prod_{i=1}^m \mathbb{Z}^-$ be the cartesian product of m copies of $\mathbb{Z}^-$ endowed with the lexicographic order. Let us take the set $Z = \{-\infty\} \cup (\mathbb{Z}^-)^m$ and define an order on Z by setting $-\infty \leq z$ for any $z \in Z$. Clearly Z is order-isomorphic to $(\omega^m + 1)^*$. Since $\mathbf{S}$ is order-isomorphic to $(\omega^m + 1)^*$ as well by Theorem 3.5.10, there is a ΠMTL-chain $\mathbf{Z} = (Z, \odot, \to_\odot, \leq, -\infty, \langle 0, \ldots, 0\rangle)$ which is isomorphic to $\mathbf{S}$. Thus we obtain the following theorem.

Theorem 3.5.12 *Let* $\mathcal{K}$ *be the class of all* ΠMTL-*chains whose universe is* $(\omega^m + 1)^*$ *for some* $m \in \mathbb{N}$. *Then the following holds:*

1. ΠMTL $\vdash \varphi$ *iff* φ *is a* $\mathbf{Z}$-*tautology for each* ΠMTL-*chain* $\mathbf{Z}$ *from* $\mathcal{K}$.

2. *The class* $\mathcal{K}$ *generates the variety of* ΠMTL-*algebras.*

Chapter 4

Structure of standard ΠMTL-chains

In this chapter we deal with the structure of standard ΠMTL-chains. We mainly concentrate on the structure of standard subdirectly irreducible ΠMTL-chains since they are the generators of the variety of ΠMTL-algebras. However, we will also deal with the general ΠMTL-chains. We show in Section 4.3 that it is possible to embed the ℓ-monoid reduct of each ΠMTL-chain $\mathbf{L}$ into a totally ordered Abelian group $\mathbf{G_L}$ by forming fractions in the same way as the positive rational numbers are constructed from positive integers. Then we use Hahn's Embedding Theorem (see [14, 15]) and embed $\mathbf{G_L}$ to a full Hahn group. A full Hahn group is a group of functions from the set of principal ℓ-subgroups of $\mathbf{G_L}$ to reals under addition. Moreover, the supports of the functions (the regions where the functions are not zero) are inversely well ordered w.r.t. the order induced by the inclusion of the principal ℓ-subgroups. Thus it is possible to make each full Hahn group totally ordered by lexicographic order. In this chapter we show how to select an arbitrary ΠMTL-chain from a full Hahn group which is order-isomorphic to $[0, 1]$. In this way we obtain a characterization of the structure of standard subdirectly irreducible ΠMTL-chains up to an isomorphism. In particular, in Section 4.5 we present a construction of a subdirectly irreducible standard ΠMTL-chain from a full Hahn group and in Section 4.6 we prove that each standard subdirectly irreducible ΠMTL-chain can be obtained in this way. Finally, in Section 4.7 we summarize our results on standard ΠMTL-chains which are not subdirectly irreducible. The results from this chapter come from our paper [23].

4.1 Complete ΠMTL-chains

Since the real unit interval $[0, 1]$ is a complete chain, we list here some important facts about complete ΠMTL-chains.

Lemma 4.1.1 *Nontrivial filters of a complete* ΠMTL-*chain* $\mathbf{L}$ *are intervals of the form* $(a, \mathbf{1}]$ *for some* $a \in L - \{\mathbf{1}\}$.

PROOF: Each filter F is an interval with an upper bound $\mathbf{1}$. Let us denote $\bigwedge F$ by a. As F is nontrivial, $a < \mathbf{1}$. Suppose that $a \in F$. If $a = \mathbf{0}$, then F is trivial (a contradiction). If $a > \mathbf{0}$, then $a^2 < a$ and $a^2 \in F$; we get a contradiction with the fact that $a = \bigwedge F$. □

Lemma 4.1.2 *Each nontrivial filter of a complete* ΠMTL-*chain* $\mathbf{L}$ *has a successor.*

PROOF: Let F be a nontrivial filter. By Lemma 4.1.1 we have $F = (a, \mathbf{1}]$ for some $a \in L - \{\mathbf{1}\}$. Since any filter greater than F must contain a, the principal filter F^a generated by a is the successor. □

Lemma 4.1.3 *Let* $\mathbf{L}$ *be a complete* ΠMTL-*chain,* $x \in L$*,* $x > \mathbf{0}$*, and* F *be a nontrivial filter. Then the equivalence class* $[x]_F$ *is a left-open and right-closed interval.*

PROOF: Firstly, we show that $[x]_F$ is an interval. Suppose that $a, b \in [x]_F$ and $a \leq c \leq b$ for some $c \in L$. Then $b \to c \geq b \to a \in F$. Hence $c \in [x]_F$.

Secondly, let us denote by m_x^F the supremum of $[x]_F$. Assume that $m_x^F \notin [x]_F$. Let s be an arbitrary element of $F - \{\mathbf{1}\}$. The element $z = m_x^F * s$ is in $[m_x^F]_F$, because $m_x^F \to z = s \in F$. Cancellativity implies $z < m_x^F$. As z is an upper bound of $[x]_F$, m_x^F cannot be the supremum of $[x]_F$ (a contradiction). Consequently, $m_x^F \in [x]_F$ and $[x]_F$ is right-closed.

Finally, let us take the element $z = \bigwedge [x]_F$; we shall prove that it does not belong to $[x]_F$. Let us take $s \in F - \{\mathbf{1}\}$. Due to cancellativity, $z * s < z$. If $z \in [x]_F$, then $z * s \in [x]_F$ which contradicts the minimality of z. Therefore $z \notin [x]_F$. □

From now on the maximum of an equivalence class $[x]_F$ will be denoted by m_x^F. From the structure of the equivalence classes it follows the following two corollaries.

Corollary 4.1.4 *Let* $\mathbf{L}$ *be a complete* ΠMTL-*chain,* F *be a nontrivial filter, and* $x > \mathbf{0}$*. Then each element* $[x]_F$ *of* $\mathbf{L}/F$ *has a predecessor.*

PROOF: By Lemma 4.1.3, $[x]_F$ is left-open and right-closed interval. Let us denote $a = \bigwedge [x]_F$. Then $[a]_F$ must be a predecessor of $[x]_F$. Moreover, $a = m_a^F$. □

Corollary 4.1.5 *Let* $\mathbf{L}$ *be a complete* ΠMTL-*chain,* $F \in \mathcal{F}$*, and* $x \in L$*. Then* $m_x^F = \max\,[m_x^F]_{F'}$ *for all* $F' \subseteq F$*.*

Lemma 4.1.6 *Let* $\mathbf{L}$ *be a complete* ΠMTL-*chain and* F *be a nontrivial filter. Then* $\mathbf{L}/F$ *is complete as well.*

PROOF: Let M be a subset of $\mathbf{L}/F$. Let us define the following subset of L:

$$M' = \bigcup_{[x]_F \in M} [x]_F \,.$$

Since $\mathbf{L}$ is complete, there is a supremum of M'. Let us denote it by m. We claim that $[m]_F$ is a supremum of M. Clearly, $[m]_F \geq [x]_F$ for all $[x]_F \in M$ because $x \leq m$ for any $x \in M'$. Suppose that there is $[m']_F < [m]_F$ such that $[m']_F \geq [x]_F$ for all $[x]_F \in M$. Without any loss of generality we can assume that m' is the maximum of $[m']_F$. Since $m = \bigvee M'$, there must be $x \in M'$ such that $x > m'$. Thus $[x]_F > [m']_F$ and $[x]_F \in M$ which gets a contradiction. □

Now we will show the crucial property of the congruence lattice of a complete ΠMTL-chain. By well-known theorem from Universal Algebra (see e.g. [5, Chapter 2, Theorem 8.4] if a ΠMTL-chain $\mathbf{L}$ is subdirectly irreducible then Con $\mathbf{L} - \{\Delta\}$ has a minimum. Thus there must also be the corresponding nontrivial minimal filter. We will denote it by F_Δ.

Theorem 4.1.7 *Let* $\mathbf{L}$ *be a complete subdirectly irreducible* ΠMTL-*chain. Then* Con $\mathbf{L}$ *is well ordered.*

PROOF: As Con $\mathbf{L}$ is isomorphic to the set of all filters $\mathcal{F}$ of $\mathbf{L}$, it is sufficient to show that $\mathcal{F}$ is well ordered. Since $\mathbf{L}$ is subdirectly irreducible, there must be the minimum nontrivial filter F_Δ. Thus F_Δ is a successor of $\{1\}$. Consequently, each $F \in \mathcal{F} - \{L\}$ has a successor by Lemma 4.1.2. Now, take an arbitrary subset $\mathcal{M}$ of $\mathcal{F}$. Let us denote by $\overline{\mathcal{M}}$ upper closure of $\mathcal{M}$, i.e., $\overline{\mathcal{M}} = \{F \in \mathcal{F} \mid (\exists D \in \mathcal{M})(F \supseteq D)\}$. Let us consider $\mathcal{N} = \mathcal{F} - \overline{\mathcal{M}}$. There are two cases. In the first case, let $\mathcal{N}$ has a maximum F_m. Then the successor of F_m is the minimum of $\mathcal{M}$. In the second case, if $\mathcal{N}$ has no maximum then $\bigcup_{F\in\mathcal{N}} F \in \mathcal{F}$ by Lemma 3.2.4 and it is the minimum of $\mathcal{M}$. □

The similar statement holds also for the equivalence classes w.r.t. a nontrivial filter.

Theorem 4.1.8 *Let* $\mathbf{L}$ *be a complete* ΠMTL-*chain and* F *be a nontrivial filter. Then* $\mathbf{L}/F$ *is i.w.o.*

PROOF: The universe of $\mathbf{L}/F$ is the set L/F of all equivalence classes $[x]_F$. Let M be a subset of L/F. We will show that M possesses a maximum. Let us denote by $\overline{M}$ its lower closure, i.e., $\overline{M} = \{z \in L/F \mid (\exists y \in M)(z \leq y)\}$. Now, there are two cases. (1) $L/F - \overline{M}$ has a minimum. Then the successor of this minimum must be a maximum of M. (2) $L/F - \overline{M}$ has no minimum. Let us denote the infimum of the union of all equivalence classes belonging to $L/F - \overline{M}$ as follows:

$$a = \bigwedge \bigcup_{[x]_F \in L/F - \overline{M}} [x]_F\,.$$

Since $[x]_F$ are left-open intervals by Lemma 4.1.3, $[a]_F$ is the maximum of M. □

4.2 Standard subdirectly irreducible ΠMTL-chains

In this section we start to investigate the structure of the standard ΠMTL-chains. We will deal with the subdirectly irreducible standard ΠMTL-chains because they are the generators of the variety of ΠMTL-algebras.

Lemma 4.2.1 *Let* $\mathbf{L}$ *be a subdirectly irreducible standard* ΠMTL-*chain. Then* Con $\mathbf{L}$ *is countable.*

PROOF: By Lemma 4.1.1 each nontrivial filter F in $\mathbf{L}$ is of the form $(a, 1]$. Thus if we identify the nontrivial filters with their infima, we obtain a subset of $[0, 1]$ which is i.w.o. since Con $\mathbf{L}$ is well ordered by Theorem 4.1.7. But any i.w.o. subset of $[0, 1]$ is countable. Thus the set of all filters $\mathcal{F}$ is countable and Con $\mathbf{L}$ as well. □

Lemma 4.2.2 *Let* $\mathbf{L}$ *be a subdirectly irreducible standard* ΠMTL-*chain and* $a, b \in [x]_{F_\Delta}$. *Then* $a * (a \to b) = a \wedge b$.

PROOF: If $a \leq b$ then the equality trivially holds. If also a or b equals 1 then the equality trivially holds. Thus suppose that $a > b$ and $a, b \neq 1$. By residuation we get $a * (a \to b) \leq b$. Suppose that $a * (a \to b) < b$. Fix an arbitrary strictly increasing sequence $\langle r_n \rangle_{n \in \mathbb{N}}$ such that $\bigvee r_n = 1$ and $r_n \in F_\Delta$ for all n. As F_Δ is a left-open interval of the type $(c, 1]$ for some $0 \leq c < 1$, there surely exists such sequence. Since $F_\Delta - \{1\}$ is an Archimedean class, we get that for each n there exists k_n such that

$$r_n^{k_n} \leq a \to b < r_n^{k_n - 1}\,.$$

Thus we obtain for all $n \in \mathbb{N}$:

$$a * r_n^{k_n} \leq a * (a \to b) < b < a * r_n^{k_n - 1}\,.$$

The last inequality holds since $a \to b$ is the maximal solution of the inequality $a * x \leq b$ and $a \to b < r_n^{k_n - 1}$.

Further, by Proposition 2.1.1 we get $\bigvee (b * r_n) = b * \bigvee r_n = b$. Hence there must be an n_0 such that $a * (a \to b) < b * r_{n_0}$. Thus we obtain

$$a * r_{n_0}^{k_{n_0}} \leq a * (a \to b) < b * r_{n_0} < a * r_{n_0}^{k_{n_0}}\,,$$

a contradiction. □

Theorem 4.2.3 *Let* **L** *be a subdirectly irreducible standard* ΠMTL-*chain. For each equivalence class w.r.t.* F_Δ *it holds*

$$[x]_{F_\Delta} = \{z \in L \mid z = m_x^{F_\Delta} * s,\ s \in F_\Delta\}\,.$$

PROOF: By Lemma 4.2.2 we have $m_x^{F_\Delta} * (m_x^{F_\Delta} \to z) = z$ for any $z \in [x]_{F_\Delta}$. Thus we can take $s = m_x^{F_\Delta} \to z \in F_\Delta$. On the other hand, $m_x^{F_\Delta} * s$ belongs to $[x]_{F_\Delta}$ for all $s \in F_\Delta$ because $m_x^{F_\Delta} \to m_x^{F_\Delta} * s = s$. □

Lemma 4.2.2 has also other important consequence. It implies that the elements belonging to $F_\Delta = [1]_{F_\Delta}$ behave like in a divisible algebra, i.e., if $x \leq y$ then there is an element z such that $y * z = x$, namely $z = y \to x$.

Theorem 4.2.4 *Let* **L** *be a standard subdirectly irreducible* ΠMTL-*chain. Then* $F_\Delta \cup \{0\}$ *is a subalgebra of* **L** *isomorphic to the standard product algebra* $[0, 1]_\Pi$.

PROOF: Firstly, F_Δ is closed under $*$ and $\to$ because $a \to b \geq b$. Let $a \in F_\Delta \cup \{0\}$. Since $a * 0 = 0$, $0 \to a = 1$, and $a \to 0 = 0$ for $a > 0$, we get that $F_\Delta \cup \{0\}$ is a subuniverse. Since $F_\Delta = [1]_{F_\Delta}$, we get by Lemma 4.2.2 that $a * (a \to b) = a \wedge b$

for all $a, b \in F_\Delta$. This equation is also trivially satisfied if a or b equals 0. Thus $(F_\Delta \cup \{0\}, *, \to, \leq, 0, 1)$ is a product algebra. Since $F_\Delta \cup \{0\}$ is order isomorphic to $[0, 1]$, $(F_\Delta \cup \{0\}, *, \to, \leq, 0, 1)$ is isomorphic to a product algebra in $[0, 1]$. But all product algebras in $[0, 1]$ are isomorphic to the standard product algebra $[0, 1]_\Pi$. □

Let $\mathbf{L}$ be a ΠMTL-chain. Then the set of all elements which cannot be expressed as a product of greater elements is denoted by E, i.e.,

$$E = \{z \in L \mid \neg(\exists x, y \in L)(z = x * y \ \& \ x, y > z)\}\,.$$

An element from E will be called a *product irreducible* element.

Lemma 4.2.5 *Let* $\mathbf{L}$ *be a standard subdirectly irreducible* ΠMTL-*chain. Then the set* E *of all product irreducible elements of* $\mathbf{L}$ *is i.w.o. Moreover,* E *is a subset of the set of all maxima of equivalence classes w.r.t.* F_Δ*, i.e.,* $E \subseteq \{m_x^{F_\Delta} \mid x \in L\}$*.*

PROOF: Firstly, we show that $E \cap F_\Delta = \{1\}$. The element 1 obviously belong to this intersection. Let $z \in F_\Delta - \{1\}$. Since $F_\Delta \cup \{0\}$ is isomorphic to the standard product algebra by Theorem 4.2.4, there must be an element $x \in F_\Delta$ such that $z = x * x$ (x is the square root of z). Thus $z \notin E$.

Secondly, we show that $E \subseteq \{m_x^{F_\Delta} \mid x \in L\}$. Let $z \in L$ and $z \neq m_z^{F_\Delta}$. Then by Theorem 4.2.3, we have $z = m_z^{F_\Delta} * s$ for some $s \in F_\Delta$. Thus $z \notin E$.

Finally, the set $\{m_x^{F_\Delta} \mid x \in L\}$ is order-isomorphic to L/F_Δ. Since L/F_Δ is i.w.o. by Theorem 4.1.8, the set E is i.w.o. as well. □

Theorem 4.2.6 *Let* $\mathbf{L}$ *be a standard subdirectly irreducible* ΠMTL-*chain,* $x \in L$*, and* E *be the set of all product irreducible elements of* $\mathbf{L}$*. If* $x \notin E$*, then*

$$x = g_1 * \cdots * g_n * s$$

for some $g_i \in E$*,* $i = 1, \ldots, n$*, and* $s \in F_\Delta$*.*

PROOF: By Theorem 4.2.3, we can write $x = m_x^{F_\Delta} * s$ for some $s \in F_\Delta$. Thus it is sufficient to show that $m_x^{F_\Delta} = g_1 * \cdots * g_n$ for some $g_i \in E$, $i = 1, \ldots, n$.

If $m_x^{F_\Delta} \in E$ then we are done. If not, $m_x^{F_\Delta}$ can be expressed as $m_x^{F_\Delta} = a * b$ for some $a, b > m_x^{F_\Delta}$. Moreover, $a, b \in \{m_y^{F_\Delta} \mid y \in L\}$. Indeed, suppose that $a = m_a^{F_\Delta} * r$ for some $r \in F_\Delta$. Then $m_x^{F_\Delta} = a * b = m_a^{F_\Delta} * r * b$. Thus $m_a^{F_\Delta} * b > m_x^{F_\Delta}$ contradicting the fact that $m_x^{F_\Delta}$ is the maximum of the equivalence class. Now a, b belong to E or can be again decomposed. In this way, we obtain a binary tree where the leafs belong to E. Moreover, each branch of the tree is strictly increasing. Since $\{m_y^{F_\triangle} \mid y \in L\}$ is i.w.o. by Theorem 4.1.8, each branch must be finite. Thus there is a finite number of leafs. Let us denote them by $g_1, \ldots, g_n$. Then $m_x^{F_\triangle} = g_1 * \cdots * g_n$. □

4.3 Fraction group

Let $\mathbf{L}$ be a ΠMTL-chain. By Lemma 3.2.2 we have that $L_0 = L - \{\mathbf{0}\}$ is a subuniverse of $\mathbf{L}$ and the subreduct with this subuniverse $\mathbf{L}_0 = (L_0, *, \rightarrow, \leq, \mathbf{1})$ is an integral cancellative residuated chain. Let us denote the ℓ-monoid reduct $(L_0, *, \leq, \mathbf{1})$ by $\mathbf{L}_0'$. The set of all filters of $\mathbf{L}_0$ will be denoted by $\mathcal{F}_0 = \mathcal{F} - \{L\}$. Since $\mathbf{L}_0'$ is also cancellative, we can extend it to an o-group of fractions $\mathbf{G_L}$ in the similar way as rationals are constructed from integers. The universe of $\mathbf{G_L}$ is $G_L = (L_0 \times L_0)/\approx$, where $(a, b) \approx (c, d)$ iff $a * d = c * b$. The group operation is defined by $(a, b)*(c, d) = (a*c, b*d)$, $(\mathbf{1}, \mathbf{1})$ is the neutral element, $(a, b)^{-1} = (b, a)$, and $(a, b) \leq (c, d)$ iff $a * d \leq b * c$. We will denote the ordered pair (a, b) by a/b or $a * b^{-1}$. Further, we identify the elements from L_0 with the corresponding elements in G_L, i.e., we will write a instead of $a/1$. Thus L_0 can be viewed as a subuniverse of the monoid reduct of $\mathbf{G_L}$. The set of all convex subgroups of $\mathbf{G_L}$ will be denoted by $\mathcal{G}$.

Lemma 4.3.1 *The congruence lattice* Con $\mathbf{L}_0$ *can be embedded into* Con $\mathbf{G_L}$.

PROOF: We will work with the chain of the filters $\mathcal{F}_0$ (resp. convex subgroups $\mathcal{G}$) instead of Con $\mathbf{L}_0$ (resp. Con $\mathbf{G_L}$). To each filter $F \in \mathcal{F}_0$ we can assign a corresponding convex subgroup $\overline{F} \in \mathcal{G}$ as follows:

$$\overline{F} = \{z \in G_L \mid (\exists y \in F)(|z| \leq |y|)\} .$$

We start with the proof that $\overline{F}$ is a convex subgroup. Clearly, $\overline{F}$ is convex. Let a/b and c/d be elements of $\overline{F}$. Then there are $y_1, y_2 \in F$ such that $|y_1| \geq |a/b|$ and $|y_2| \geq |c/d|$. Since $y_1 * y_2 \leq 1$ and $y_1 * y_2 \in F$, we get

$$|y_1 * y_2| = y_1^{-1} * y_2^{-1} = |y_1| * |y_2| \geq |a/b| * |c/d| \geq |(a * c)/(b * d)| .$$

Thus $(a * c)/(b * d)$ belongs to $\overline{F}$.

Finally, we have to show that the mapping assigning $\overline{F}$ to F is injective and order-preserving. Let F, F' be two different filters in $\mathcal{F}_0$. Since $\mathcal{F}_0$ is linearly ordered, one of the filters contains the other. Without any loss of generality suppose that $F' \subseteq F$. Let us take an arbitrary element $z \in F - F'$. Then $z \notin \overline{F'}$ since there is no $y \in F'$ such that $y \leq z$. Thus $\overline{F} \supsetneq \overline{F'}$. □

From now on, we will denote by $\overline{F}$ the convex subgroup corresponding to F.

Now we show an example demonstrating the fact that Con $\mathbf{L}_0$ need not be isomorphic to Con $\mathbf{G_L}$. Let $\mathbf{L}$ be the integral cancellative residuated chain from Example 3.2.10. Let us take its submonoid $\mathbf{S} = (S, *, \leq, 1)$ generated by $\langle -1, 0\rangle$ and $\langle -1, -1\rangle$. Then $\mathbf{S}$ is i.w.o. by Lemma 3.3.1. Hence such a submonoid is in fact a residuated chain if we define a residuum in $\mathbf{S}$ as $a \to_S b = \max\{c \in S \mid a * c \leq b\}$. It is obvious that $\mathbf{S}$ has only trivial filters since there is $n \in \mathbb{N}$ such that $\langle -1, 0\rangle^n \leq \langle k, r\rangle$ for all $\langle k, r\rangle \in S$. On the other hand the group of fractions $\mathbf{G_S}$ has one nontrivial convex subgroup. It is a subgroup generated by the fraction $\langle -1, -1\rangle/\langle -1, 0\rangle$. Indeed, we have for all $n \in \mathbb{N}$ the following:

$$(\langle -1, -1\rangle/\langle -1, 0\rangle)^n = \langle -n, -n\rangle/\langle -n, 0\rangle > \langle -1, 0\rangle\,,$$

since $\langle -n, -n\rangle > \langle -1, 0\rangle * \langle -n, 0\rangle = \langle -n-1, 0\rangle$.

Lemma 4.3.2 *Let* $\mathbf{L}$ *be a subdirectly irreducible standard* ΠMTL-*chain, then its group of fractions* $\mathbf{G_L}$ *has the same property and the minimal nontrivial congruence is determined by the convex subgroup* $\overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$.

PROOF: Since $\mathbf{L}$ is subdirectly irreducible, there must be a nontrivial minimal congruence in $\mathbf{L}$. Thus we have a nontrivial minimal filter F_Δ. Firstly, we will show that for any $a/b \in \overline{F_\Delta}$ either $a/b \in F_\Delta$ or $b/a \in F_\Delta$. Without any loss of generality we can suppose that $a/b \leq 1$, i.e., $a \leq b$. Then there exists $y \in F_\Delta$ such that $|a/b| \leq |y|$ and $y \leq a/b$. If we multiply this inequality by b, we get $b*y \leq b*(a/b) = a$. Since $y \in F_\Delta$, we obtain $y \leq b \to a$ which implies $b \to a \in F_\Delta$. Thus $m_a^{F_\Delta} = m_b^{F_\Delta}$. By Theorem 4.2.3 we get $a = m_a^{F_\Delta} * r$ and $b = m_a^{F_\Delta} * s$ for some $r, s \in F_\Delta$. Thus $a/b = (m_a^{F_\Delta} * r)/(m_a^{F_\Delta} * s) = r/s$. Since elements from F_Δ satisfy the divisibility condition by Lemma 4.2.2, $r/s = s \to r \in F_\Delta$. Thus $\overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$.

Secondly, we will prove that $\overline{F_\Delta}$ is the minimal nontrivial convex subgroup. Let us take an arbitrary element $z \in \overline{F_\Delta} - \{1\}$. We will show that z generates $\overline{F_\Delta}$. We can suppose that $z \leq 1$ (if not take z^{-1}). From the previous paragraph we have $z \in F_\Delta$. Since F_Δ is the minimal nontrivial filter, z generates F_Δ and it must generate also $\overline{F_\Delta}$. □

Corollary 4.3.3 *Let* $\mathbf{L}$ *be a subdirectly irreducible standard* ΠMTL-*chain and* F_Δ *be its minimal nontrivial filter. Then the o-group* $(\overline{F_\Delta}, *, \leq, 1)$ *is isomorphic to* $(\mathbb{R}, +, \leq, 0)$.

PROOF: Since $(F_\Delta, *, \leq, 1)$ is isomorphic to $((0,1], \cdot, \leq, 1)$ by Theorem 4.2.4, $(F_\Delta, *, \leq, 1)$ is also isomorphic to $(\mathbb{R}^-, +, \leq, 0)$. Let us denote this isomorphism by Φ. Since $\overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$, we define a mapping $\Psi : \overline{F_\Delta} \to \mathbb{R}$ by

$$\Psi(x) = \begin{cases} \Phi(x)\,, & x \leq 1\,, \\ -\Phi(x^{-1})\,, & x > 1\,. \end{cases}$$

We claim that Ψ is an isomorphism between $\overline{F_\Delta}$ and $(\mathbb{R}, +, \leq, 0)$. The mapping Ψ is clearly onto and order-preserving. The fact that Ψ is an isomorphism can be easily checked. The cases when $x, y \leq 1$ or $x, y > 1$ are trivial. We only show the case when $x \leq 1$, $y > 1$, and $x * y \leq 1$. The other cases are similar. Then $\Psi(x * y) = \Phi(x * y)$ and $\Psi(x) + \Psi(y) = \Phi(x) - \Phi(y^{-1})$. Since $x * y \leq 1$, we get $x \leq y^{-1}$. Thus $\Phi(x) \leq \Phi(y^{-1})$ and $\Phi(x) - \Phi(y^{-1}) \leq 0$. Hence there is $z \in F_\Delta$ such that $\Phi(z) = \Phi(x) - \Phi(y^{-1})$. Since

$$\Phi(z * y^{-1}) = \Phi(z) + \Phi(y^{-1}) = \Phi(x) - \Phi(y^{-1}) + \Phi(y^{-1}) = \Phi(x)\,,$$

we obtain that $z * y^{-1} = x$. Consequently, $z = x * y$ and $\Psi(x * y) = \Psi(x) + \Psi(y)$. Similarly the case for $x * y > 1$. Thus $\overline{F_\Delta}$ is isomorphic to $(\mathbb{R}, +, \leq, 0)$. □

Now we show the relation between principal filters of a ΠMTL-chain $\mathbf{L}$ and the principal convex subgroups of its group of fractions $\mathbf{G_L}$.

Lemma 4.3.4 *Let* $\mathbf{L}$ *be a* ΠMTL-*chain,* $F^b \in \mathcal{P}$, $F^b \neq L$, *and* V^b *be the principal convex subgroup of* $\mathbf{G_L}$ *generated by* b. *Then* $\overline{F^b} = V^b$.

PROOF: The case when $b = \mathbf{1}$ is trivial. Let $F^b \neq \{\mathbf{1}\}$. We will show that $\overline{F^b}$ is a successor of V_b. As V_b is the greatest convex subgroup not containing b, we obtain $V_b \subseteq \overline{F^b}$. Since V_b is the predecessor of V^b, it is sufficient to prove that $\overline{F^b} \subseteq V^b$. Let $x \in \overline{F^b}$. By Lemma 4.3.1, there is $y \in F^b$ such that $|x| \leq |y|$. Further by Lemma 3.2.5, there exists $n \in \mathbb{N}$ such that $b^n \leq y$. Since $b, y \leq \mathbf{1}$, we get $|y| \leq |b^n| = |b|^n$. Thus $|x| \leq |b|^n$. Finally by Lemma 2.1.5, we obtain $x \in V^b$. □

Corollary 4.3.5 *Let* $\mathbf{L}$ *be a* ΠMTL-*chain and* $F^b \in \mathcal{P}$ *be a nontrivial principal filter. Then* $\overline{F^b}/V_b$ *is isomorphic (as an o-group) to a subgroup of the additive group of real numbers.*

PROOF: By Lemma 4.3.4 we have $\overline{F^b}/V_b = V^b/V_b$. Since V^b/V_b is an Archimedean *o*-group, it is isomorphic to a subgroup of the additive group of real numbers by Hölder's Theorem (see [15, Corollary 4.1.4]). □

Lemma 4.3.6 *The congruence lattice* Con $\mathbf{G_L}$ *can be embedded into* Con $\mathbf{L}'_0$.

PROOF: Let $\theta \in$ Con $\mathbf{G_L}$. Since $L_0 \subseteq G_L$, we can map θ to the restriction $\theta\restriction_{\mathbf{L}'_0}$. Clearly, $\theta\restriction_{\mathbf{L}'_0}$ belongs to Con $\mathbf{L}'_0$. Thus we have to show that this mapping is injective and order-preserving. If $\theta_1 \subseteq \theta_2$ then trivially $\theta_1\restriction_{\mathbf{L}'_0} \subseteq \theta_2\restriction_{\mathbf{L}'_0}$. Thus the mapping is order-preserving. Now, assume that $\theta_1 \subsetneq \theta_2$. Let us denote by V_1 (resp. V_2) the convex subgroup corresponding to θ_1 (resp. θ_2). Then there must be an element a/b in V_2 such that $a/b \notin V_1$ and $a, b \in L_0$. Thus $a\,\theta_2\restriction_{\mathbf{L}'_0} b$. But it is not true that $a\,\theta_1\restriction_{\mathbf{L}'_0} b$. Hence $\theta_1\restriction_{\mathbf{L}'_0} \subsetneq \theta_2\restriction_{\mathbf{L}'_0}$. □

Although Con $\mathbf{G_L}$ can be embedded into Con $\mathbf{L}'_0$, they need not be isomorphic. Let $\mathbf{Z}^- = (\mathbb{Z}^-, +, \leq, 0)$ be the *o*-monoid of non-positive integers with the usual addition. Then the lexicographic product $\mathbf{Z} = \prod_{i=1}^{3} \mathbf{Z}^-$ is an *o*-monoid which is clearly cancellative and i.w.o. Thus $\mathbf{Z}$ can be enriched by a residuum and become a cancellative residuated chain. The group of fractions $\mathbf{G_Z}$ has two nontrivial convex subgroups; V_1 generated by $\langle 0, 0, -1\rangle$ and V_2 generated by $\langle 0, -1, 0\rangle$. Obviously $V_1 \subseteq V_2$. Let θ_1, θ_2 be the congruences corresponding to V_1 and V_2 respectively. Now we define an equivalence in $\mathbf{Z}$ as follows: $\langle a, b, c\rangle \approx \langle a', b', c'\rangle$ iff either $a = 0$ and $\langle a, b, c\rangle\, \theta_1\restriction_{\mathbf{Z}} \langle a', b', c'\rangle$ or $a \neq 0$ and $\langle a, b, c\rangle\, \theta_2\restriction_{\mathbf{Z}} \langle a', b', c'\rangle$. Then it can be easily shown that $\approx$ is an ℓ-monoid congruence but there is no corresponding convex subgroup in $\mathbf{G_Z}$.

Let $\mathbf{L}$ be a ΠMTL-chain, $\theta \in$ Con $\mathbf{G_L}$, and V be its corresponding convex subgroup. In order to make the notation more transparent, we will use the expression $\mathbf{L}'_0/V$ instead of $\mathbf{L}'_0/\theta\restriction_{\mathbf{L}'_0}$. Also the equivalence class $[x]_{\theta\restriction_{\mathbf{L}'_0}}$ will be denoted by $[x]'_V$. Thus we are also able to distinguish between an element of $\mathbf{G_L}/V$ (it is an equivalence class $[x]_V$) and an element of $\mathbf{L}'_0/V$ (it is an equivalence class $[x]'_V$).

Lemma 4.3.7 *Let* $\mathbf{L}$ *be a complete subdirectly irreducible* ΠMTL-*chain and* V *be a nontrivial convex subgroup of* $\mathbf{G_L}$. *Then the elements of* $\mathbf{L}'_0/V$ *(i.e., the equivalence classes w.r.t. the congruence determined by* V*) are left-open and right-closed intervals.*

PROOF: Let $[x]'_V \in \mathbf{L}'_0/V$ be an equivalence class, $a, b \in [x]'_V$, and $c \in L_0$ such that $a < c < b$. Then $a * b^{-1} \in V$. Since $a * b^{-1} < c * b^{-1} < 1$, $c * b^{-1} \in V$. Thus $c \in [x]'_V$.

Since $\mathbf{L}$ is complete, there must be a supremum of $[x]'_V$. Let us denote it by m. The supremum m is the maximum of $[x]'_V$ as well. Indeed, $V \supseteq \overline{F_\Delta}$ by Lemma 4.3.2. Suppose that $m \notin [x]'_V$. Then $m * s \notin [x]'_V$ for any $s \in F_\Delta \subseteq V$. But $m * s < m$ (a contradiction with the fact that m is supremum).

Finally, there is no minimum since for any $a \in [x]'_V$ and $s \in F_\Delta - \{\mathbf{1}\}$ we have $a * s < a$ and $a * s \in [x]'_V$. □

Theorem 4.3.8 *Let* $\mathbf{L}$ *be a* ΠMTL*-chain and* $F^b \in \mathcal{P}$ *be a nontrivial principal filter. Then* F^b/V_b *is isomorphic (as an o-monoid) to a submonoid of* V^b/V_b*.*

PROOF: Since F^b is a subalgebra of ℓ-monoid reduct of V^b, we get by Third Isomorphism Theorem (see [5, Chapter 2, Theorem 6.18]) that F^b/V_b is isomorphic (as an o-monoid) to V^b_θ/V_b where $V^b_\theta = \{a \in V^b \mid F^b \cap [a]_{V_b} \neq \emptyset\}$. The isomorphism assigns to $[x]'_{V_b}$ the equivalence class $[x]_{V_b}$. Since V^b_θ/V_b is a submonoid of V^b/V_b by [5, Chapter 2, Lemma 6.17], we are done. □

Corollary 4.3.9 *Let* $\mathbf{L}$ *be a* ΠMTL*-chain and* $F^b \in \mathcal{P}$ *be a nontrivial principal filter. Then* F^b/V_b *is isomorphic (as an o-monoid) to a submonoid of* $(\mathbb{R}^-, +, \leq, 0)$*.*

PROOF: From Theorem 4.3.8 it follows that F^b/V_b is isomorphic (as an o-monoid) to a submonoid of V^b/V_b. By Hölder's Theorem we get an isomorphism Φ between V^b/V_b and an additive submonoid of real numbers.

Since $x \leq \mathbf{1}$ for any $x \in F^b$, we obtain $\Phi(x) \leq 0$. Thus F^b/V_b is isomorphic to a submonoid of $(\mathbb{R}^-, +, \leq, 0)$. □

Hahn's Embedding Theorem

Firstly, we recall what is a full Hahn group. Let Γ be a totally ordered set. Let us denote the set of all functions $f : \Gamma \to \mathbb{R}$ such that $\mathrm{supp}\, f = \{\gamma \in \Gamma \mid f(\gamma) \neq 0\}$ is inversely well-ordered (i.e., each subset of supp f has a maximum) by $\mathbf{V}(\Gamma)$. The set $\mathbf{V}(\Gamma)$ forms an o-group under addition $(f + g)(\gamma) = f(\gamma) + g(\gamma)$ and $f > 0$ provided that $f(\max(\mathrm{supp}\, f)) > 0$. Such a group is called a *full Hahn group*.

Secondly, let us recall that $\mathbf{G_L}$ can be extended to a divisible o-group $\hat{\mathbf{G}}_\mathbf{L}$, which is unique up to an isomorphism, such that Con $\mathbf{G_L} \cong$ Con $\hat{\mathbf{G}}_\mathbf{L}$. By divisible we mean here that for any element $g \in \hat{\mathbf{G}}_\mathbf{L}$ and any $n \in \mathbb{N}$ there is an element f such that $f^n = g$. Moreover, each convex subgroup V of $\mathbf{G_L}$ generates a convex subgroup $\hat{V}$ of $\hat{\mathbf{G}}_\mathbf{L}$ such that $\hat{V} \cap G_L = V$ (for details see [14, Chapter 4, Lemma A]). Thus we can assume for our purposes that $\mathbf{G_L}$ is already divisible. Consequently, $\mathbf{G_L}$ and all its convex subgroups can be viewed as vector spaces over rationals.

Let $\mathbf{L}$ be a ΠMTL-chain and $\Gamma(G_L)$ be the chain of all values of $\mathbf{G_L}$. The o-group $\mathbf{G_L}$ can be embedded into the full Hahn group $\mathbf{V}(\Gamma(G_L))$ by Hahn's

Embedding Theorem (see [15, Theorem 4.C] or [14, Chapter 4, Theorem 16]). Thus we obtain the following result.

Theorem 4.3.10 *Let* $\mathbf{L}$ *be a* ΠMTL*-chain and* $\Gamma(G_L)$ *be the chain of all values of* $\mathbf{G_L}$*. Then* $\mathbf{L}_0'$ *can be embedded (as an ordered monoid) into* $\mathbf{V}(\Gamma(G_L))$*.*

Let $V_\gamma \in \Gamma(G_L)$. We identify γ with V_γ and denote the successor of V_γ by V^γ. The embedding from Hahn's Embedding Theorem assigns to each $g \in G_L$ the function $\hat{g}$ defined as follows: $\hat{g}(\gamma) = \rho_\gamma(\pi_\gamma(g)) \in \mathbb{R}$ for each $\gamma \in \Gamma(G_L)$, where π_γ is the projection of the vector space G_L onto the subspace V^γ and ρ_γ is an order-preserving homomorphism from V^γ into $\mathbb{R}$ whose kernel is V_γ. Such a homomorphism exists by Hölder's Theorem (see Corollary 4.3.5).

Lemma 4.3.11 *Let* $\mathbf{L}$ *be a* ΠMTL*-chain,* $F \in \mathcal{P}$ *be a nontrivial principal filter,* $V^\gamma = \overline{F}$*, and* $g \in F$*. Then the corresponding function* $\hat{g}$ *from Hahn's Embedding Theorem maps all* $\alpha > \gamma$ *to* 0*.*

PROOF: As F is principal, V^γ is a principal as well by Lemma 4.3.4. Thus there is a value V_γ which is the predecessor of V^γ and $\gamma \in \Gamma(G_L)$. Since $\alpha > \gamma$, we get $V_\alpha \supseteq V^\gamma \supseteq F$. It follows that $g \in V^\alpha$ and also $g \in V_\alpha$. Thus $g = \pi_\alpha(g)$ and $\rho_\alpha(g) = 0$ as g belongs to the kernel V_α. □

At this point we know that each ΠMTL-chain $\mathbf{L}$ can be embedded into a full Hahn group $\mathbf{V}(\Gamma(G_L))$. Now we are going to describe which functions from $\mathbf{V}(\Gamma(G_L))$ correspond to the original elements from L. Firstly, we will prove several useful results about i.w.o. additive submonoids of negative reals.

4.4 Additive submonoids of $\mathbb{R}^-$

Let A, B be totally ordered sets. We denote by $A \times B$ the cartesian product of A, B endowed with the cartesian order, i.e., $(a,b) \leq_c (c,d)$ iff $a \leq c$ and $b \leq d$.

Lemma 4.4.1 *Let* A, B *be i.w.o. sets and* C *be a totally ordered set such that there is a surjective order-preserving mapping* $\phi : A \times B \to C$*. Then* C *is i.w.o.*

PROOF: Suppose that C is not. Let $S \subseteq C$ such that S has no maximum. We show that $\phi^{-1}(S)$ must contain infinitely many maximal elements. Let π_1, π_2 be the projections from $A \times B$. Let us denote $a_0 = \max \pi_1(\phi^{-1}(S))$ and $b_0 = \max(\{a_0\} \times B) \cap \phi^{-1}(S)$. The element (a_0, b_0) is a maximal element. Let M be the set of all maximal elements of $\phi^{-1}(S)$. Suppose that M is finite, then $\phi(M)$

is also finite and one of its elements must be the maximum of S. Thus M must be infinite.

Finally, we will prove that $\phi^{-1}(S)$ cannot posses infinitely many maximal elements which gets a contradiction. As M is infinite, one of the projections to A or B must be also infinite. Without any loss of generality suppose that there is a strictly decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$, $a_n \in \pi_1(M)$, such that $(a_n, b_n) \in \phi^{-1}(S)$. Since all (a_n, b_n) are incomparable, we get that $a_1 > a_2 > a_3 \ldots$ and $b_1 < b_2 < b_3 \ldots$. Thus $\langle b_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence but it is a contradiction since $\{b_n\}_{n\in\mathbb{N}}$ is i.w.o. and has a maximum. □

Throughout this section, let $\mathbf{R} = (R, +, \leq, 0)$ be an i.w.o. submonoid of the additive monoid of negative reals $(\mathbb{R}^-, +, \leq, 0)$. Since $R \subseteq \mathbb{R}$ is i.w.o., R must be countable. Note that by Lemma 4.4.1, if A is a subset of R then $A + A = \{a + b \mid a, b \in A\}$ is also i.w.o. since $+$ is a surjective order-preserving mapping from $A \times A$ onto $A + A$. Indeed, $(a, b) \leq_c (c, d)$ implies $a + b \leq c + d$.

Let us introduce the set of all solutions of the equation $c = a + b$ for given $c \in R$ and $a, b \neq 0$.

$$\mathcal{T}_c = \{(a, b) \in R^2 \mid a + b = c,\ a, b \neq 0\}\,.$$

Observe that if $(a, b) \in \mathcal{T}_c$ then $a, b > c$.

Lemma 4.4.2 *There is a unique minimal set of generators G of* $\mathbf{R}$.

PROOF: Let G be the following set:

$$G = \{g \in R \mid \mathcal{T}_g = \emptyset\}\,.$$

It is obvious that each set of generators of $\mathbf{R}$ must contain G because the elements of G cannot be expressed as a sum of other elements from R. Thus it is sufficient to prove that G generates $\mathbf{R}$. Let $c \in R$. We will show that c can be generated from G. Either $c \in G$ and we are done or $c = a + b$ for some $(a, b) \in \mathcal{T}_c$. Now a belongs either to G or can be written as a sum of greater elements. Similarly for b. In this way we obtain a binary tree. Since each branch of this tree is strictly increasing and R is i.w.o., each branch must be finite. Thus the tree is finite and the leafs belong to G. Let us denote the leafs by $g_1, \ldots, g_n$. Then $c = \sum_{i=1}^{n} g_i$ and the proof is done. □

Lemma 4.4.3 *Let $c \in R$. Then the set $\mathcal{T}_c$ is finite.*

PROOF: Suppose that there are infinitely many solutions. Then we can select a sequence of solutions $a_n + b_n = c$ such that $a_m < a_n$ for $m > n$ and $a_n \geq b_n$ for all n. Since $a_m < a_n$ for $m > n$ and $a_n + b_n = a_m + b_m$, we get $0 < a_n - a_m = b_m - b_n$. Thus $b_m > b_n$.

Now, as $\langle a_n \rangle_{n\in\mathbb{N}}$ is strictly decreasing sequence, we obtain that $\langle b_n \rangle_{n\in\mathbb{N}}$ is strictly increasing. Hence $\{b_n\}_{n\in\mathbb{N}}$ has no maximum, a contradiction with the fact that R is i.w.o. □

Lemma 4.4.4 *Let G be an i.w.o. set of negative reals and $\mathbf{R}_G$ be the submonoid generated by G. Then there is an $n \in \mathbb{N}$ such that whenever $c \in R_G$ can be expressed as $c = \sum_{i=1}^{k} a_i$ ($a_i \neq 0$, $a_i \in R_G$), we have $k < n$.*

PROOF: Let $g = \max G - \{0\}$. Since $\mathbf{R}_G$ is submonoid of $(\mathbb{R}^-, +, 0)$, there exists $n \in \mathbb{N}$ such that $ng < c$. As $\sum_{i=1}^{k} a_i \leq ng < c$ for all $k \geq n$, we are done. □

Lemma 4.4.5 *Let G be an i.w.o. set of negative reals. Then the submonoid $\mathbf{R}_G$ of $(\mathbb{R}^-, +, 0)$ generated by G is i.w.o.*

PROOF: Let M be a subset of R_G and $c \in M$. Suppose that M has no maximum. By Lemma 4.4.4 there is $n \in \mathbb{N}$ such that if $c = \sum_{i=1}^{k} a_i$, $a_i \neq 0$, then $k < n$. It is enough to consider only the sums of generators with less than n summands since all longer sums are surely less than c. But all such elements belongs to $G + G + \cdots + G$ (n-times) which is i.w.o. by Lemma 4.4.1. □

Thanks to Lemma 4.4.4, we can introduce the following notion.

Definition 4.4.6 Let $c \in R$. For $c < 0$, we say that $\varrho(c)$ is the *rank* of c if $\varrho(c)$ is the maximal natural number such that $c = \sum_{i=1}^{\varrho(c)} a_i$ for some $a_i \neq 0$. For $c = 0$, we define $\varrho(0) = 0$.

Notice that if g belongs to the minimal set of generators of R, then $\varrho(g) = 1$.

Since we use the notion of the rank in a subsequent proof by induction, we need the following lemma.

Lemma 4.4.7 *Let $c \in R$ and $(a, b) \in \mathcal{T}_c$. Then $\varrho(c) > \varrho(a), \varrho(b)$.*

PROOF: There is a solution such that $a = \sum_{i=1}^{\varrho(a)} a_i$. We get

$$c = a + b = \left(\sum_{i=1}^{\varrho(a)} a_i \right) + b\,.$$

Thus $\varrho(c) > \varrho(a)$. The case for $\varrho(b)$ is analogous. □

4.5 Construction method

Now we are going to present a method how to construct a standard subdirectly irreducible ΠMTL-chain and in the subsequent section we will show that each standard subdirectly irreducible ΠMTL-chain can be constructed by this method. Let Γ be a countable totally ordered set with minimal element $\gamma_0 \in \Gamma$ and $\mathbf{V}(\Gamma)$ be the full Hahn group. We define also an addition for subsets of $\mathbf{V}(\Gamma)$. Let $A, B \subseteq \mathbf{V}(\Gamma)$. Then

$$A + B = \{f + g \in V(\Gamma) \mid f \in A,\ g \in B\}\,.$$

Note that if A and B are i.w.o. then $A + B$ is i.w.o. by Lemma 4.4.1.

Let us take an arbitrary at most countable well-ordered subset $C \subseteq \Gamma$ such that $\gamma_0 \in C$. There exists an ordinal τ such that the elements of C can be indexed by all ordinals $\alpha < \tau$. Let us assign to each $0 < \alpha < \tau$, an i.w.o. submonoid $\mathbf{R}_\alpha = (R_\alpha, +, \leq, 0)$ of $(\mathbb{R}^-, +, \leq, 0)$. Let $\mathbf{R}_0 = (\mathbb{R}^-, +, \leq, 0)$.

For $\alpha = 0$ let us define the following set:

$$A_0 = \{f \in V(\Gamma) \mid f(\gamma_0) \leq 0,\ f \restriction (\gamma_0, \rightarrow) = 0\}\,.$$

Observe that A_0 forms a submonoid of $\mathbf{V}(\Gamma)$ isomorphic to $\mathbf{R}_0$. For each $0 < \alpha < \tau$ and $c \in R_\alpha - \{0\}$ choose a set $A_\alpha(c)$ which is an arbitrary countable i.w.o. subset of $\{f \in V(\Gamma) \mid f(\alpha) = c,\ f\restriction (\alpha, \rightarrow) = 0\}$ and $A_\alpha(c) \neq \emptyset$ provided that $\varrho(c) = 1$, i.e., c belongs to the minimal set of generators of $\mathbf{R}_\alpha$ (see Lemma 4.4.2). Furthermore, let us define the following two sets:

$$A_\alpha = \bigcup_{c \in R_\alpha - \{0\}} A_\alpha(c) \text{ for } \alpha > 0\,, \qquad A = \bigcup_{\alpha < \tau} A_\alpha\,.$$

Note that if $f \in A_\alpha$, $g \in A_\beta$, and $\alpha < \beta$, then $g < f$.

Definition 4.5.1 Let $\mathbf{L}_A = (L_A, +, \leq, 0)$ be the submonoid of $\mathbf{V}(\Gamma)$ generated by A, i.e., $\mathbf{L}_A = \mathbf{Sg}(A)$.

In order to investigate the structure of $\mathbf{L}_A$ we will need also other subsets of A. Let us define

$$B_0 = \{0\}\,, \qquad B_\alpha = B_0 \cup \bigcup_{0 < \gamma \leq \alpha} A_\gamma\,, \qquad B = \bigcup_{\alpha < \tau} B_\alpha\,.$$

Note that $B = (A - A_0) \cup \{0\}$. Further, observe that B_α contains only functions mapping all $\beta > \alpha$ to 0. Moreover, if $\alpha < \beta$ then $B_\alpha \subseteq B_\beta$.

Lemma 4.5.2 *Let $g \in \mathbf{Sg}(B_\alpha)$ and $f \in \mathbf{Sg}(B)$. Then if $f \geq g$ then $f \in \mathbf{Sg}(B_\alpha)$.*

PROOF: Suppose that $f \notin \mathbf{Sg}(B_\alpha)$. Then $f = \sum_{i=1}^n h_i$ for some $h_i \in B$ and there is at least one j such that $h_j \notin B_\alpha$. Thus h_j belongs to A_β for some $\beta > \alpha$. From the definition of A_β we have $h_j < h$ for any $h \in \mathbf{Sg}(B_\alpha)$ since $h_j(\beta) < 0$ and $h(\beta) = 0$. As $f = \sum_{i=1}^n h_i \leq h_j$, we get $f < g$ (a contradiction). □

Lemma 4.5.3 *The submonoid $\mathbf{Sg}(B)$ of $\mathbf{V}(\Gamma)$ is countable and i.w.o.*

PROOF: Firstly, let $\alpha > 0$. As R_α is countable and $A_\alpha(c)$ for all $c < 0$ as well, the union A_α is countable. Since C is countable, we get that τ must a countable ordinal. Thus the set B is also countable. Consequently, $\mathbf{Sg}(B)$ is countable because its cardinality is less or equal to the cardinality of the set of all finite sequences of elements from B which is countable.

Secondly, we will show by transfinite induction that $\mathbf{Sg}(B_\alpha)$ is i.w.o. for each $\alpha < \tau$. Clearly $\mathbf{Sg}(B_0)$ is i.w.o. Let us suppose that for all $\gamma < \beta$ we have $\mathbf{Sg}(B_\gamma)$ is i.w.o. We will denote the union $\bigcup_{\gamma<\beta} B_\gamma$ by $B_{<\beta}$. We will show that $\mathbf{Sg}(B_{<\beta})$ is i.w.o. Let M be an arbitrary subset of $\mathbf{Sg}(B_{<\beta})$ and $f \in M$. Thus $f = \sum_{i=1}^n g_i$ for some $g_i \in B_{\gamma_i}$ and $\gamma_i < \beta$. Since B_{γ_i} are ordered by inclusion, there is $j \in \{1, \ldots, n\}$ such that $B_{\gamma_j} \supseteq B_{\gamma_i}$ for all $i = 1, \ldots, n$. Thus f belongs to $\mathbf{Sg}(B_{\gamma_j})$ which is i.w.o. Hence $\max M = \max M \cap \mathbf{Sg}(B_{\gamma_j})$ by Lemma 4.5.2.

Let $c \in R_\beta$. Then we will denote by $S_\beta(c)$ the set of all functions from $\mathbf{Sg}(B_\beta)$ which map β to c, i.e.,

$$S_\beta(c) = \{f \in \mathbf{Sg}(B_\beta) \mid f(\beta) = c\}\,.$$

Clearly, $S_\beta(0) = \mathbf{Sg}(B_{<\beta})$. Thus we have

$$\mathbf{Sg}(B_\beta) = \bigcup_{c \in R_\beta} S_\beta(c)\,.$$

Let $c \in R_\beta - \{0\}$ such that $\varrho(c) = 1$. It can be seen that

$$S_\beta(c) = A_\beta(c) + \mathbf{Sg}\left(B_{<\beta}\right)\,.$$

For $c \in R_\beta - \{0\}$ such that $\varrho(c) > 1$, we get that:

$$S_\beta(c) = \left(A_\beta(c) + \mathbf{Sg}\left(B_{<\beta}\right)\right) \cup O_\beta(c)\,,$$

where

$$O_\beta(c) = \bigcup_{(a,b)\in\mathcal{T}_c} (S_\beta(a) + S_\beta(b)) \ .$$

Now, we prove by induction on the ranks that $S_\beta(c)$ is i.w.o. for all $c \in R_\beta$. By Lemma 4.4.1, we get that $S_\beta(c)$ is i.w.o. for all $c \in R_\beta$ such that $\varrho(c) = 1$. Now let $\varrho(c) = n$. Then $A_\beta(c) + \mathbf{Sg}(B_{<\beta})$ is i.w.o. by Lemma 4.4.1. Let $(a, b) \in \mathcal{T}_c$. Since $\varrho(a), \varrho(b) < \varrho(c)$ by Lemma 4.4.7, we obtain that $S_\beta(a)$ and $S_\beta(b)$ are i.w.o. by induction assumption. Thus $S_\beta(a) + S_\beta(b)$ is i.w.o. for all $(a, b) \in \mathcal{T}_c$ by Lemma 4.4.1. Since $\mathcal{T}_c$ is finite, we obtain that $O_\beta(c)$ is i.w.o. Hence $S_\beta(c)$ is i.w.o.

Let M be an arbitrary subset of $\mathbf{Sg}(B_\beta)$. Clearly, if $\mathbf{Sg}(B_{<\beta}) \cap M \neq \emptyset$, then $\max M = \max \mathbf{Sg}(B_{<\beta}) \cap M$ since $\mathbf{Sg}(B_{<\beta})$ is i.w.o. by induction assumption. Let $\mathbf{Sg}(B_{<\beta}) \cap M = \emptyset$, i.e., for any $f \in M$, we have $f(\beta) < 0$. Since R_β is i.w.o., $c = \max\{f(\beta) \mid f \in M\}$ exists. Since $S_\beta(c)$ is i.w.o., $\max M$ exists and $\max M = \max M \cap S_\beta(c)$. Thus $\mathbf{Sg}(B_\beta)$ is i.w.o.

Finally, let M be an arbitrary subset of $\mathbf{Sg}(B)$ and $f \in M$. Since $f = \sum_{i=1}^{n} g_i$ for some $g_i \in B_\alpha$ and some $\alpha < \tau$, f belongs to $\mathbf{Sg}(B_\alpha)$ which is i.w.o. Thus $\max M = \max M \cap \mathbf{Sg}(B_\alpha)$. □

Let us define the following relation on L_A.

$$f \sim g \text{ iff } -|f - g| \in A_0 \,.$$

It can be easily seen that $\sim$ is an equivalence. We show that each equivalence class w.r.t. $\sim$ has a maximum. Let $f = \sum_{i=1}^{n} g_i$ for some $g_i \in A - \{0\}$. If one of g_i is from A_0 then f cannot be maximum. Without any loss of generality suppose that $g_n \in A_0$. Then $f \sim f'$ for $f' = \sum_{i=1}^{n-1} g_i$ and moreover $f' > f$. Thus if $[f]_\sim$ has a maximum then this maximum belongs to $\mathbf{Sg}(B)$. Since $\mathbf{Sg}(B)$ is i.w.o. by Lemma 4.5.3, $\max\,[f]_\sim = \max\,[f]_\sim \cap \mathbf{Sg}(B)$. Let us denote this maximum by f_m. Each equivalence class $[f]_\sim$ forms an interval. Let $f < g < h$ such that $f \sim h$. Then $f - h < g - h < 0$ and $f - h \in A_0$. Since all function between 0 and $f - h$ belong to A_0, $g - h$ also belongs to A_0. Thus $g \sim h$.

Further, it is obvious that each element g of $[f]_\sim$ can be decompose as $g = f_m + z$ for some $z \in A_0$. Indeed, since $-|g - f_m| = g - f_m \in A_0$, we can set $z = g - f_m$.

Let us define a structure $\mathbf{L}_A = (L_A, +, \to, \leq, 0)$ where $(L_A, +, \leq, 0)$ is the ordered submonoid of $\mathbf{V}(\Gamma)$ and the operation $\to$ is defined as follows:

$$f \to g = \max\{h \in L_A \mid f + h \leq g\} \,.$$

We have to prove the existence of $f \to g$. Suppose that $M = \{h \in L_A \mid f+h \leq g\}$ has no maximum for some $f, g \in L_A$. Let us denote the maximum of $M \cap \mathbf{Sg}(B)$ by k. Thus $f + k < g$. Since M has no maximum, there is $z \in L_A - \mathbf{Sg}(B)$ such that $f + z < g$ and $z > k$. According to the latter paragraph, we can write $z = z_m + s$ for some $s \in A_0$. Moreover, $z_m \in \mathbf{Sg}(B)$. Hence $f + z_m > g$. Since $f + z_m + s < g$, we get $f + z_m \sim g$. Let us set $t = g - (f + z_m) \in A_0$. Then $f + z_m + t = g$ and $z_m + t = \max M$.

Lemma 4.5.4 $\mathbf{L}_A$ *is an integral cancellative residuated chain. Moreover,* $\mathbf{L}_A$ *is subdirectly irreducible.*

PROOF: Since $\mathbf{L}_A$ is a submonoid of the group $\mathbf{V}(\Gamma)$, $\mathbf{L}_A$ is obviously cancellative. The definition of $\to$ ensures that $(+, \to)$ is a residuated pair. Moreover, as 0 is the greatest element of L_A, $\mathbf{L}_A$ is integral.

The irreducibility of $\mathbf{L}_A$ follows from the fact that A_0 forms a minimal non-trivial filter. □

The final step in the construction of a general ΠMTL-chain is to add a bottom element. Let us introduce the following set:

$$L'_A = L_A \cup \{-\infty\}\,.$$

Further, we extend the definitions of the operations of $\mathbf{L}_A$. It is done as follows: $a + -\infty = -\infty$, $-\infty \to a = 0$, and $a \to -\infty = -\infty$ for $a > -\infty$. Then $\mathbf{L}'_A = (L'_A, +, \to, \leq, -\infty, 0)$ is a ΠMTL-chain extending $\mathbf{L}_A$ by Lemma 3.2.3.

Let us denote the set of maxima of the equivalence classes w.r.t. $\sim$ by M. Since M is a subset of $\mathbf{Sg}(B)$, we get this set is countable and i.w.o. Thus $M \cup \{-\infty\}$ can be order-embedded into $\mathbb{Q} \cap [0,1]$ by a morphism Φ such that $\Phi(-\infty) = 0$ and $\Phi(0) = 1$. Moreover, each element $f_m \in M$ has a predecessor $\mathrm{pr}(f_m)$. The equivalence class $[f_m]_\sim$ is order-isomorphic to $\mathbb{R}^-$ and can be linearly mapped onto $(\Phi(\mathrm{pr}(f_m)), \Phi(f_m)]$. Thus L'_A is order-isomorphic to $[0,1]$. By means of Φ we can define a ΠMTL-chain in $[0,1]$ and obtain the following theorem.

Theorem 4.5.5 *The* ΠMTL-*chain* $\mathbf{L}'_A = (L'_A, +, \to, \leq, -\infty, 0)$ *is isomorphic to a standard subdirectly irreducible* ΠMTL-*chain.*

4.6 Structural theorem

We are ready to prove that each standard subdirectly irreducible ΠMTL-chain is isomorphic to some $\mathbf{L}'_A$. We start with a standard subdirectly irreducible

ΠMTL-chain $\mathbf{L}$. Let $\Gamma(G_L)$ be the chain of all values of its fraction group $\mathbf{G_L}$. Let $V_\gamma \in \Gamma(G_L)$ then its successor will be denoted by V^γ. The collection of all principal filters of $\mathbf{L}$ will be denoted by $\mathcal{P}$. To each principal filter F there is a corresponding principal convex subgroup $\overline{F}$ by Lemma 4.3.4. Moreover the predecessor of $\overline{F}$ is a value of $\mathbf{G_L}$. Let C be the subset of $\Gamma(G_L)$ of all V_γ such that $V^\gamma = \overline{F}$ for $F \in \mathcal{P}$. The set C is countable and well ordered since Con $\mathbf{L}$ is countable by Lemma 4.2.1 and well ordered by Theorem 4.1.7. Thus the elements of C can be indexed by ordinals $\alpha < \tau$ for some countable ordinal τ. We identify γ with V_γ and denote the minimum of C by γ_0. Clearly, V^{γ_0} corresponds to the minimal nontrivial convex subgroup of $\mathbf{G_L}$, i.e., $V^{\gamma_0} = F_\Delta \cup F_\Delta^{-1}$ by Lemma 4.3.2.

For each $\alpha < \tau$, let $\mathbf{K}_\alpha$ be the submonoid of $(\mathbb{R}, +, \leq, 0)$ which is isomorphic to V^α/V_α (see Corollary 4.3.5). Let us denote the isomorphism from V^α/V_α to $\mathbf{K}_\alpha$ by Ψ. Let $F^\alpha \in \mathcal{P}$ such that $\overline{F^\alpha} = V^\alpha$. By Theorem 4.3.8 there is an embedding ν from F^α/V_α into V^α/V_α such that $\nu([z]'_{V_\alpha}) = [z]_{V_\alpha}$ for $[z]'_{V_\alpha} \in F^\alpha/V_\alpha$. Let $\mathbf{R}_\alpha = \Psi(\nu(F^\alpha/V_\alpha))$ be the submonoid of $\mathbf{K}_\alpha$ which is isomorphic to F^α/V_α.

Lemma 4.6.1 *The monoid $\mathbf{R}_\alpha$ is countable and i.w.o.*

PROOF: Since F^α is an interval of the form $(a, 1]$ by Lemma 4.1.1 and the elements of F^α/V_α are equivalence classes of the form $(b, c]$ by Lemma 4.3.7, there can be only countable many such equivalence classes covering $(a, 1]$. Since the maxima of the equivalence classes of F^α/V_α belong to the set $\{m_x^{F_\Delta} \mid x \in L\}$ which is i.w.o., R_α is i.w.o. as well. □

By Theorem 4.3.10 there is an embedding from $\mathbf{L}'_0$ to $\mathbf{V}(\Gamma(G_L))$. Let us denote it by Φ. Clearly the set $W = \{g \in V(\Gamma(G_L)) \mid g \restriction (\gamma_0, \rightarrow) = 0\}$ forms a convex subgroup of $\mathbf{V}(\Gamma(G_L))$ and W is isomorphic to $(\mathbb{R}, +, \leq, 0)$. Let us define $W^- = \{f \in W \mid f(\gamma_0) \leq 0\}$.

Lemma 4.6.2 *The following equalities hold: $\Phi(\overline{F_\Delta}) = W$ and $\Phi(F_\Delta) = W^-$.*

PROOF: By Corollary 4.3.3, $\overline{F_\Delta}$ is isomorphic to $(\mathbb{R}, +, \leq, 0)$. Let us denote this isomorphism by $\Psi : \overline{F_\Delta} \to \mathbb{R}$. Since $V^{\gamma_0} = \overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$, we have $f \restriction (\gamma_0, \rightarrow) = 0$ for all $f \in \Phi(\overline{F_\Delta})$ by Lemma 4.3.11. Thus $\Phi(\overline{F_\Delta}) \subseteq W$. If we identify the functions from W with the corresponding real numbers, we can write $\Phi(\overline{F_\Delta}) \subseteq \mathbb{R}$. Since Φ is an embedding, $\Phi \circ \Psi^{-1} : \mathbb{R} \to \mathbb{R}$ is an order-preserving automorphism on reals. By Hion's Lemma [15, Lemma 4.1.6], the only order-preserving automorphisms on reals are the multiplications by a positive real number. Thus $\Phi(\overline{F_\Delta}) = W$. Since $\overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$, we obtain $\Phi(F_\Delta) = W^-$. □

Now, let $\alpha > 0$, $c \in R_\alpha - \{0\}$, and $[z]'_{V_\alpha} \in F^\alpha/V_\alpha$ be the equivalence class corresponding to c, i.e., $\Psi(\nu([z]'_{V_\alpha})) = \Psi([z]_{V_\alpha}) = c$. Let us define the set

$$A_\alpha(c) = \Phi([z]'_{V_\alpha} \cap E)\,,$$

where E is the set of product irreducible elements from Lemma 4.2.5. Since $E \subseteq [0,1]$, each $A_\alpha(c)$ is countable and i.w.o. by Lemma 4.2.5. Further, let $g \in A_\alpha(c)$ then $g = \Phi(x)$ for some $x \in [z]'_{V_\alpha} \subseteq F^\alpha$. Thus $g \restriction (\alpha, \rightarrow) = 0$ by Lemma 4.3.11. Further, we have to show that $g(\alpha) = c$. From Hahn's Embedding Theorem we have $g(\alpha) = \rho_\alpha(\pi_\alpha(x))$ where π_α is the projection of x onto V^α and ρ_α is the order-preserving homomorphism from V^α to $\mathbf{K}_\alpha$ whose kernel is V_α. Since $x \in F^\alpha \subseteq V^\alpha$, we get $\pi_\alpha(x) = x$. Clearly, $[z]_{V_\alpha} \in V^\alpha/V_\alpha$ is the equivalence class corresponding to c. As $x \in [z]'_{V_\alpha} \subseteq [z]_{V_\alpha}$, we obtain $\rho_\alpha(x) = c$. Finally, we will show that $\varrho(c) = 1$ implies $A_\alpha(c) \neq \emptyset$. Suppose that $\varrho(c) = 1$ and $[z]'_{V_\alpha} \cap E = \emptyset$. Let m be the maximum of $[z]'_{V_\alpha}$. Then $m \notin E$ and there are $a, b \in L$ such that $a * b = m$ and $a, b \neq m$. Clearly, $a, b > m$. Then $[z]'_{V_\alpha} = [m]'_{V_\alpha} = [a * b]'_{V_\alpha} = [a]'_{V_\alpha} * [b]'_{V_\alpha}$. Since m is the maximum of $[z]'_{V_\alpha}$, we have $[a]'_{V_\alpha}, [b]'_{V_\alpha} > [z]'_{V_\alpha}$. Thus there are $u, v \in R_\alpha$ corresponding to $[a]'_{V_\alpha}, [b]'_{V_\alpha}$ such that $u, v > c$ and $u + v = c$. Obviously $u, v < 0$. Thus $\varrho(c) > 1$ which gets a contradiction.

For $\alpha = 0$, let us set $A_0 = W^-$. Finally, let

$$A_\alpha = \bigcup_{c \in R_\alpha - \{0\}} A_\alpha(c) \text{ for } \alpha > 0\,, \qquad A = \bigcup_{\alpha < \tau} A_\alpha\,.$$

Now we have defined everything that is needed in Definition 4.5.1 of $\mathbf{L}_A$.

Theorem 4.6.3 *The standard subdirectly irreducible* ΠMTL-*chain* $\mathbf{L}$ *is isomorphic (as a* ΠMTL-*chain) to* $\mathbf{L}'_A$.

PROOF: Let Φ be the order-preserving embedding from $\mathbf{L}'_0$ to $\mathbf{V}(\Gamma(G_L))$. Let us define a mapping $\Psi : L \to L'_A$ by $\Psi(0) = -\infty$ and $\Psi(x) = \Phi(x)$ for $x > 0$. We have to show that $\Psi(x) \in L'_A$ for each $x > 0$. Clearly, $\Phi(1) = 0 \in L'_A$. Let $0 < x < 1$. Then $x \in F^x$. Since $\overline{F^x} = V^x$ by Lemma 4.3.4, there is $\alpha < \tau$ such that $\overline{F^x} = V^\alpha$. If $x \in E$ then $\Phi(x) \in A_\alpha(c) \subseteq L'_A$ where $c \in R_\alpha$ is the real number corresponding to the equivalence class $[x]'_{V_\alpha}$. If $x \notin E$ then $x = g_1 * \cdots * g_n * s$ for some $g_i \in E$, $i = 1, \ldots, n$, and $s \in F_\Delta$ by Lemma 4.2.6. Since Φ is embedding, we have $\Phi(x) = \Phi(g_1 * \cdots * g_n * s) = \Phi(g_1) + \cdots + \Phi(g_n) + \Phi(s)$. Further, $\Phi(g_i) \in L'_A$, $i = 1, \ldots, n$ because $g_i \in E$. Since $\Phi(s) \in W^- = A_0 \subseteq L'_A$ by Lemma 4.6.2 and L'_A is closed under $+$, we get $\Phi(x) \in L'_A$. Thus Ψ is an o-monoid embedding from $\mathbf{L} \to \mathbf{L}'_A$.

Now it is sufficient to prove that Ψ is onto. Let $h \in L'_A$ and $h \neq -\infty$. Then $h = \sum_{i=1}^n f_i$ for some $f_i \in A_{\alpha_i}$. By definition of A_{α_i}, there are elements $g_i \in L$

such that $\Psi(g_i) = f_i$. Let us take $x = g_1 * \cdots * g_n$. Since Ψ is embedding, we have $\Psi(x) = \Psi(g_1) + \cdots + \Psi(g_n) = \sum_{i=1}^{n} f_i = h$. Thus Ψ is onto. Since $\mathbf{L}$ and $\mathbf{L}'_A$ are isomorphic as *o*-monoids and the residuum is determined by the monoid operation and the order, Ψ must be also a ΠMTL-isomorphism. □

4.7 Standard subdirectly reducible ΠMTL-chains

In this section we will study the structure of the standard ΠMTL-chains which are not subdirectly irreducible.

Lemma 4.7.1 *Let* $\mathbf{L}$ *be a complete subdirectly reducible* ΠMTL-*chain and* F *be a nontrivial filter. Then* $\mathbf{L}/F$ *is subdirectly irreducible and* Con $\mathbf{L}/F$ *is well ordered.*

PROOF: Since each $F \in \mathcal{F}$, $F \neq \{1\}$, has a successor by Lemma 4.1.2, $\mathbf{L}/F$ is subdirectly irreducible. As $\mathbf{L}/F$ is complete by Lemma 4.1.6, Con $\mathbf{L}/F$ is well ordered by Theorem 4.1.7. □

For a standard subdirectly irreducible ΠMTL-chain $\mathbf{L}$ we have that Con $\mathbf{L}$ is well ordered. For subdirectly reducible we have the following result. Let ω^* be the inversely ordered set of natural numbers.

Theorem 4.7.2 *Let* $\mathbf{L}$ *be a standard subdirectly reducible* ΠMTL-*chain. Then* Con $\mathbf{L} - \{\Delta\}$ *has the same order type as a subset of the lex. product* $\omega^* \times \alpha$ *where* α *is a countable ordinal.*

PROOF: Let θ_n be a strictly decreasing sequence such that $\bigcap_{n\in\mathbb{N}} \theta_n = \Delta$. The sequence has obviously order type ω^*. Since Con $\mathbf{L}/\theta_n \cong [\theta_n, \nabla]$ is well ordered for all $n \in \mathbb{N}$ by Lemma 4.7.1, we get that $(\theta_{n+1}, \theta_n] \subseteq [\theta_{n+1}, \nabla]$ is well ordered. Moreover, Con $\mathbf{L}/\theta_n$ is countable by the same reasoning as in the proof of Lemma 4.2.1. Thus $(\theta_{n+1}, \theta_n]$ are countable as well. Let α be the supremum of all ordinals order-isomorphic to the intervals $(\theta_{n+1}, \theta_n]$. Since all such intervals are countable, α is countable as well. Thus the proof is done. □

If $\mathbf{L}$ is not subdirectly irreducible then the fraction group $\mathbf{G_L}$ has the same property.

Lemma 4.7.3 *Let* **L** *be a standard subdirectly reducible* ΠMTL-*chain and* Δ *be the minimum of* Con $\mathbf{G_L}$. *Then*

$$\bigcap_{F\in\mathcal{F},\ F\neq\{1\}} \overline{F} = \Delta\,.$$

PROOF: Suppose that there is a nontrivial convex subgroup V such that $V \subseteq \overline{F}$ for all $F \in \mathcal{F}$, $F \neq \{1\}$. Thus there is an element $a/b \in V$ such that $a < b$ and $a/b > x$ for all $x \in L - \{1\}$. Thus we get $b * x < b * (a/b) = a$. Let $\langle r_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $\bigvee r_n = 1$. Since we have $b * r_n < a$ for all $n \in \mathbb{N}$, we obtain

$$b = b * \bigvee r_n = \bigvee (b * r_n) \leq a\,,$$

a contradiction with the fact that $a < b$. □

By Lemma 4.7.3 there is no minimal nontrivial congruence in Con $\mathbf{G_L}$. Thus $\mathbf{G_L}$ cannot be subdirectly irreducible.

Theorem 4.7.4 *Let* **L** *be a standard* ΠMTL-*chain which is subdirectly reducible. Then* $\mathbf{G_L}$ *is subdirectly reducible as well.*

If **L** is subdirectly irreducible, we showed that the set of all maxima of the equivalence classes w.r.t. all nontrivial filters is i.w.o. This is not the case if **L** is subdirectly reducible.

Proposition 4.7.5 *Let* **L** *be a standard subdirectly reducible* ΠMTL-*chain. The set of the maxima of the equivalence classes w.r.t. all nontrivial filters, i.e.,*

$$M = \{m^F_x \in L \mid x \in L,\ F \in \mathcal{F},\ F \neq \{1\}\}$$

is dense in L.

PROOF: Let $m_1, m_2 \in M$ and $m_1 > m_2$. Then $z = m_1 \to m_2$ belongs to the principal filter F^z. Let F_z be the predecessor of F^z. Since **L** is subdirectly reducible, there exists a filter F such that $F \subsetneq F_z$. Let s be an element from $F_z - F$ and $m = \max\ [m_1 * s]_F$. Then $m_1 \to m_1 * s = s \notin F$. Thus $m_1 \notin [m_1 * s]_F$ and consequently $m < m_1$. Since $s > z$, we have $m_1 * s > m_2$. Thus $m > m_2$ and the proof is done. □

Let **L** be a standard subdirectly reducible ΠMTL-chain. It is well-known fact from Universal Algebra that if we have an indexed family $\theta_i \in$ Con **L** such that $\bigcap_{i\in I} \theta_i = \Delta$. Then the natural homomorphism

$$\nu : \mathbf{L} \to \prod_{i\in I} \mathbf{L}/\theta_i$$

defined by $\nu(x)(i) = [x]_{\theta_i}$ is a subdirect embedding (see [5, Lemma 8.2]). Since there is no minimum in Con $\mathbf{L} - \{\Delta\}$, there must be a decreasing sequence θ_n such that $\bigcap_{n\in\mathbb{N}} \theta_n = \Delta$ and $\theta_n \neq \Delta$. In the following theorem we show that the structure of each factor $\mathbf{L}/\theta_n$ can be described by means of Theorem 4.6.3. Thus we obtain a characterization of the structure of $\mathbf{L}$ since $\mathbf{L}$ can be subdirectly embedded into the direct product $\prod_{n\in\mathbb{N}} \mathbf{L}/\theta_n$.

Theorem 4.7.6 *Let* $\mathbf{L}$ *be a standard* ΠMTL*-chain,* $\theta \in \text{Con } \mathbf{L}$*, and* $\theta \neq \Delta$*. Then* $\mathbf{L}/\theta$ *is isomorphic to some* $\mathbf{L}'_A/\theta_m$ *where* $\theta_m = \min(\text{Con } \mathbf{L}'_A - \{\Delta\})$*.*

PROOF: By Theorem 4.1.8 we have that $\mathbf{L}/\theta$ is i.w.o. Moreover, it is countable since the elements of $\mathbf{L}/\theta$ are left-open, right-closed intervals covering $[0, 1]$ by Lemma 4.1.3. Thus we can use the same construction as in Section 3.3 and extend $\mathbf{L}/\theta$ to a ΠMTL-chain $\mathbf{D}$ which is order-isomorphic to $[0, 1]$. Further, it is easy to see that

$$F_\Delta = \{\langle \mathbf{1}, r\rangle \in D \mid r \in (0, 1]\}$$

is a minimal nontrivial filter in $\mathbf{D}$. Hence $\mathbf{D}$ is subdirectly irreducible.

Finally, $\mathbf{D}$ is isomorphic to some $\mathbf{L}'_A$ by Theorem 4.6.3. Let θ_m be the congruence determined by F_Δ. Then $\mathbf{L}/\theta \cong \mathbf{D}/\theta_m \cong \mathbf{L}'_A/\theta_m$. □

Note that from the construction in the proof of the Theorem 4.7.6 it follows that if $\mathbf{L}$ is a complete ΠMTL-chain, θ is a nontrivial congruence such that $\mathbf{L}/\theta$ is countable, then $\mathbf{L}/\theta$ can be embedded into a standard ΠMTL-chain.

Chapter 5

Concluding remarks

The thesis hopefully shows that ΠMTL is a fuzzy logic interesting not only for logicians but also for people working on residuated lattices since ΠMTL-algebras are tightly connected to cancellative residuated lattices.

In the first part of the thesis we solved the open problem whether ΠMTL satisfies Standard Completeness. Moreover, the presented construction in the proof gives us a method or at least a hint how we can try to solve similar problems in other logics. For instance, it can be shown that a straightforward modification of this method works also for MTL.

Although the structure of the standard BL-algebras is completely characterized, the structure of the standard MTL-algebras not at all. Moreover, this task seems to be very difficult. Since we described the structure of standard ΠMTL-algebras, we made a first step towards a solution of this task. Furthermore, as the monoidal operation in a standard ΠMTL-algebra is in fact a left-continuous t-norm, the characterization of the standard ΠMTL-algebras also sheds some light on the structure of cancellative left-continuous t-norms.

Our future aim is to use these results and try to solve some complexity questions concerning ΠMTL (for example, it is still not known whether ΠMTL is decidable). Another possible future task is to characterize subvarieties of the variety of ΠMTL-algebras.

5.1 List of contributions

Now we are going to summarize the achieved results. The particular contributions of the thesis are the following.

1. We solved the open problem mentioned in the previous section and proved Standard Completeness Theorem for ΠMTL.

2. From the construction in the proof of Standard Completeness Theorem we got even more and showed that ΠMTL is complete w.r.t. the class of standard ΠMTL-algebras with finitely many Archimedean classes or equivalently with finite congruence lattices.

3. We also extend Standard Completeness Theorem for finite theories, i.e., we proved that if T is a finite theory over ΠMTL and φ is a formula then T proves φ if and only if $e(\varphi) = \mathbf{1}$ for each standard ΠMTL-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of T.

4. From the construction in the proof of Standard Completeness Theorem, we obtained that the variety of ΠMTL-algebras is generated by ΠMTL-chains whose monoid reducts are finitely generated. Moreover, we described the order type of such ΠMTL-chains.

5. The structure of standard subdirectly irreducible ΠMTL-algebras was characterized up to an isomorphism.

6. Finally, we also described the structure of standard ΠMTL-algebras which are not subdirectly irreducible. Each such ΠMTL-algebra is isomorphic to a subdirect product of subdirectly irreducible ΠMTL-chains. Since we show that the structure of each ΠMTL-chain appearing in the subdirect product can be described in the same way as the structure of a standard ΠMTL-algebra, we obtain also a structural theorem for standard subdirectly reducible ΠMTL-algebras.

Bibliography

[1] P. Aglianò, F. Montagna: *Varieties of BL-algebras I: General Properties.* J. Pure Appl. Algebra, 181:105–129, 2003.

[2] P. Bahls, J. Cole, N. Galatos, P. Jipsen, C. Tsinakis: *Cancellative Residuated Lattices.* Algebra Universalis 50(1):83-106, 2003.

[3] G. Birkhoff: *Lattice Theory.* Amer. Math. Soc. Colloquium Publications (3rd edition), 1995.

[4] K. Blount, C. Tsinakis: *The Structure of Residuated Lattices.* Internat. J. Algebra Comput. 13(4):437–461, 2003.

[5] S. Burris, H. P. Sankappanavar: *A Course in Universal Algebra.* Graduate Texts in Mathematics 78, Springer-Verlag, 1981.

[6] C. C. Chang: *A New Proof of the Completeness of the Łukasiewicz Axioms.* Trans. AMS, 93:74–90, 1959.

[7] R. Cignoli, I. M. L. D'Ottaviano, D. Mundici: *Algebraic Foundations of Many-Valued Reasoning.* Kluwer, Dordrecht, 1999.

[8] D. Cox, J. Little, D. O'Shea: *Ideals, Varieties, and Algorithms.* Springer (2nd edition), 1996.

[9] M. Dummett: *A Propositional Calculus with Denumerable Matrix.* J. Symb. Logic 24:97–106, 1959.

[10] R. Cignoli, F. Esteva, L. Godo, A. Torrens: *Basic Fuzzy Logic is the Logic of Continuous T-norms and Their Residua.* Soft Computing 4:106–112, 2000.

[11] F. Esteva, G. Godo: *Monoidal T-norm Based Logic: Towards a Logic for Left-continuous T-norms.* Fuzzy Sets and Systems 124(3):271–288, 2001.

[12] F. Esteva, G. Godo, P. Hájek: *A Complete Many-valued Logic with Product Conjunction.* Archive for Mathematical Logic, 35:191–208, 1996.

[13] F. Esteva, J. Gispert, G. Godo, F. Montagna: *On the Standard Completeness of some Axiomatic Extensions of the Monoidal T-norm Logic.* Studia Logica 71(2):199–226, 2002.

[14] L. Fuchs: *Partially Ordered Algebraic Systems.* Pergamon Press, Oxford, 1963.

[15] A. M. W. Glass: *Partially Ordered Groups.* Series in Algebra, Vol. 7, World Scientific, Singapore, 1999.

[16] K. Gödel: *Zum Intuizionistischen Aussagenkalkül.* Anzeiger Akademie der Wissenschaften Wien, 69:65–66, 1932.

[17] P. Hájek: *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.

[18] P. Hájek: *Basic Fuzzy Logic and BL-algebras.* Soft Computing 2:124–128, 1998.

[19] P. Hájek: *Observations on the Monoidal T-norm Logic.* Fuzzy Sets and Systems 132:107–112, 2002.

[20] J. Hart, L. Rafter, C. Tsinakis: *The Structure of Commutative Residuated Lattices.* Internat. J. Algebra Comput. 12(4):509-524, 2002.

[21] U. Höhle: *Commutative, Residuated ℓ-monoids.* In Non–Classical Logics and Their Applications to Fuzzy Subsets, U. Höhle, E.P. Klement (eds.), Kluwer Academic Publisher, Dordrecht, pp. 53–106, 1995.

[22] R. Horčík: *Standard Completeness Theorem for* ΠMTL. Archive for Mathematical Logic (accepted).

[23] R. Horčík: *Structure of Commutative Cancellative Residuated Lattices in* $(0, 1]$. Algebra Universalis (submitted).

[24] R. Horčík, P. Cintula: *Product Łukasiewicz Logic.* Archive for Mathematical Logic, 43(4):477–503, May 2004.

[25] R. Horčík: *Residuated Fuzzy Logics with Additional Connectives and Their Validation Sets.* Fuzzy Sets and Systems, 143(1):75–87, April 2004.

[26] R. Horčík, M. Navara: *Validation Sets in Fuzzy Logics.* Kybernetika, 38(3):319–326, June 2002.

[27] K. Hrbacek, T. Jech: *Introduction to Set Theory.* Monographs and Textbooks in Pure and Applied Mathematics, 3^{rd} ed., Marcel Dekker, New York, 1999.

[28] S. Jenei, F. Montagna: *A Proof of Standard Completeness for Esteva and Godo's Logic MTL.* Studia Logica 70(2):183–192, 2002.

[29] S. Jenei, F. Montagna: *A General Method for Constructing Left-Continuous T-norms.* Fuzzy Sets and Systems 136(3):263–282, 2003

[30] S. Jenei, F. Montagna: *On the Continuity Points of Left-continuous T-norms.* Archive for Mathematical Logic 42:797–810, 2003.

[31] S. Jenei: *How to Construct Left-continuous Triangular Norms—State of the Art.* Fuzzy Sets and Systems 143(1):27–45, 2004.

[32] P. Jipsen, C. Tsinakis: *A Survey of Residuated Lattices.* Ordered Algebraic Structures (J. Martinez, editor), Kluwer Academic Publishers, Dordrecht, 19–56, 2002.

[33] E. P. Klement, R. Mesiar, E. Pap: *Triangular Norms.* Trends in Logic, Volume 8. Kluwer, Dordrecht, 2000.

[34] J. Łukasiewicz: *O logice trojwartosciowej (On Three-valued Logic).* Ruch filozoficzny 5:170-171, 1920.

[35] J. Łukasiewicz. *Bemerkungen zu mehrwertigen Systemen des Aussagenkalkül.* Comptes Rendus Séances Société des Sciences et Lettres Varsovie cl. III, 23:51–77, 1930.

[36] P. S. Mostert, A. L. Shields: *On the Structure of Semigroups on a Compact Manifold with Boundary.* Annals of Math., 65:117–143, 1957.

[37] M. Ward, R. P. Dilworth: *Residuated Lattices.* Transactions of the AMS 45:335-354, 1939.

[38] L. A. Zadeh: *Fuzzy Sets.* Inform. Control, 8:338–353, 1965.