# STRUCTURE OF COMMUTATIVE CANCELLATIVE INTEGRAL RESIDUATED LATTICES ON $(\mathbf{0}, \mathbf{1}]$

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Abstract. ΠMTL-algebras were introduced as an algebraic counterpart of the cancellative extension of monoidal t-norm based logic. It was shown that they form a variety generated by ΠMTL-chains on the real interval $[0, 1]$. In this paper the structure of these generators is investigated. The results illuminate the structure of cancellative integral commutative residuated chains, because every such algebra belongs to the quasivariety generated by the zero-free subreducts on $(0, 1]$ of all ΠMTL-chains on $[0, 1]$.

It turns out that for many logical calculi (e.g. fuzzy logics or substructural logics) the corresponding algebras of truth values form a residuated lattice, i.e., a lattice ordered monoid endowed with a residuum. Such algebras were first introduced by Ward and Dilworth in [23] as a generalization of ideal lattices of rings. Recently, the structure of residuated lattices was studied in the following papers [2, 5, 13, 20]. In this paper, we will concentrate on commutative cancellative residuated lattices because of the reasons which follow.

In [7], Esteva and Godo introduced so-called monoidal t-norm based logic (MTL). This logic is algebraizable in the sense of Blok and Pigozzi (see [4]) and its equivalent algebraic semantics is the class of MTL-algebras. MTL-algebras form a variety generated by bounded, commutative, integral residuated chains. Properties of this variety and some of its subvarieties were investigated in [8, 12, 14, 16]. One of the subvarieties is the class of ΠMTL-algebras introduced by Hájek in [12] and studied in [8, 14]. It was shown in [14] that this variety is generated by the members whose lattice reduct is the lattice $([0, 1], \min, \max)$ on the real unit interval $[0, 1]$. Such members are called standard ΠMTL-chains. Thus, more knowledge of their structure is desirable, and this is the topic of the present paper. It can be shown that each standard ΠMTL-chain is isomorphic to the ordinal sum of the two-element Boolean algebra $\mathbf{2}$ and a commutative, cancellative, integral residuated lattice on $(0, 1]$. Thus if we want to characterize the structure of standard ΠMTL-chains, it is sufficient to investigate the structure of commutative, cancellative, integral residuated lattices on $(0, 1]$.

The second reason is that the monoidal operation of a standard ΠMTL-chain is a left-continuous cancellative t-norm (for the definition of a t-norm see [21]). While the class of continuous t-norms is completely characterized, we have only a few results about the class of left-continuous t-norms (see [18]). So far mainly construction methods for left-continuous t-norms were published e.g. [17, 19]. In this paper, we partially contribute to this characterization task. As we give an algebraic characterization of the standard ΠMTL-chains, we also shed some light on the structure of cancellative left-continuous t-norms.

We mainly focus on the structure of standard subdirectly irreducible ΠMTL-chains since they are the generators of the variety of ΠMTL-algebras. However, we will also deal with the general ΠMTL-chains. Let $\mathbf{L}$ be a standard subdirectly irreducible ΠMTL-chain and $\mathbf{C}$ the corresponding commutative, cancellative, integral residuated chain on $(0, 1]$ such that the ordinal sum of $\mathbf{2}$ and $\mathbf{C}$ is isomorphic to $\mathbf{L}$. We show in Section 5 that it is possible to embed the $\ell$-monoidal reduct of $\mathbf{C}$ into a totally ordered Abelian group $\mathbf{G_C}$ by forming fractions in the same way in which the integers are constructed from the natural numbers. Then we use Hahn's Embedding Theorem (see [9, 10]) and embed $\mathbf{G_C}$ into the full Hahn group $\mathbf{V}(\Gamma(\mathbf{G_C}))$. We prove that $\mathbf{C}$ is isomorphic to a certain kind of residuated submonoid of $\mathbf{V}(\Gamma(\mathbf{G_C}))$, and that each residuated submonoid

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of $\mathbf{V}(\Gamma(\mathbf{G_C}))$ of this particular kind is isomorphic to a subdirectly irreducible, commutative, cancellative, integral residuated chain on $(0,1]$. In this way we obtain a characterization of the structure up to isomorphism. In particular, in Section 7 we introduce the discussed class of residuated submonoids of $\mathbf{V}(\Gamma(\mathbf{G_C}))$ and in Section 8 we prove that each subdirectly irreducible, commutative, cancellative, integral residuated chain on $(0,1]$ belongs to this class. Finally, in Section 9 we summarize our results on chains which are not subdirectly irreducible.

Throughout the text we use the following notation. The set of non-positive reals (resp. integers) will be denoted by $\mathbb{R}^-$ (resp. $\mathbb{Z}^-$). Further, i.w.o. stands for inversely well ordered, i.e., each non-empty subset has a maximum. We will also use without comment the alternative signature for a lattice using the lattice order $\leq$ instead of $\wedge, \vee$. Let $\mathbf{L}$ be an algebra, $L$ its universe, and $G \subseteq L$. Then the subalgebra generated by $G$ will be denoted by $\mathbf{Sg}(G)$. The congruence lattice of $\mathbf{L}$ will be denoted by $\mathrm{Con}\,\mathbf{L}$ and the minimum and the maximum congruence will be denoted by $\Delta, \nabla$ respectively. Given a monoid $(M, *, \mathbf{1})$ and $a \in M$, the symbol $a^n$ stands for $a * \cdots * a$ ($n$-times). For $n = 0$ we define $a^n = \mathbf{1}$.

## 1. Preliminaries

A *commutative residuated lattice* (CRL) $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is an algebraic structure, where $(L, *, \mathbf{1})$ is a commutative monoid, $(L, \wedge, \vee)$ is a lattice, and $(*, \rightarrow)$ forms a residuated pair, i.e.,

$$x * y \leq z \text{ iff } x \leq y \rightarrow z\,.$$

The operation $\rightarrow$ is called a *residuum*. It follows from the definition that $*$ is order-preserving, i.e., $a \leq b$ implies $a * c \leq b * c$. The residuum is decreasing in the first argument and increasing in the second one. Further, the inequality $a * x \leq b$ has a greatest solution for $x$ (namely $a \rightarrow b$). In particular, the residuum is uniquely determined by $*$ and $\leq$. It is well known that the class $\mathcal{CRL}$ of all commutative residuated lattices forms a variety (see e.g. [20]).

The existence of the residuum has the following consequence which will be useful for us later.

**Proposition 1.1** ([20]). *Let $\mathbf{L} \in \mathcal{CRL}$. The operation $*$ preserves all existing joins in each argument, i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subseteq L$ then $\bigvee_{x \in X, y \in Y} x * y$ exists and*

$$\Big(\bigvee X\Big) * \Big(\bigvee Y\Big) = \bigvee_{x \in X, y \in Y} x * y\,.$$

Now we introduce several well-known subvarieties of residuated lattices.

- A CRL $\mathbf{L}$ is said to be *integral* if $\mathbf{1}$ is the top element of $L$. In this case we have that $x \leq y$ implies $x \rightarrow y = \mathbf{1}$. The class $\mathcal{ICRL}$ of integral CRLs is easily seen to be a variety defined by $x \leq \mathbf{1}$.
- A CRL $\mathbf{L}$ is said to be *cancellative* if for any $x, y, z \in L$, $x * z = y * z$ implies $x = y$. It was shown in [2] that the class $\mathcal{C}an\mathcal{CRL}$ of cancellative CRLs forms a variety defined by $x = y \rightarrow (y * x)$.
- Finally, a totally ordered CRL is referred to as a *commutative residuated chain* (CRC). A residuated lattice is called *representable* if it is a subdirect product of CRCs. This is equivalent to the demand that it satisfies $\mathbf{1} \leq (x \rightarrow y) \vee (y \rightarrow x)$ (see [13]; the additional axiom provided there turns out to be redundant), whence the representable CRLs also form a variety.

The class of all cancellative integral CRCs will be denoted by $\mathcal{C}an\mathcal{ICRC}$ and the variety generated by this class by $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$ (i.e., the variety of representable cancellative integral CRLs). The superscript $\mathcal{C}$ refers to the fact that this variety is generated by chains.

**Definition 1.2.** An algebra $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is called an MTL-*algebra* if it satisfies the following conditions:

(1) $(L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is a representable integral CRL and
(2) $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice.

A totally ordered MTL-algebra is called an MTL-*chain*.

Observe that if a CRL $\mathbf{L}$ possesses a bottom element $\mathbf{0}$ then we have $a * \mathbf{0} = \mathbf{0}$ for any $a \in L$.

MTL-algebras were introduced by Esteva and Godo in [7] as the algebras of truth values for the monoidal t-norm based logic (MTL for short). In this paper we deal with a subvariety of MTL-algebras introduced by Hájek in [12].

**Definition 1.3.** A $\Pi$MTL-*algebra* $\mathbf{L} = (L, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is an MTL-algebra satisfying the following identities:

(1) $\neg\neg z \to [(x * z \to y * z) \to (x \to y)] = \mathbf{1}$,
(2) $x \wedge \neg x = \mathbf{0}$,

where $\neg x = x \to \mathbf{0}$. A totally ordered $\Pi$MTL-algebra is called a $\Pi$MTL-*chain*. The variety of all $\Pi$MTL-algebras will be denoted by $\mathcal{PMTL}$.

Since zero-free reducts of $\Pi$MTL-algebras are integral representable CRLs, the variety $\mathcal{PMTL}$ is generated by the class of all $\Pi$MTL-chains.

**Theorem 1.4** (Subdirect Representation Theorem)**.** *Each* $\Pi$MTL-*algebra is isomorphic to a subdirect product of* $\Pi$MTL-*chains.*

Thanks to the previous theorem, it is sufficient to investigate the structure of $\Pi$MTL-chains, if we want to understand the structure of $\Pi$MTL-algebras. In [12] the following result was proved.

**Lemma 1.5.** *An* MTL-*chain* $\mathbf{L}$ *is a* $\Pi$MTL-*chain if and only if for any* $x, y, z \in L$, $z \neq \mathbf{0}$, *we have* $x * z = y * z$ *implies* $x = y$.

Observe that by Lemma 1.5 we obtain for $a, b, c \in L$, $c \neq \mathbf{0}$, that $a < b$ implies $a * c < b * c$, in particular $a^2 < a$ and $a * b < a$ for $a \neq \mathbf{0}, \mathbf{1}$ and $b < \mathbf{1}$. Furthermore, we get $a * c \to b * c = a \to b$. Moreover, due to Lemma 1.5 it can be shown that there is a connection between cancellative integral CRCs and $\Pi$MTL-chains. First, we need the notion of an ordinal sum. This notion was introduced in [1] for BL-chains and it was generalized to the case of MTL-chains in [22] (where integral CRCs are called totally ordered semihoops).

**Definition 1.6.** Let $\langle I, \leq \rangle$ be a totally ordered set. Let $\{\mathbf{A}_i \mid i \in I\}$ be a family of integral CRCs sharing the same top element, say $\mathbf{1}$, and such that for $i \neq j$, $A_i \cap A_j = \{\mathbf{1}\}$. Then $\bigoplus_{i \in I} \mathbf{A}_i$ (the *ordinal sum* of the family) is the integral CRC whose universe is $\bigcup_{i \in I} A_i$ and whose operations are:

$$x * y = \begin{cases} x *_{A_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{\mathbf{1}\} \text{ with } i > j, \\ x & \text{if } x \in A_i \setminus \{\mathbf{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

$$x \to y = \begin{cases} x \to_{A_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j, \\ \mathbf{1} & \text{if } x \in A_i \setminus \{\mathbf{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

For every $i \in I$, $\mathbf{A}_i$ is called a *component* of the ordinal sum.

If in addition $I$ has a minimum, say $i_0$, and $\mathbf{A}_{i_0}$ is bounded, then the ordinal sum $\bigoplus_{i \in I} \mathbf{A}_i$ forms an MTL-chain.

The following proposition describing $\Pi$MTL-chains by means of an ordinal sum was proved in [22, Proposition 26].

**Proposition 1.7.** *Let* $\mathbf{L}$ *be a nontrivial* $\Pi$MTL-*chain and* $\mathbf{2}$ *the two-element boolean algebra. Then* $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ *where* $\mathbf{C}$ *is the zero-free subreduct of* $\mathbf{L}$ *whose domain is* $L \setminus \{\mathbf{0}\}$.

**Proposition 1.8.** *Let* $\mathcal{K}$ *be the class of zero-free subreducts of* $\Pi$MTL-*chains* $\mathbf{L}$ *whose domain is* $L \setminus \{\mathbf{0}\}$. *Then* $\mathcal{K} = \mathcal{C}an\mathcal{ICRC}$.

*Proof.* Let $\mathbf{C}$ be a zero-free subreduct of a $\Pi$MTL-chain $\mathbf{L}$ such that $C = L \setminus \{\mathbf{0}\}$. Since $x*z = y*z$ implies $x = y$ for any $x, y, z \in L \setminus \{\mathbf{0}\}$ by Lemma 1.5 and $\mathbf{1}$ is the top element, $\mathbf{C}$ forms an integral cancellative CRC.

Conversely, let $\mathbf{C}$ be an integral cancellative CRC and $\mathbf{2}$ the two-element boolean algebra. Then it can be easily verified that $\mathbf{L} = \mathbf{2} \oplus \mathbf{C}$ is a $\Pi$MTL-chain and $\mathbf{C}$ is its zero-free subreduct whose domain is $L \setminus \{\mathbf{0}\}$. □

It follows from Propositions 1.7 and 1.8, that each $\Pi$MTL-chain $\mathbf{L}$ is isomorphic to $\mathbf{2} \oplus \mathbf{C}$ for some integral cancellative CRC $\mathbf{C}$. Thus it is sufficient to investigate the structure of integral cancellative CRCs if we want to characterize the structure of $\Pi$MTL-chains.

In order to study the structure of $\Pi$MTL-chains, we have to work with congruences. Let $\mathbf{L}$ be a CRL. A subalgebra $\mathbf{S}$ of $\mathbf{L}$ is called *convex* if for all $x, y \in S$ and $z \in L$ we have: $x \leq z \leq y$ implies $z \in S$. It is easy to see that the class of all convex subalgebras ordered by set-inclusion forms a lattice.

**Theorem 1.9** ([13])**.** *Let* $\mathbf{L}$ *be a CRL. Then* Con $\mathbf{L}$ *is isomorphic to the lattice of all convex subalgebras of* $\mathbf{L}$*. The isomorphism is established via the assignments* $\theta \mapsto F_\theta$ *and* $F \mapsto \theta_F$*, where*

$$F_\theta = \{a \in L \mid \langle a, \mathbf{1} \rangle \in \theta\},$$

*and*

$$\theta_F = \{\langle a, b \rangle \in L \times L \mid (a \to b) \wedge \mathbf{1} \in F \text{ and } (b \to a) \wedge \mathbf{1} \in F\}.$$

In the case when a bottom element $\mathbf{0}$ is in the signature (in particular if $\mathbf{L}$ is an MTL-algebra) then $F_\theta$ need not be a subalgebra. Nevertheless, it is a subalgebra of the zero-free reduct of $\mathbf{L}$. Thus we will call $F_\theta$ in this case a *filter*. We will use the same terminology also for integral CRLs. In other words:

> When $\mathbf{L}$ is an integral CRL or an MTL-algebra, then the *filters* of $\mathbf{L}$ are just the upward closed submonoids of $(L, *, \leq, \mathbf{1})$.

Let $\mathbf{L}$ be an MTL-algebra or integral CRL. Then the collection of all filters of $\mathbf{L}$ will be denoted by $\mathcal{F}_{\mathbf{L}}$. Let $F \in \mathcal{F}_{\mathbf{L}}$. Then we write simply $\mathbf{L}/F$ instead of $\mathbf{L}/\theta_F$. The equivalence class containing an element $x \in L$ with respect to a filter $F$ will be denoted by $[x]_F = \{a \in L \mid a \; \theta_F \; x\}$. Observe also that if $\mathbf{L}$ is totally ordered then only one of the terms in the definition of $\theta_F$ is important because for all $x, y$ either $x \leq y$ or $y \leq x$, thus either $(x \to y) \wedge \mathbf{1} = x \to y = \mathbf{1}$ or $(y \to x) \wedge \mathbf{1} = y \to x = \mathbf{1}$. Moreover, $\mathcal{F}_{\mathbf{L}}$ forms a chain in this case since filters are upward closed.

**Lemma 1.10.** *Let* $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ *be a* $\Pi$MTL*-chain. Then* Con $\mathbf{C} \cong$ Con $\mathbf{L} \setminus \nabla$*. In particular,* $\mathbf{L}$ *is subdirectly irreducible iff* $\mathbf{C}$ *is.*

**Definition 1.11.** The structure $\mathbf{M} = (M, *, \leq, \mathbf{1})$ is called an $o$-monoid if the following conditions are satisfied:

(1) $(M, *, \mathbf{1})$ is a commutative monoid,
(2) $(M, \leq, \mathbf{1})$ is a chain and $\mathbf{1}$ is its top element,
(3) $x \leq y$ implies $x * z \leq y * z$ for all $x, y, z \in M$.

Clearly, for any integral CRC $\mathbf{L} = (L, *, \to, \leq, \mathbf{1})$ the structure $(L, *, \leq, \mathbf{1})$ is an $o$-monoid. Notice that each $o$-monoid is also an $\ell$-monoid (i.e., a monoid whose universe is lattice ordered and the monoidal operation distributes over the join) because its universe forms a chain.

As will be seen later on, the filters in integral CRCs are also related to so-called Archimedean classes (see [9]).

**Definition 1.12.** Let $\mathbf{M}$ be an $o$-monoid, $a, b$ elements of $M$, and $\sim$ an equivalence on $M$ defined as follows:

$$a \sim b \text{ iff there exists an } n \in \mathbb{N} \text{ such that } a^n \leq b \leq a \text{ or } b^n \leq a \leq b.$$

Then for any $a \in M$ the equivalence class $[a]_\sim$ is called an *Archimedean class.*

Archimedean classes correspond to the subsets of $M$ where the elements behave like in an Archimedean $o$-monoid, i.e., for any pair of elements $x, y \in [a]_\sim$ there is an $n \in \mathbb{N}$ such that $y^n \leq x$.

From Theorem 1.4 it follows that ΠMTL-chains generate the variety of ΠMTL-algebras. We proved (see [14] for details) that it is sufficient to consider only ΠMTL-chains on the real unit interval $[0,1]$. A ΠMTL-chain is called *standard* if its underlying lattice is the chain $([0,1], \min, \max)$ on the real unit interval $[0,1]$. The following theorem justifies the investigation of the structure of standard ΠMTL-chains.

**Theorem 1.13.** *Standard* ΠMTL*-chains with finitely many Archimedean classes generate the variety of* ΠMTL*-algebras as a quasivariety.*

The following corollary and its proof is due to James Raftery.

**Corollary 1.14.** *The variety* $\mathcal{CanICRL^C}$ *is generated as a quasivariety by the class of all cancellative integral CRCs with finitely many Archimedean classes whose underlying lattice is* $((0,1], \min, \max)$.

*Proof.* Let $\Phi$ be a quasi-identity in the language $\{*, \to, \wedge, \vee, \mathbf{1}\}$. We need to show that $\Phi$ holds in $\mathcal{CanICRL^C}$ iff it holds in all cancellative integral CRCs of the form $((0,1], *, \to, \min, \max, 1)$ with finitely many Archimedean classes. The implication from left to right is trivial. Conversely, suppose that $\Phi$ fails in some algebra belonging to $\mathcal{CanICRL^C}$. Then since $\mathcal{CanICRL^C}$ is a variety, there is a subdirectly irreducible algebra $\mathbf{A}$, still belonging to $\mathcal{CanICRL^C}$, such that $\Phi$ fails in $\mathbf{A}$. But a subdirectly irreducible representable CRL is a chain, so $\mathbf{A}$ is a chain. Let $\vec{x} = x_1, \dots, x_n$ be the variables occurring in $\Phi$ and consider the formula

$$[(x_1 \approx x_1 \to (x_1 * x_1)) \ \& \ \cdots \ \& \ (x_n \approx x_n \to (x_n * x_n))] \implies \Phi\,.$$

Clearly, this formula is logically equivalent to a quasi-identity $\overline{\Phi}$, and $\overline{\Phi}$ also fails in $\mathbf{A}$ (because $\mathbf{A}$ is cancellative). Then $\mathbf{2} \oplus \mathbf{A}$ is a ΠMTL-chain. Since $\overline{\Phi}$ fails in $\mathbf{A}$, it also fails in $\mathbf{2} \oplus \mathbf{A}$ (because $\mathbf{A}$ embeds in a zero-free subreduct of $\mathbf{2} \oplus \mathbf{A}$). Then $\overline{\Phi}$ must fail in some standard ΠMTL-chain $\mathbf{B} \cong \mathbf{2} \oplus \mathbf{C}$ with finitely many Archimedean classes by Theorem 1.13. Thus, there exist $\vec{b} = b_1, \dots, b_n \in B$ such that $\overline{\Phi}[\vec{b}]$ is false in $\mathbf{B}$. This means that $b_i = b_i \to (b_i * b_i)$ for all $i$ and $\Phi[\vec{b}]$ is false in $\mathbf{B}$. It follows that $b_i \neq 0$ for all $i$ (because $0 \neq 1 = 0 \to (0 * 0)$). So $\Phi[\vec{b}]$ is also false in the zero-free subreduct $\mathbf{C}$. The subreduct $\mathbf{C}$ belongs to $\mathcal{CanICRC}$ by Proposition 1.8, has finitely many Archimedean classes, and since it does not satisfy $\Phi$, the proof is complete. $\square$

A ΠMTL-algebra $\mathbf{L}$ satisfying moreover the following identity:

$$x * (x \to y) = x \wedge y\,,$$

is called a *product algebra*. A product algebra is called *standard* if its lattice reduct is $([0,1], \min, \max)$. All standard product algebras are isomorphic to the algebra $[\mathbf{0},\mathbf{1}]_\Pi = ([0,1], \cdot, \to_\Pi, \min, \max, 0, 1)$, where $\cdot$ is the ordinary product of reals and $\to_\Pi$ is the corresponding residuum. By Propositions 1.7 and 1.8 we have $[\mathbf{0},\mathbf{1}]_\Pi \cong \mathbf{2} \oplus (\mathbf{0},\mathbf{1}]_\Pi$ where $(\mathbf{0},\mathbf{1}]_\Pi$ is the corresponding integral cancellative CRC. The algebra $(\mathbf{0},\mathbf{1}]_\Pi$ is an example of an Archimedean integral cancellative CRC. Now we show that there are also examples which are not Archimedean. First, we define a lexicographic product of two integral cancellative CRCs.

**Definition 1.15.** Let $\mathbf{A} = (A, *_A, \to_A, \leq_A, \mathbf{1}_A)$ and $\mathbf{B} = (B, *_B, \to_B, \leq_B, \mathbf{1}_B)$ be integral cancellative CRCs. Then the *lexicographic product* of $\mathbf{A}$ and $\mathbf{B}$ is the algebra $\mathbf{A} \overrightarrow{\times} \mathbf{B} = (A \times B, *, \to, \leq, \langle \mathbf{1}_A, \mathbf{1}_B \rangle)$ where $\leq$ is the lexicographic order, i.e., $\langle a, b\rangle \leq \langle c, d\rangle$ iff $a <_A c$ or $[a = c$ and $b \leq_B d]$ and the operations are defined as follows:

$$\langle a, b\rangle * \langle c, d\rangle = \langle a *_A c, b *_B d\rangle\,,$$

$$\langle a, b\rangle \to \langle c, d\rangle = \begin{cases} \langle a \to_A c, \mathbf{1}_B\rangle & \text{if } a * (a \to_A c) <_A c\,, \\ \langle a \to_A c, b \to_B d\rangle & \text{otherwise.} \end{cases}\,.$$

**Proposition 1.16.** *Let* $\mathbf{A} = (A, *_A, \to_A, \leq_A, \mathbf{1}_A)$ *and* $\mathbf{B} = (B, *_B, \to_B, \leq_B, \mathbf{1}_B)$ *be integral cancellative CRCs. Then* $\mathbf{A} \overrightarrow{\times} \mathbf{B} = (A \times B, *, \to, \leq, \langle \mathbf{1}_A, \mathbf{1}_B\rangle)$ *is an integral cancellative CRC.*

*Proof.* It is obvious that $(A\times B,*,\langle\mathbf{1}_A,\mathbf{1}_B\rangle)$ is a commutative cancellative monoid and $(A\times B,\leq)$ is a chain whose top element is $\langle\mathbf{1}_A,\mathbf{1}_B\rangle$. Thus it is sufficient to check that

$$\langle a,b\rangle*\langle c,d\rangle\leq\langle x,y\rangle\quad\text{iff}\quad\langle a,b\rangle\leq\langle c,d\rangle\to\langle x,y\rangle\,.$$

Assume that $\langle a,b\rangle*\langle c,d\rangle\leq\langle x,y\rangle$. Since $\langle a,b\rangle*\langle c,d\rangle=\langle a*_A c,b*_B d\rangle$, we have that either $a*_A c<_A x$ or $[a*_A c=x$ and $b*_B d\leq_B y]$. In any case we have $a\leq_A c\to_A x$. Thus if $\langle c,d\rangle\to\langle x,y\rangle=\langle c\to_A x,\mathbf{1}_B\rangle$ then obviously $\langle a,b\rangle\leq\langle c,d\rangle\to\langle x,y\rangle$. Assume that $\langle c,d\rangle\to\langle x,y\rangle=\langle c\to_A x,d\to_B y\rangle$. Then $c*_A(c\to_A x)=x$ by the definition of $\to$. Now, there are two cases:

(1) Suppose that $a*_A c<_A x$. If $a=c\to_A x$ then $c*_A a=c*_A(c\to_A x)=x$ which is a contradiction with our assumption. Thus $a<c\to_A x$ which implies $\langle a,b\rangle\leq\langle c\to_A x,d\to_B y\rangle$.
(2) Let $a*_A c=x$ and $b*_B d\leq_B y$. Then $b\leq_B d\to_B y$. Thus we obtain $\langle a,b\rangle\leq\langle c\to_A x,d\to_B y\rangle$.

Conversely, assume that $\langle a,b\rangle\leq\langle c,d\rangle\to\langle x,y\rangle$. Then $a\leq_A c\to_A x$. If $\langle c,d\rangle\to\langle x,y\rangle=\langle c\to_A x,\mathbf{1}_B\rangle$ then $c*_A(c\to_A x)<_A x$. Thus $c*_A a\leq c*_A(c\to_A x)<_A x$ showing that $\langle a,b\rangle*\langle c,d\rangle=\langle a*_A c,b*_B d\rangle\leq\langle x,y\rangle$. Now, suppose that $\langle c,d\rangle\to\langle x,y\rangle=\langle c\to_A x,d\to_B y\rangle$. Then $c*_A(c\to_A x)=x$. If $a<_A c\to_A x$ then by cancellativity $c*_A a<c*_A(c\to_A x)=x$ showing that $\langle a*_A c,b*_B d\rangle\leq\langle x,y\rangle$. If $a=c\to_A x$ then $b\leq_B d\to_B y$. Consequently, $c*_A a=c*_A(c\to_A x)=x$ and $b*_B d\leq y$. Hence we have $\langle a*_A c,b*_B d\rangle\leq\langle x,y\rangle$ also in this case. □

**Example 1.17.** Let $\mathbf{R}^-=(\mathbb{R}^-,+,\Rightarrow,\min,\max,0)$ be the cancellative integral CRC arising from the negative cone of the additive group of reals, i.e., $x\Rightarrow y=(y-x)\wedge 0$. Similarly, let $\mathbf{Z}^-=(\mathbb{Z}^-,+,\Rightarrow,\min,\max,0)$ be the cancellative integral CRC arising from the negative cone of the additive group of integers. Then $\mathbf{Z}^-\overrightarrow{\times}\mathbf{R}^-$ is an integral cancellative CRC which is not Archimedean since $\langle -1,0\rangle<\langle 0,-1\rangle^n$ for all $n\in\mathbb{N}$.

## Abelian totally ordered groups

We also recall several facts about Abelian totally ordered groups ($o$-groups) useful in the sequel. For details see [9, 10]. Since each $o$-group forms a cancellative CRC (if we set $x\to y=y-x$), its congruence lattice is characterized by Theorem 1.9. Hence the congruences on an $o$-group $\mathbf{G}$ are completely determined by the convex subalgebras. Let $V$ be a convex subalgebra of $\mathbf{G}$. Then the corresponding quotient $o$-group will be denoted by $\mathbf{G}/V$. The equivalence class of $x\in G$ w.r.t. $V$ will be denoted by $[x]_V$.

A convex subalgebra generated by an element $g$ is said to be *principal* and we denote it by $V^g$. The principal convex subalgebras are characterized in the following lemma (see [10, Lemma 3.1.5]).

**Lemma 1.18.** *If* $\mathbf{G}$ *is an o-group and* $g\in G$, *then*

$$V^g=\{f\in G\mid |f|\leq|g|^n\text{ for some }n\in\mathbb{N}\}\,,$$

*where* $|g|=g\vee g^{-1}$.

Let $\mathbf{G}$ be an $o$-group. Then the collection $\mathcal{CS}_{\mathbf{G}}$ of all convex subalgebras is a chain. Thus any non-empty union of a system of convex subalgebras of $\mathbf{G}$ is again a convex subalgebra. For each $g\in G\setminus\{\mathbf{1}\}$ the principal convex subalgebra $V^g$ has a predecessor, namely the largest convex subalgebra not containing $g$, i.e., the union of all such convex subalgebras. This predecessor will be denoted by $V_g$. Following the terminology from [10], $V_g$ will be called *value* of $g$. The set of all values of $\mathbf{G}$ will be denoted by $\Gamma(\mathbf{G})$.

## 2. Filters and Archimedean classes

In this section we will prove several basic statements about general integral CRCs which will be useful in the sequel.

**Lemma 2.1.** *Let* $\mathbf{L}$ *be an integral CRC. Then any non-empty union of filters of* $\mathbf{L}$ *is again a filter.*

*Proof.* Recall that $\mathcal{F}_{\mathbf{L}}$ forms a chain. A union of a non-empty chain of subalgebras of any algebra is always a subalgebra, and a union of upward closed sets is upward closed. Since filters of an integral CRL are just the upward closed subalgebras, a non-empty union of filters of $\mathbf{L}$ is again a filter. □

The next trivial result characterizes the *principal* filters, i.e., the filters generated by a single element. A principal filter $F$ generated by $b$ will be denoted by $F^b$. The set of all principal filters of an integral CRL $\mathbf{L}$ will be denoted by $\mathcal{P}_{\mathbf{L}}$.

**Lemma 2.2.** *Let $\mathbf{L}$ be an integral CRL, and $b \in L$. Then the principal filter $F^b$ is of the form:*

$$F^b = \{z \in L \mid (\exists n \in \mathbb{N})(b^n \leq z)\}\,.$$

Let $\mathbf{L}$ be an integral CRC and $F^b \in \mathcal{P}_{\mathbf{L}}$. Then by Lemma 2.1 the union of all filters not containing $b$ is a filter. Clearly, it is the largest filter not containing $b$. Thus we obtain the following lemma.

**Lemma 2.3.** *Let $\mathbf{L}$ be an integral CRC and $F \in \mathcal{P}_{\mathbf{L}}$ a principal filter. Then $F$ has a predecessor.*

Following the notation from $o$-groups we will denote the predecessor of $F^b$ by $F_b$. Further, we list several easy results about Archimedean classes.

**Lemma 2.4.** *Let $\mathbf{M} = (M, *, \leq, \mathbf{1})$ be an o-monoid and $a, b \in M$. Then the Archimedean classes of $\mathbf{M}$ have the following properties:*

1. $[a]_\sim$ *is closed under* $*$.
2. $[a]_\sim$ *is convex.*
3. $[a * b]_\sim = [\min\{a, b\}]_\sim$.

*Moreover, if $\mathbf{M}$ is cancellative, then $[a]_\sim$ has no minimum for $a \neq \mathbf{1}$.*

*Proof.*

1. Suppose that $x, y \in [a]_\sim$. Without any loss of generality suppose that $x \leq y$. Then $x^2 \leq x * y \leq x$, thus $x * y$ belongs to $[a]_\sim$.
2. Suppose that $x, y \in [a]_\sim$, $z \in M$, and $x < z < y$. Then there is an $n$ such that $y^n \leq x < z < y$. Thus $z \in [a]_\sim$.
3. Without any loss of generality suppose that $a \leq b$. Then $a^2 \leq a * b \leq a$. Thus $a * b \in [a]_\sim = [\min\{a, b\}]_\sim$.

If $\mathbf{M}$ is cancellative and $x \in [a]_\sim$ then $x^2 \in [a]_\sim$. From cancellativity it follows $x^2 < x$ showing that $[a]_\sim$ has no minimum. □

Note that even if $\mathbf{M}$ is cancellative then $[a]_\sim$ may contain a maximum. For instance the integral cancellative CRC $\mathbf{Z}^- \overrightarrow{\times} \mathbf{R}^-$ from Example 1.17 has three Archimedean classes: $\{\langle 0, 0\rangle\}$, $\{\langle 0, x\rangle \in \mathbb{Z}^- \times \mathbb{R}^- \mid x < 0\}$, and $\{\langle x, y\rangle \in \mathbb{Z}^- \times \mathbb{R}^- \mid x < 0,\ y \leq 0\}$. The element $\langle -1, 0\rangle$ is the maximum of the last one. On the other hand the second one has no maximum. Observe also that $M/{\sim}$ can be linearly ordered as follows: $[a]_\sim < [b]_\sim$ iff $a \notin [b]_\sim$ and $a < b$.

As we mentioned in the previous section, the Archimedean classes are related to the filters. This connection is described by the next proposition.

**Proposition 2.5.** *Let $\mathbf{L}$ be an integral CRC and $(\mathcal{C}_{\mathbf{L}}, \leq)$ the chain of all its Archimedean classes. Then the chain $\mathcal{C}_{\mathbf{L}}$ is dually-isomorphic to the chain of all principal filters $\mathcal{P}_{\mathbf{L}}$. Let $C \in \mathcal{C}_{\mathbf{L}}$. The dual-isomorphism $\Phi : \mathcal{C}_{\mathbf{L}} \to \mathcal{P}_{\mathbf{L}}$ is defined as follows:*

$$\Phi(C) = F^b\,, \text{ for some } b \in C\,.$$

*Proof.* First, we have to show that the definition of $\Phi$ is independent of the choice of $b$. We prove that $F^b = F^c$ for $b, c \in C$. Without any loss of generality we can assume that $b \leq c$. Then $F^c \subseteq F^b$. Let $x \in F^b$. By Lemma 2.2 we have $n \in \mathbb{N}$ such that $b^n \leq x$. As $b, c$ belong to the same Archimedean class, there exists $m \in \mathbb{N}$ such that $c^m \leq b^n$. Thus $c^m \leq x$ and $x \in F^c$. Consequently, $F^b = F^c$.

Secondly, we prove that $\Phi$ is order-reversing and injective. Let $C_1, C_2 \in \mathcal{C}_{\mathbf{L}}$ such that $C_1 < C_2$. Further, let $b \in C_1$ and $c \in C_2$. Since $b < c$, we get $F^b \supseteq F^c$. Moreover, we show that $F^b \neq F^c$. Since $C_1$ and $C_2$ are disjoint, we have $b \notin C_2$, i.e., $b < c^n$ for all $n \in \mathbb{N}$. Thus $b \notin F^c$.

Finally, we show that $\Phi$ is onto. Let $F^b \in \mathcal{P}_{\mathbf{L}}$. By Lemma 2.3, $F^b$ has a predecessor $F_b$. We show that $C = F^b \setminus F_b$ is the Archimedean class such that $\Phi(C) = F^b$. Clearly, $b$ belongs to $C$ because $b \notin F_b$. Thus it is sufficient to show that $C$ is an Archimedean class. Let $x, y \in C$ such that $x \leq y$. We will show that $F^b = F^y$. Since $y \in C = F^b \setminus F_b$, we get $F^y \supseteq F^b$ (because it must be larger than $F_b$ and $F^b$ is the successor of $F_b$). Conversely, there is $n \in \mathbb{N}$ such that $b^n \leq y$ (because $y \in F^b$). Let $z \in F^y$. Then there is $m \in \mathbb{N}$ such that $y^m \leq z$. Hence $b^{nm} \leq y^m \leq z$ showing that $z \in F^b$. Since $x \in F^b = F^y$, there is $n \in \mathbb{N}$ such that $y^n \leq x$, i.e., $x \sim y$. □

From the last paragraph of the proof of Proposition 2.5, it follows that the inverse isomorphism between $\mathcal{C}_{\mathbf{L}}$ and $\mathcal{P}_{\mathbf{L}}$ is $\Phi^{-1}(F^b) = F^b \setminus F_b$ where $F_b$ is the predecessor of $F^b$.

## 3. Complete cancellative integral CRCs

The main aim of this paper is to investigate the structure of standard ΠMTL-chains whose underlying lattice is complete. If $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ is a complete ΠMTL-chain where $\mathbf{C} \in \mathcal{CanICRC}$, then $\mathbf{C}$ is not complete since it has no minimum. However, $\mathbf{C}$ is almost complete, i.e. $\bigvee S$ exists for all $\emptyset \neq S \subseteq C$. Thus we call $\mathbf{C}$ in this case *complete* as well. Notice that $\bigwedge S$ exists for $S \subseteq C$ if $S$ has a lower bound in $C$.

Let $\mathbf{L}$ be an integral CRC and $F \in \mathcal{F}_{\mathbf{L}}$. Then $F$ is said to be *nontrivial* if $F \neq \{\mathbf{1}\}$.

**Lemma 3.1.** *Let $\mathbf{C} \in \mathcal{CanICRC}$ and let $F \in \mathcal{F}_{\mathbf{C}} \setminus \{C\}$ be a nontrivial filter. If $\mathbf{C}$ is complete, then $F$ is an interval of the form $(a, \mathbf{1}] = \{x \in C \mid a < x \leq \mathbf{1}\}$ for some $a \in C \setminus \{\mathbf{1}\}$.*

*Proof.* Each filter $F$ is an interval. It follows from the fact that $F$ is an upward closed set. If $F$ is nontrivial then $\bigwedge F < \mathbf{1}$ and $\bigwedge F$ exists because $F$ has a lower bound. Let us denote $\bigwedge F$ by $a$. The element $a$ cannot belong to $F$. For if $a \in F$ then $a^2 \in F$ and cancellativity implies $a^2 < a$ (a contradiction with the fact that $a = \bigwedge F$). □

**Lemma 3.2.** *Let $\mathbf{C}$ be a complete cancellative integral CRC. Then each nontrivial filter $F \in \mathcal{F}_{\mathbf{C}} \setminus \{C\}$ has a successor.*

*Proof.* Let $F$ be a nontrivial filter. By Lemma 3.1 we have $F = (a, \mathbf{1}]$ for some $a \in C \setminus \{\mathbf{1}\}$. Since any filter greater than $F$ must contain $a$, the principal filter $F^a$ generated by $a$ is the successor. □

**Lemma 3.3.** *Let $\mathbf{C} \in \mathcal{CanICRC}$, $x \in C$, and let $F \in \mathcal{F}_{\mathbf{C}}$ be a nontrivial filter. If $\mathbf{C}$ is complete, then the equivalence class $[x]_F$ is convex, has a maximum and no minimum.*

*Proof.* First, it is obvious that the lemma holds when $F = C$ because in this case we have $[x]_F = C$. Assume that $F \neq C$. It is well known that every algebra with a lattice reduct has the property that its congruence classes are convex sets with respect to the lattice order.

Secondly, we will show that there is no minimum. Let $i_x^F = \bigwedge [x]_F$. We will prove that the infimum exists (i.e. we have to show that $[x]_F$ has a lower bound). If $[x]_F \neq F$ then $[x^2]_F < [x]_F$ (i.e. $x^2$ is the lower bound). If $[x]_F = F$ then there is $F' \supsetneq F$ and any $y \in F' \setminus F$ is the lower bound. Suppose that $i_x^F \in [x]_F$. Then we would have $i_x^F * s < i_x^F$ for all $s \in F \setminus \{\mathbf{1}\}$ by cancellativity. Since $s \,\theta_F\, \mathbf{1}$, it follows $i_x^F * s \,\theta_F\, i_x^F \,\theta_F\, x$. Thus $i_x^F * s \in [x]_F$ but $i_x^F * s < \bigwedge [x]_F$ (a contradiction).

Finally, let us denote by $m_x^F$ the supremum of $[x]_F$. Assume that $m_x^F \notin [x]_F$. Let $s \in F \setminus \{\mathbf{1}\}$. Then $m_x^F * s \in [m_x^F]_F$. Since $m_x^F * s < m_x^F$, $m_x^F$ cannot be the supremum of $[x]_F$ (a contradiction). Consequently, $m_x^F$ is the maximum of $[x]_F$. □

From now on, the maximum of an equivalence class $[x]_F$ will be denoted by $m_x^F$. The next two corollaries follow from the latter lemma.

**Corollary 3.4.** *Let $\mathbf{C} \in \mathcal{CanICRC}$ and let $F \in \mathcal{F}_{\mathbf{C}} \setminus \{C\}$ be a nontrivial filter. If $\mathbf{C}$ is complete, then each element $[x]_F$ of $\mathbf{C}/F$ has a predecessor.*

*Proof.* By Lemma 3.3, $[x]_F$ has no minimum. Let us denote $a = \bigwedge [x]_F$. Then $[a]_F$ must be a predecessor of $[x]_F$. Moreover, $a = m_a^F$. $\square$

**Corollary 3.5.** *Let* $\mathbf{C} \in \mathcal{C}an\mathcal{ICRC}$*,* $F \in \mathcal{F}_{\mathbf{C}}$*, and* $x \in C$*. If* $\mathbf{C}$ *is complete, then* $m_x^F = \max\, [m_x^F]_{F'}$ *for all* $F' \subseteq F$*,* $F' \in \mathcal{F}_{\mathbf{C}}$*.*

**Lemma 3.6.** *Let* $\mathbf{C} \in \mathcal{C}an\mathcal{ICRC}$ *and* $F \in \mathcal{F}_{\mathbf{C}}$*. If* $\mathbf{C}$ *is complete, then* $\mathbf{C}/F$ *is complete as well.*

*Proof.* Let $\emptyset \neq M \subseteq C/F$. Let us define the following subset of $C$:

$$M' = \bigcup_{[x]_F \in M} [x]_F\,.$$

Since $\mathbf{C}$ is complete, there is a supremum of $M'$. Let us denote it by $m$. Observe that $m = \bigvee_{x\in M'} m_x^F$. We claim that $[m]_F$ is a supremum of $M$. Clearly, $[m]_F \geq [x]_F$ for all $[x]_F \in M$ because $x \leq m$ for any $x \in M'$. Suppose $[y]_F \geq [x]_F$ for all $[x]_F \in M$. Then $m_y^F \geq m_x^F$ for all $x \in M'$. Thus $m_y^F \geq \bigvee_{x\in M'} m_x^F = m$. Consequently, $[y]_F = [m_y^F]_F \geq [m]_F$. $\square$

Now we will show the crucial property of the congruence lattice of a complete integral cancellative CRC. By well-known theorem from Universal Algebra if an integral CRL $\mathbf{L}$ is subdirectly irreducible then Con $\mathbf{L}\backslash\{\Delta\}$ has a minimum. Thus there must also be the corresponding nontrivial minimum filter. We will denote it by $F_\Delta$.

**Theorem 3.7.** *Let* $\mathbf{C} \in \mathcal{C}an\mathcal{ICRC}$*. If* $\mathbf{C}$ *is complete and subdirectly irreducible, then* Con $\mathbf{C}$ *is well ordered.*

*Proof.* As Con $\mathbf{C}$ is isomorphic to the chain $\mathcal{F}_{\mathbf{C}}$ of all filters of $\mathbf{C}$, it is sufficient to show that $\mathcal{F}_{\mathbf{C}}$ is well ordered. Since $\mathbf{C}$ is subdirectly irreducible, there must be a minimum nontrivial filter $F_\Delta$. Let $M$ be a non-empty set of filters, and let $F = \bigcap M$. Then $F$ is a filter, and it suffices to prove that $F \in M$. If not, then all filters in $M$ properly contain $F$ and $F \neq C$, therefore they contain the successor $s(F)$ of $F$ (this successor exists: if $F = \{\mathbf{1}\}$ then $s(F) = F_\Delta$, otherwise it exists by Lemma 3.2). Thus $F = \bigcap M \supseteq s(F) \supsetneq F$, which is a contradiction. $\square$

A similar result holds also for the equivalence classes w.r.t. a nontrivial filter.

**Theorem 3.8.** *Let* $\mathbf{C} \in \mathcal{C}an\mathcal{ICRC}$ *and* $F \in \mathcal{F}_{\mathbf{C}}$ *a nontrivial filter. If* $\mathbf{C}$ *is complete then* $\mathbf{C}/F$ *is i.w.o.*

*Proof.* If $F = C$ then $C/F$ is a singleton which is obviously i.w.o. Assume that $F \neq C$. The quotient algebra $\mathbf{C}/F$ is complete by Lemma 3.6. Thus any non-empty subset $M$ of $C/F$ has a supremum, $[m]_F$ say. Clearly, it suffices to show that $[m]_F \in M$. Suppose not. Then $[m]_F > [x]_F$ for all $[x]_F \in M$. By Corollary 3.4 $[m]_F$ has a predecessor $p([m]_F)$, and $p([m]_F) \geq [x]_F$ for all $[x]_F \in M$. Therefore, $[m]_F > p([m]_F) \geq \bigvee M = [m]_F$, which is a contradiction. $\square$

## 4. Standard subdirectly irreducible ΠMTL-chains

In this section we start to investigate the structure of the standard ΠMTL-chains. We will deal with the subdirectly irreducible standard ΠMTL-chains because they are the generators of the variety of ΠMTL-algebras. Let $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ be a standard ΠMTL-chain. Recall that $\mathbf{L}$ is subdirectly irreducible iff $\mathbf{C}$ is by Lemma 1.10.

Let $((0,1], *, \to, \min, \max, 1)$ be an integral cancellative CRC on $(0,1]$. Then we denote it by $(\mathbf{0},\mathbf{1}]_*$.

**Lemma 4.1.** *Let* $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$*. If* $(\mathbf{0},\mathbf{1}]_*$ *is subdirectly irreducible, then* Con $(\mathbf{0},\mathbf{1}]_*$ *is at most countable.*

*Proof.* By Lemma 3.1 each nontrivial filter $F \in \mathcal{F}_{(\mathbf{0},\mathbf{1}]_*}$ such that $F \neq (0,1]$ is of the form $(a,1]$. Thus if we identify the nontrivial filters with their infima, we obtain a subset of $(0,1]$ which i.w.o. since Con $(\mathbf{0},\mathbf{1}]_*$ is well ordered by Theorem 3.7. But any i.w.o. subset of $(0,1]$ is at most countable. Thus the set of all filters $\mathcal{F}_{(\mathbf{0},\mathbf{1}]_*}$ is at most countable and Con $(\mathbf{0},\mathbf{1}]_*$ as well. $\square$

**Lemma 4.2.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible, then for all $a, b \in [x]_{F_\Delta}$ we have $a * (a \to b) = a \wedge b$.*

*Proof.* If $a \leq b$ then the equality trivially holds. Also, if $a$ or $b$ equals 1 then the equality trivially holds. Thus suppose that $a > b$ and $a, b \neq 1$, i.e. $a \to b < 1$ and $a \to b \in F_\Delta$. By residuation we get $a * (a \to b) \leq b$. Suppose that $a * (a \to b) < b$. Fix an arbitrary strictly increasing sequence $\langle r_n \rangle_{n\in\mathbb{N}}$ such that $\bigvee r_n = 1$ and $r_n \in F_\Delta$ for all $n$. As $F_\Delta$ is a left-open interval of the type $(c, 1]$ for some $0 \leq c < 1$, there surely exists such a sequence. Since $F_\Delta \setminus \{1\}$ is an Archimedean class, we get that for each $n$ there exists $k_n$ such that

$$r_n^{k_n} \leq a \to b < r_n^{k_n - 1}\,.$$

Thus we obtain for all $n \in \mathbb{N}$:

$$a * r_n^{k_n} \leq a * (a \to b) < b < a * r_n^{k_n - 1}\,.$$

The last inequality holds since $a \to b$ is the maximal solution of the inequality $a * x \leq b$.

Further, by Proposition 1.1 we get $\bigvee (b * r_n) = b * \bigvee r_n = b$. Hence there must be an $n_0$ such that $a * (a \to b) < b * r_{n_0}$. Thus we obtain

$$a * r_{n_0}^{k_{n_0}} \leq a * (a \to b) < b * r_{n_0} < a * r_{n_0}^{k_{n_0}}\,,$$

a contradiction. □

**Theorem 4.3.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$ and $x \in (0, 1]$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible, then each $z \in [x]_{F_\Delta}$ can be expressed uniquely as $z = m_x^{F_\Delta} * s$ for some $s \in F_\Delta$ and we have*

$$[x]_{F_\Delta} = \{m_x^{F_\Delta} * s \mid s \in F_\Delta\}\,.$$

*Proof.* Let $z \in [x]_{F_\Delta}$. By Lemma 4.2 we have $m_x^{F_\Delta} * (m_x^{F_\Delta} \to z) = z$. Thus we can take $s = m_x^{F_\Delta} \to z \in F_\Delta$. The uniqueness of $s$ follows from cancellativity (if $z = m_x^{F_\Delta} * s_1 = m_x^{F_\Delta} * s_2$ then $s_1 = s_2$).

On the other hand, $m_x^{F_\Delta} * s$ belongs to $[x]_{F_\Delta}$ for all $s \in F_\Delta$ because $m_x^{F_\Delta} \to m_x^{F_\Delta} * s = s$. □

Lemma 4.2 has also other important consequences. It implies that the elements belonging to $F_\Delta = [1]_{F_\Delta}$ behave like in a divisible algebra, i.e., if $x \leq y$ then there is an element $z$ such that $y * z = x$, namely $z = y \to x$.

Let $\mathbf{R}^- = (\mathbb{R}^-, +, \Rightarrow, \min, \max, 0)$ be the cancellative integral CRC from Example 1.17.

**Theorem 4.4.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible, then $\mathbf{F}_\Delta = (F_\Delta, *, \to, \min, \max, 1)$ is a subalgebra of $(\mathbf{0},\mathbf{1}]_*$ isomorphic to $\mathbf{R}^-$.*

*Proof.* The filter $F_\Delta$ is a subuniverse of $(\mathbf{0},\mathbf{1}]_*$ (see Theorem 1.9). By Lemma 4.2 the subalgebra $\mathbf{F}_\Delta$ is divisible. Since $\mathbf{F}_\Delta$ is cancellative, divisible and Archimedean, it follows from [9, Theorem 2, Page 165] that the $\ell$-monoidal reduct of $\mathbf{F}_\Delta$ is isomorphic to a sub-$\ell$-monoid $\mathbf{S}$ of $\mathbf{R}^-$. Thus $\mathbf{F}_\Delta \cong \mathbf{S}$ (as CRLs), because residua are uniquely determined by the monoid operation and order in both algebras.

Now, it is sufficient to show that $S = \mathbb{R}^-$. Suppose not. Then there is an element $y \in \mathbb{R}^- \setminus S$. Let $M = \{x \in S \mid x \leq y\}$. As $\mathbf{S}$ is complete, we get $\bigvee M \in S$ and $\bigwedge (S \setminus M) \in S$. Thus $\bigvee M < y < \bigwedge (S \setminus M)$ by our assumption. However, it is a contradiction with the fact that $\mathbf{S}$ is dense. Thus $S = \mathbb{R}^-$. □

**Lemma 4.5.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible, then the set $E_*$ of all elements which cannot be expressed as a product of greater elements, i.e., the set*

$$E_* = \{z \in (0, 1] \mid \neg(\exists x, y \in (0, 1])(z = x * y \ \&\ x, y > z)\}$$

*is at most countable and i.w.o.*

*Proof.* First, we show that $E_* \cap F_\Delta = \{1\}$. The element 1 obviously belongs to this intersection. Let $z \in F_\Delta \setminus \{1\}$. Since $\mathbf{F}_\Delta$ is isomorphic to $\mathbf{R}^-$ by Theorem 4.4, there must be an element $x \in F_\Delta$ such that $z = x * x$. Thus $z \notin E_*$.

Secondly, we show that $E_* \subseteq \{m_x^{F_\Delta} \mid x \in (0,1]\}$. Let $z \in (0,1] \setminus F_\Delta$ and $z \neq m_z^{F_\Delta}$. Then by Theorem 4.3, we have $z = m_z^{F_\Delta} * s$ for some $s \in F_\Delta$. Thus $z \notin E_*$.

Finally, the set $\{m_x^{F_\Delta} \mid x \in (0,1]\}$ is order-isomorphic to $(\mathbf{0},\mathbf{1}]_*/F_\Delta$. Since $(\mathbf{0},\mathbf{1}]_*/F_\Delta$ is i.w.o. by Theorem 3.8, the set $E_*$ is i.w.o. as well. Moreover, $E_*$ is at most countable because any i.w.o. subset of $(0,1]$ is at most countable. □

**Theorem 4.6.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{CanICRC}$ and $x \in (0,1]$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible, then $x = g_1 * \cdots * g_n * s$ for some $g_i \in E_*$, $i = 1,\ldots,n$, and $s \in F_\Delta$.*

*Proof.* By Theorem 4.3, we can write $x = m_x^{F_\Delta} * s$ for some $s \in F_\Delta$. Thus it is sufficient to show that $m_x^{F_\Delta} = g_1 * \cdots * g_n$ for some $g_i \in E_*$, $i = 1,\ldots,n$.

If $m_x^{F_\Delta} \in E_*$ then we are done. If not, $m_x^{F_\Delta}$ can be expressed as $m_x^{F_\Delta} = a*b$ for some $a, b > m_x^{F_\Delta}$. Moreover, $a, b \in \{m_y^{F_\Delta} \mid y \in (0,1]\}$. Indeed, suppose that $a = m_a^{F_\Delta} * r$ for some $r \in F_\Delta$ with $r < 1$. Then $m_x^{F_\Delta} = a * b = m_a^{F_\Delta} * r * b$. Thus $m_a^{F_\Delta} * b > m_x^{F_\Delta}$ contradicting the fact that $m_x^{F_\Delta}$ is the maximum of the equivalence class $[x]_{F_\Delta}$. Now $a, b$ belong to $E_*$ or can be again decomposed. In this way, we obtain a binary tree where the leaves belong to $E_*$. Moreover, each branch of the tree is strictly increasing. Since $\{m_y^{F_\Delta} \mid y \in (0,1]\}$ is i.w.o. by Theorem 3.8, each branch must be finite. Thus there is a finite number of leaves. Let us denote them by $g_1,\ldots,g_n$. Then $m_x^{F_\Delta} = g_1 * \cdots * g_n$. □

## 5. Fraction group

Let $\mathbf{C} \in \mathcal{CanICRC}$. Let us denote the $\to$-free reduct $(C, *, \leq, \mathbf{1})$ by $\mathbf{C}_*$. Since $\mathbf{C}_*$ is a cancellative $o$-monoid, we can extend it to an $o$-group of fractions $\mathbf{G_C}$ in the same way as integers are constructed from natural numbers. The universe of $\mathbf{G_C}$ is the set $G_\mathbf{C} = (C \times C)/\approx$, where $(a,b) \approx (c,d)$ iff $a * d = c * b$. The group operation is defined $(a,b) * (c,d) = (a * c, b * d)$, $(\mathbf{1},\mathbf{1})$ is the neutral element, $(a,b)^{-1} = (b,a)$, and $(a,b) \leq (c,d)$ iff $a * d \leq b * c$. We will denote the ordered pair $(a,b)$ by $a/b$ or $a * b^{-1}$. Further, we identify the elements from $C$ with the corresponding elements in $G_\mathbf{C}$, i.e., we will write $a$ instead of $a/\mathbf{1}$. Thus $C$ can be viewed as a subuniverse of the $\ell$-monoidal reduct of $\mathbf{G_C}$.

**Lemma 5.1.** *Let $\mathbf{C} \in \mathcal{CanICRC}$. Then* Con $\mathbf{C}$ *can be embedded into* Con $\mathbf{G_C}$.

*Proof.* We will work with the chain of the filters $\mathcal{F}_\mathbf{C}$ (resp. convex subalgebras $\mathcal{CS}_{\mathbf{G_C}}$) instead of Con $\mathbf{C}$ (resp. Con $\mathbf{G_C}$). To each filter $F \in \mathcal{F}_\mathbf{C}$ we can assign a corresponding convex subalgebra $\overline{F} \in \mathcal{CS}_{\mathbf{G_C}}$ as follows:

$$\overline{F} = \{z \in G_\mathbf{C} \mid (\exists y \in F)(|z| \leq |y|)\}\,.$$

We start with the proof that $\overline{F}$ is a convex subalgebra. Clearly, $\overline{F}$ is convex. Let $a/b$ and $c/d$ be elements of $\overline{F}$. Then there are $y_1, y_2 \in F$ such that $|y_1| \geq |a/b|$ and $|y_2| \geq |c/d|$. Since $y_1 * y_2 \leq 1$ and $y_1 * y_2 \in F$, we get

$$|y_1 * y_2| = y_1^{-1} * y_2^{-1} = |y_1| * |y_2| \geq |a/b| * |c/d| \geq |(a * c)/(b * d)|\,.$$

Thus $(a * c)/(b * d)$ belongs to $\overline{F}$. We have to also show that $\overline{F}$ is closed under taking inverse elements. If $|a/b| \leq |y|$ for some $y \in F$, then $|b/a| = |a/b| \leq y$.

Finally, we have to show that the mapping assigning $\overline{F}$ to $F$ is injective and order-preserving. Let $F, F'$ be two different filters in $\mathcal{F}_\mathbf{C}$. Since $\mathcal{F}_\mathbf{C}$ is linearly ordered, one of the filters contains the other. Without any loss of generality suppose that $F' \subseteq F$. Let us take an arbitrary element $z \in F \setminus F'$. Then $z \notin \overline{F'}$ since there is no $y \in F'$ such that $y \leq z$. Thus $\overline{F} \supsetneq \overline{F'}$. □

From now on, we will denote by $\overline{F}$ the convex subalgebra corresponding to $F \in \mathcal{F}_\mathbf{C}$.

Next we show an example demonstrating the fact that Con $\mathbf{C}$ need not be isomorphic to Con $\mathbf{G_C}$. Let $\mathbf{C} = \mathbf{Z}^- \overrightarrow{\times} \mathbf{R}^-$ be the integral cancellative CRC from Example 1.17. Let us take its submonoid $\mathbf{S}$ generated by $\langle -1, 0\rangle$ and $\langle -1, -1\rangle$. It is known that each finitely generated submonoid of an $o$-monoid is i.w.o. (this follows also from our Theorem 6.5). Hence such a submonoid is in fact residuated if we define a residuum in $\mathbf{S}$ as $a \to_S b = \max\{c \in S \mid a + c \leq b\}$. It is obvious that $\mathbf{S}$ has only trivial filters since for all $\langle k, r\rangle \in S$ there is $n \in \mathbb{N}$ such that $\langle -1, 0\rangle^n \leq \langle k, r\rangle$. On the other hand the group of fractions $\mathbf{G_S}$ has one nontrivial convex subalgebra. It is a subalgebra generated by the fraction $\langle -1,-1\rangle/\langle -1,0\rangle$. Indeed, we have for all $n \in \mathbb{N}$ the following:

$$(\langle -1,-1\rangle/\langle -1,0\rangle)^n = \langle -n,-n\rangle/\langle -n,0\rangle > \langle -1,0\rangle\,,$$

since $\langle -n,-n\rangle > \langle -1,0\rangle + \langle -n,0\rangle = \langle -n-1,0\rangle$.

**Lemma 5.2.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible, then its group of fractions $\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*}$ is subdirectly irreducible as well and the minimum nontrivial congruence is determined by the convex subalgebra $\overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$, where*

$$F_\Delta^{-1} = \{x^{-1} \in G_{(\mathbf{0},\mathbf{1}]_*} \mid x \in F_\Delta\}\,.$$

*Proof.* Since $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible, there must be a minimum nontrivial congruence of $(\mathbf{0},\mathbf{1}]_*$. Thus we have a nontrivial minimum filter $F_\Delta$. First, we will show that for any $a/b \in \overline{F_\Delta}$ either $a/b \in F_\Delta$ or $b/a \in F_\Delta$. Without any loss of generality we can suppose that $a/b \leq 1$, i.e., $a \leq b$. Then there exists $y \in F_\Delta$ such that $|a/b| \leq |y|$ and $y \leq a/b$. If we multiply this inequality by $b$, we get $b * y \leq b * (a/b) = a$. Hence $y \leq b \to a$. Since $y \in F_\Delta$, we get $b \to a \in F_\Delta$. Thus $m_a^{F_\Delta} = m_b^{F_\Delta}$. By Theorem 4.3 we get $a = m_a^{F_\Delta} * r$ and $b = m_a^{F_\Delta} * s$ for some $r, s \in F_\Delta$. Thus $a/b = (m_a^{F_\Delta} * r)/(m_a^{F_\Delta} * s) = r/s$. Since elements from $F_\Delta$ satisfy the divisibility condition by Lemma 4.2, $r/s = s \to r \in F_\Delta$. Thus $\overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$.

Secondly, we will prove that $\overline{F_\Delta}$ is the minimum nontrivial convex subalgebra. Let us take an arbitrary element $z \in \overline{F_\Delta} \setminus \{1\}$. We will show that $z$ generates $\overline{F_\Delta}$. We can suppose that $z \leq 1$ (if not take $z^{-1}$). From the previous paragraph we have $z \in F_\Delta$. Since $F_\Delta$ is the minimum nontrivial filter, $z$ generates $F_\Delta$ and it must generate also $\overline{F_\Delta}$. □

**Corollary 5.3.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$ such that $(\mathbf{0},\mathbf{1}]_*$ is subdirectly irreducible and let $F_\Delta$ be its minimum nontrivial filter. Then the structure $(\overline{F_\Delta}, *, \leq, 1)$ is isomorphic to $(\mathbb{R}, +, \leq, 0)$.*

*Proof.* The structure $(F_\Delta, *, \leq, 1)$ is isomorphic to $(\mathbb{R}^-, +, \leq, 0)$ by Theorem 4.4. Let us denote the corresponding isomorphism by $\Phi$. Since $\overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$, we define a mapping $\Psi : \overline{F_\Delta} \to \mathbb{R}$ by

$$\Psi(x) = \begin{cases} \Phi(x)\,, & x \leq 1\,, \\ -\Phi(x^{-1})\,, & x > 1\,. \end{cases}$$

We claim that $\Psi$ is an isomorphism between $\overline{F_\Delta}$ and $(\mathbb{R}, +, \leq, 0)$. The mapping $\Psi$ is clearly one-to-one, onto, and order-preserving. The fact that $\Psi$ is an isomorphism can be easily checked. The cases when $x, y \leq 1$ or $x, y > 1$ are trivial. We only show the case when $x \leq 1$, $y > 1$, and $x * y \leq 1$. The other cases are similar. Then $\Psi(x * y) = \Phi(x * y)$ and $\Psi(x) + \Psi(y) = \Phi(x) - \Phi(y^{-1})$. Since $x * y \leq 1$, we get $x \leq y^{-1}$. Thus $\Phi(x) \leq \Phi(y^{-1})$ and $\Phi(x) - \Phi(y^{-1}) \leq 0$. Hence there is $z \in F_\Delta$ such that $\Phi(z) = \Phi(x) - \Phi(y^{-1})$. Since

$$\Phi(z * y^{-1}) = \Phi(z) + \Phi(y^{-1}) = \Phi(x) - \Phi(y^{-1}) + \Phi(y^{-1}) = \Phi(x)\,,$$

we obtain that $z * y^{-1} = x$. Consequently, $z = x * y$ and $\Psi(x * y) = \Psi(x) + \Psi(y)$. Similarly the case for $x * y > 1$. Thus $\overline{F_\Delta}$ is isomorphic to $(\mathbb{R}, +, \leq, 0)$. □

Now we show the relation between principal filters of an integral cancellative CRC $\mathbf{C}$ and the principal convex subalgebras of its group of fractions $\mathbf{G_C}$.

**Lemma 5.4.** *Let $\mathbf{C} \in \mathcal{C}an\mathcal{ICRC}$, $F^b \in \mathcal{P}_\mathbf{C}$, and let $V^b$ be the principal convex subalgebra of $\mathbf{G_L}$ generated by $b$. Then $\overline{F^b} = V^b$.*

*Proof.* The case when $b = \mathbf{1}$ is trivial. Let $F^b \neq \{\mathbf{1}\}$. We will show that $\overline{F^b}$ is a successor of $V_b$. As $V_b$ is the greatest convex subgroup not containing $b$, we obtain $V_b \subseteq \overline{F^b}$. Since $V_b$ is the predecessor of $V^b$, it is sufficient to prove that $\overline{F^b} \subseteq V^b$. Let $x \in \overline{F^b}$. Then there is $y \in F^b$ such that $|x| \leq |y|$. Further by Lemma 2.2, there exists $n \in \mathbb{N}$ such that $b^n \leq y$. Since $b, y \leq \mathbf{1}$, we get $|y| \leq |b^n| = |b|^n$. Thus $|x| \leq |b|^n$. Finally by Lemma 1.18, we obtain $x \in V^b$. □

**Corollary 5.5.** *Let $\mathbf{C} \in \mathcal{C}an\mathcal{ICRC}$ and let $F^b \in \mathcal{P}_\mathbf{C}$ be a nontrivial principal filter. Then $\overline{F^b}/V_b$ is isomorphic (as an o-group) to a subgroup of the additive group of real numbers.*

*Proof.* By Lemma 5.4 we have $\overline{F^b}/V_b = V^b/V_b$. Since $V^b/V_b$ is an ordered Archimedean group, it is isomorphic to a subgroup of the additive group of real numbers by Hölder's Theorem (see [10, Corollary 4.1.4]). $\square$

**Lemma 5.6.** *Let* $\mathbf{C} \in \mathcal{CanICRC}$ *and let* $\mathbf{C}_*$ *be the* $\to$*-free reduct of* $\mathbf{C}$*. Then the congruence lattice* Con $\mathbf{G_C}$ *can be embedded into* Con $\mathbf{C}_*$*.*

*Proof.* Let $\theta \in$ Con $\mathbf{G_C}$. Since $C \subseteq G_{\mathbf{C}}$, we can map $\theta$ to the restriction $\theta\restriction_{\mathbf{C}_*}$. Clearly, $\theta\restriction_{\mathbf{C}_*}$ belongs to Con $\mathbf{C}_*$. Thus it suffices to show that this mapping is injective and order-preserving. If $\theta_1 \subseteq \theta_2$ then trivially $\theta_1\restriction_{\mathbf{C}_*} \subseteq \theta_2\restriction_{\mathbf{C}_*}$. Thus the mapping is order-preserving. Now, assume that $\theta_1 \subsetneq \theta_2$. Let us denote by $V_1$ (resp. $V_2$) the convex subalgebra corresponding to $\theta_1$ (resp. $\theta_2$). Then there must be an element $a/b$ in $V_2$ such that $a/b \notin V_1$ and $a, b \in C$. Thus $a\ \theta_2\restriction_{\mathbf{C}_*}\ b$. But it is not true that $a\ \theta_1\restriction_{\mathbf{C}_*}\ b$. Hence $\theta_1\restriction_{\mathbf{C}_*} \subsetneq \theta_2\restriction_{\mathbf{C}_*}$. $\square$

Although Con $\mathbf{G_C}$ can be embedded into Con $\mathbf{C}_*$, they need not be isomorphic. Let $\mathbf{Z}^-$ be the integral cancellative CRC from Example 1.17. Consider the lexicographic product $\mathbf{C} = \mathbf{Z}^- \overrightarrow{\times} \mathbf{Z}^- \overrightarrow{\times} \mathbf{Z}^-$. Then its group of fractions $\mathbf{G_C}$ has two nontrivial convex subalgebras; $V_1$ generated by $\langle 0, 0, -1\rangle$ and $V_2$ generated by $\langle 0, -1, 0\rangle$. Obviously $V_1 \subseteq V_2$. Let $\theta_1, \theta_2$ be the congruences corresponding to $V_1$ and $V_2$ respectively. Now we define an equivalence in $\mathbf{C}$ as follows: $\langle a, b, c\rangle \approx \langle a', b', c'\rangle$ iff either $a = 0$ and $\langle a, b, c\rangle\ \theta_1\restriction_{\mathbf{C}_*}\ \langle a', b', c'\rangle$ or $a \neq 0$ and $\langle a, b, c\rangle\ \theta_2\restriction_{\mathbf{C}_*}\ \langle a', b', c'\rangle$. Then it can be easily shown that $\approx$ is an $\ell$-monoidal congruence but there is no corresponding convex subalgebra in $\mathbf{G_C}$.

Let $\mathbf{C} \in \mathcal{CanICRC}$, $\theta \in$ Con $\mathbf{G_C}$, and $V$ its corresponding convex subalgebra. In order to make the notation more transparent, we will use the expression $\mathbf{C}_*/V$ instead of $\mathbf{C}_*/\theta\restriction_{\mathbf{C}_*}$. Also the equivalence class $[x]_{\theta\restriction_{\mathbf{C}_*}}$ will be denoted by $[x]^*_V$. Thus we are able to distinguish between an element of $\mathbf{G_C}/V$ (it is an equivalence class $[x]_V$) and an element of $\mathbf{C}_*/V$ (it is an equivalence class $[x]^*_V$).

**Theorem 5.7.** *Let* $\mathbf{C} \in \mathcal{CanICRC}$ *and let* $F^b \in \mathcal{P}_{\mathbf{C}}$ *be a nontrivial principal filter. Then* $F^b/V_b$ *is isomorphic (as an* $\ell$*-monoid) to a submonoid of* $V^b/V_b$*.*

*Proof.* Since $F^b$ is a subalgebra of the $\ell$-monoidal reduct of $V^b$, we get by the Third Isomorphism Theorem (see [6, Page 49, Theorem 6.18]) that $F^b/V_b$ is isomorphic (as an $\ell$-monoid) to $V^b_\theta/V_b$ where $V^b_\theta = \{a \in V^b \mid F^b \cap [a]_{V_b} \neq \emptyset\}$. The isomorphism assigns to $[x]^*_{V_b}$ the equivalence class $[x]_{V_b}$. Since $V^b_\theta/V_b$ is a submonoid of $V^b/V_b$, we are done. $\square$

**Corollary 5.8.** *Let* $\mathbf{C} \in \mathcal{CanICRC}$ *and let* $F^b \in \mathcal{P}_{\mathbf{C}}$ *be a nontrivial principal filter. Then* $F^b/V_b$ *is isomorphic (as an* $\ell$*-monoid) to a submonoid of* $(\mathbb{R}^-, +, \leq, 0)$*.*

*Proof.* From Theorem 5.7 it follows that $F^b/V_b$ is isomorphic (as an $\ell$-monoid) to a submonoid of $V^b/V_b$. By Hölder's Theorem we get an isomorphism $\Phi$ between $V^b/V_b$ and the additive monoid of real numbers. Since $x \leq \mathbf{1}$ for any $x \in F^b$, we obtain $\Phi(x) \leq 0$. Thus $F^b/V_b$ is isomorphic to a submonoid of $(\mathbb{R}^-, +, \leq, 0)$. $\square$

## Hahn's Embedding Theorem

Now, we recall what a full Hahn group is (for details see [10]). Let $\Gamma$ be a totally ordered set. Let us denote the set of all functions $f : \Gamma \to \mathbb{R}$ such that supp $f = \{\gamma \in \Gamma \mid f(\gamma) \neq 0\}$ is i.w.o. by $V(\Gamma)$. The set $V(\Gamma)$ forms an $o$-group $\mathbf{V}(\Gamma)$ under addition $(f+g)(\gamma) = f(\gamma)+g(\gamma)$ and $f > 0$ provided that $f(\max(\text{supp } f)) > 0$. Such an $o$-group is called a *full Hahn group.*

Let $\mathbf{C} \in \mathcal{CanICRC}$ and let $\Gamma(\mathbf{G_C})$ be the chain of all values of $\mathbf{G_C}$. Then the $o$-group $\mathbf{G_C}$ can be embedded into the full Hahn group $\mathbf{V}(\Gamma(\mathbf{G_C}))$ by Hahn's Embedding Theorem (see [10, Theorem 4.C, Page 70]). Thus we obtain the following result.

**Theorem 5.9.** *Let* $\mathbf{C} \in \mathcal{CanICRC}$ *and let* $\Gamma(\mathbf{G_C})$ *be the chain of all values of* $\mathbf{G_C}$*. Then* $\mathbf{C}_*$ *can be embedded (as an* $\ell$*-monoid) into* $\mathbf{V}(\Gamma(\mathbf{G_C}))$*.*

In order to prove the next lemma, we have to also recall the definition of the embedding from Hahn's embedding theorem. First, it is known (see [9, Lemma A, Page 56]) that each $o$-group $\mathbf{G}$ can be extended to a divisible $o$-group $\hat{\mathbf{G}}$ in such a way that Con $\mathbf{G} \cong$ Con $\hat{\mathbf{G}}$ and for each convex subalgebra $V \in \mathcal{CS}_{\mathbf{G}}$ we have $V = G \cap \hat{V}$ where $\hat{V}$ is the corresponding convex subalgebra in $\hat{\mathbf{G}}$. By divisible we mean here that for any element $g \in \hat{\mathbf{G}}$ and any $n \in \mathbb{N}$ there is an element $f \in \hat{\mathbf{G}}$ such that $f^n = g$. Thus $\hat{\mathbf{G}}$ and all its convex subalgebras can be viewed as vector spaces over rationals.

Let $V_\gamma \in \Gamma(\mathbf{G})$. We identify $\gamma$ with $V_\gamma$ and denote the successor of $V_\gamma$ by $V^\gamma$. Let us denote the corresponding convex subalgebras in $\hat{\mathbf{G}}_{\mathbf{C}}$ by $\hat{V}^\gamma$ and $\hat{V}_\gamma$ respectively. The embedding from Hahn's Embedding Theorem assigns to each $g \in G$ the function $g'$ defined as follows: $g'(\gamma) = \rho_\gamma(\pi_\gamma(g)) \in \mathbb{R}$ for each $\gamma \in \Gamma(\mathbf{G})$, where $\pi_\gamma$ is the projection of the vector space $\hat{\mathbf{G}}$ onto the subspace $\hat{V}^\gamma$ and $\rho_\gamma$ is an order-preserving homomorphism from $\hat{V}^\gamma$ into $\mathbb{R}$ whose kernel is $\hat{V}_\gamma$. Such a homomorphism exists by Hölder's Theorem.

**Lemma 5.10.** *Let $\mathbf{C} \in \mathcal{CanICRC}$, $F \in \mathcal{P}_{\mathbf{C}}$ a nontrivial principal filter, and $g \in F$. Further, let $V^\gamma = \overline{F}$ be the principal convex subalgebra in $\mathbf{G}_{\mathbf{C}}$. Then the corresponding function $g' \in \mathbf{V}(\Gamma(\mathbf{G}_{\mathbf{C}}))$ maps all $\alpha > \gamma$ to 0.*

*Proof.* As $F$ is principal, $V^\gamma$ is principal as well by Lemma 5.4. Thus there is a value $V_\gamma \in \Gamma(\mathbf{G}_{\mathbf{C}})$ which is the predecessor of $V^\gamma$. Since $\alpha > \gamma$, we get $V_\alpha \supseteq V^\gamma \supseteq F$. It follows that $g \in V^\alpha$ and also $g \in V_\alpha$. Thus $g = \pi_\alpha(g)$ and $\rho_\alpha(g) = 0$ as $g$ belongs to the kernel $\hat{V}_\alpha$. □

At this point we know that each $\mathbf{C} \in \mathcal{CanICRC}$ can be embedded as an $\ell$-monoid into the full Hahn group $\mathbf{V}(\Gamma(\mathbf{G}_{\mathbf{C}}))$. Now it remains to describe which functions from $\mathbf{V}(\Gamma(\mathbf{G}_{\mathbf{C}}))$ correspond to the original elements from $C$. First, we will prove several useful results about i.w.o. monoids.

## 6. Inversely well ordered monoids

Let $A, B$ be totally ordered sets. We denote by $A \times B$ the cartesian product of $A, B$ endowed with the cartesian order, i.e., $(a, b) \leq_c (c, d)$ iff $a \leq c$ and $b \leq d$.

**Lemma 6.1.** *Let $A, B$ be i.w.o. sets and $C$ a totally ordered set such that there is a surjective order-preserving mapping $\Phi : A \times B \to C$. Then $C$ is i.w.o.*

*Proof.* Let $\emptyset \neq S \subseteq C$, $a_0 = \max \pi_1(\Phi^{-1}(S))$, and $b_0 = \max \pi_2((\{a_0\} \times B) \cap \Phi^{-1}(S))$. Both maxima exist since $A$ and $B$ are i.w.o. Clearly, $(a_0, b_0)$ is a maximal element of $\Phi^{-1}(S)$. Thus the set $M$ of maximal elements of $\phi^{-1}(S)$ is non-empty. Moreover, $M$ is finite, as the product of two i.w.o. sets cannot contain infinite antichains (by Dickson's Lemma, a direct product of two well quasi orders is a well quasi order). Hence $\max(\Phi(M))$ exists and is the maximum of $C$. □

Let $\mathbf{M} = (M, +, \leq, \mathbf{0})$ be an $o$-monoid. Note that by Lemma 6.1, if $A \subseteq M$ is i.w.o. then $A + A = \{a + b \mid a, b \in A\}$ is i.w.o. as well since $+$ is a surjective order-preserving mapping from $A \times A$ onto $A + A$. Indeed, $(a, b) \leq_c (c, d)$ implies $a + b \leq c + d$. Let us further introduce the set of all solutions of the equation $c = a + b$ for a given $c \in M$ and $a, b \neq \mathbf{0}$.

$$\mathcal{T}_c = \{(a, b) \in M^2 \mid a * b = c,\ a, b > c\}.$$

Observe that if $(a, b) \in \mathcal{T}_c$ then $a, b \neq \mathbf{0}$.

**Theorem 6.2.** *Let $\mathbf{M} = (M, +, \leq, \mathbf{0})$ be an o-monoid which is i.w.o. Then $\mathbf{M}$ has a unique minimal set of generators $G$.*

*Proof.* Let $G$ be the following set:

$$G = \{g \in M \mid \mathcal{T}_g = \emptyset\}.$$

It is obvious that each set of generators of $\mathbf{M}$ must contain $G$ because the elements of $G$ cannot be expressed as a sum of other elements from $M$. Thus it is sufficient to prove that $G$ generates $\mathbf{M}$. Let $c \in M$. We will show that $c$ can be generated from $G$. Either $c \in G$ and we are done or $c = a + b$ for some $(a, b) \in \mathcal{T}_c$. Now $a$ either belongs to $G$ or can be written as a sum of greater elements. Similarly for $b$. In this way we obtain a binary tree. Since each branch of this tree is strictly increasing and $\mathbf{M}$ is i.w.o., each branch must be finite. Thus the tree is finite and the leaves belong to $G$. Let us denote the leaves by $g_1, \ldots, g_n$. Then $c = \sum_{i=1}^{n} g_i$ and the proof is done. $\square$

**Lemma 6.3.** *Let $\mathbf{M} = (M, +, \leq, \mathbf{0})$ be an Archimedean o-monoid and $G$ its generating set. If $G$ is i.w.o. then $M$ is i.w.o. as well.*

*Proof.* If $G = \emptyset$ or $G = \{\mathbf{0}\}$ then $M = \{\mathbf{0}\}$ which is clearly i.w.o. Thus assume that there is $x \in G$ such that $x \neq \mathbf{0}$. Let $\emptyset \neq S \subseteq M$, $c \in S$, and $g = \max G \setminus \{\mathbf{0}\}$. Since $\mathbf{M}$ is Archimedean, there exists $n_c \in \mathbb{N}$ such that $n_c\, g \leq c$. Consequently, for any $m \geq n_c$, $a_i \in G \setminus \{\mathbf{0}\}$ we have

$$\sum_{i=1}^{m} a_i \leq mg \leq n_c\, g \leq c\,.$$

Thus it is enough to consider only the sums of generators with less than $n_c$ summands since all longer sums are surely less than or equal to $c$. But all such elements belongs to $G + G + \cdots + G$ ($n_c$-times) which is i.w.o. by Lemma 6.1. Hence $S$ has a maximum. $\square$

**Lemma 6.4.** *Let $\mathbf{M} = (M, +, \leq, \mathbf{0})$ be an o-monoid, $G$ its generating set, and $\mathcal{C}_{\mathbf{M}}$ the chain of Archimedean classes of $\mathbf{M}$. If $G$ is i.w.o. then $\mathcal{C}_{\mathbf{M}}$ is i.w.o. as well.*

*Proof.* If $G = \emptyset$ then $M = \{\mathbf{0}\}$ and $\mathcal{C}_{\mathbf{M}}$ is obviously i.w.o. Assume that $G \neq \emptyset$. Then for each $C \in \mathcal{C}_{\mathbf{M}}$ we have $C \cap G \neq \emptyset$. Indeed, let $x \in C$. Then $x = \sum_{i=1}^{n} g_i$ for some $g_i \in G$. By Lemma 2.4 we get $[x]_\sim = [\min\{g_1, \ldots, g_n\}]_\sim$. Thus $\min\{g_1, \ldots, g_n\} \in C \cap G$.

Let $\emptyset \neq \mathcal{S} \subseteq \mathcal{C}_{\mathbf{M}}$ and $H = \bigcup_{C \in \mathcal{S}} (C \cap G)$. Since $\emptyset \neq H \subseteq G$, it has a maximum, say $g_0$. We claim that $[g_0]_\sim = \max(\mathcal{S})$. Let $[x]_\sim \in \mathcal{S}$. Then $x = \sum_{i=1}^{n} g_i$ for some $g_i \in G$. By Lemma 2.4 we get $[x]_\sim = [\min\{g_1, \ldots, g_n\}]_\sim$. Thus $\min\{g_1, \ldots, g_n\} \in H$. As $\min\{g_1, \ldots, g_n\} \leq g_0$, we obtain $[x]_\sim \leq [g_0]_\sim$. $\square$

**Theorem 6.5.** *Let $\mathbf{M} = (M, +, \leq, \mathbf{0})$ be an o-monoid and $G$ its generating set. If $G$ is i.w.o. then $M$ is i.w.o. as well.*

*Proof.* Let $\mathcal{C}_{\mathbf{M}}$ be the chain of Archimedean classes of $\mathbf{M}$. By Lemma 6.4 the chain $\mathcal{C}_{\mathbf{M}}$ is i.w.o. Thus there is an ordinal $\tau$ such that the elements of $\mathcal{C}_{\mathbf{M}}$ can be indexed by ordinals $0 \leq \alpha < \tau$ in such a way that for any $C_\alpha, C_\beta \in \mathcal{C}_{\mathbf{M}}$ we have $\alpha < \beta$ iff $C_\beta < C_\alpha$. Consider the following sets:

$$G_\alpha = G \cap C_\alpha\,, \qquad G_{\leq\alpha} = \bigcup_{\beta\leq\alpha} G_\beta\,, \qquad G_{<\alpha} = \bigcup_{\beta<\alpha} G_\beta\,.$$

Clearly $G_{<\tau} = G$. We will prove by transfinite induction that $M = \mathrm{Sg}(G_{<\tau})$ is i.w.o.

(1) If $\tau = 1$ then $\mathbf{M}$ has only one Archimedean class. Thus $M$ is i.w.o. by Lemma 6.3.
(2) Let $\tau = \alpha + 1$. By the induction assumption we have that $\mathrm{Sg}(G_{<\alpha})$ is i.w.o. Since the o-monoid $\mathbf{Sg}(G_\alpha)$ is Archimedean, $\mathrm{Sg}(G_\alpha)$ is i.w.o. by Lemma 6.3. Let $x \in \mathrm{Sg}(G_{<\tau})$. Then $x = \sum_{i=1}^{n} g_i + \sum_{j=1}^{k} h_j$ where $g_i \in G_{<\alpha}$ and $h_j \in G_\alpha$. Thus $\mathrm{Sg}(G_{<\tau}) = \mathrm{Sg}(G_{<\alpha}) + \mathrm{Sg}(G_\alpha)$. Consequently, $\mathrm{Sg}(G_{<\tau})$ is i.w.o. by Lemma 6.1.
(3) Let $\tau$ be a limit ordinal. Then $\mathrm{Sg}(G_{\leq\beta}) = \mathrm{Sg}(G_{<\beta+1})$ is i.w.o. for all $\beta < \tau$ by the induction assumption. Let $\emptyset \neq S \subseteq \mathrm{Sg}(G_{<\tau})$ and $x \in S$. Then $x = \sum_{i=1}^{n} g_i$ where $g_i \in G_{\beta_i}$ and $\beta_i < \tau$. Thus $x \in \mathrm{Sg}(G_{\leq\beta})$ for $\beta = \max\{\beta_1, \ldots, \beta_n\}$. Moreover, $[x]_\sim = [\min\{g_1, \ldots, g_n\}]_\sim = C_\beta$ by Lemma 2.4. Let $y \in S$ such that $y \geq x$. Then $y = \sum_{j=1}^{k} h_j$ for $h_j \in G$. Thus $[\min\{h_1, \ldots, h_k\}]_\sim = [y]_\sim \geq [x]_\sim = C_\beta$. Consequently, $\min\{h_1, \ldots, h_k\} \in G_\alpha$ for some $\alpha \leq \beta$. Hence $y \in \mathrm{Sg}(G_{\leq\beta})$. Since $\mathrm{Sg}(G_{\leq\beta})$ is i.w.o. by the induction assumption, we get $\max(S) = \max(S \cap \mathrm{Sg}(G_{\leq\beta}))$.

$\square$

## 7. Construction method

In this section we shall present a method of constructing standard subdirectly irreducible ΠMTL-chains. In the next section we will show that each standard subdirectly irreducible ΠMTL-chain can be constructed by this method. Let $\mathbf{V}(\Gamma)$ be a full Hahn group. The negative cone of $\mathbf{V}(\Gamma)$ will be denoted by $V(\Gamma)^-$. We define for each $\gamma \in \Gamma$ the following set:

$$(\gamma, \rightarrow) \quad = \quad \{\alpha \in \Gamma \mid \alpha > \gamma\}\,.$$

**Definition 7.1.** Let us define an $o$-monoid $\mathbf{C}_G$ by the following steps:

(1) Let $\Gamma$ be an at most countable totally ordered set with a minimum $\gamma_0 \in \Gamma$ and $\mathbf{V}(\Gamma)$ the corresponding full Hahn group.
(2) Let $G \subseteq V(\Gamma)^-$ be any at most countable i.w.o. set.
(3) $G_0 = \{f \in V(\Gamma) \mid f(\gamma_0) \leq 0,\ f{\restriction}(\gamma_0, \rightarrow) = 0\}$.

Then $\mathbf{C}_G = (C_G, +, \leq, 0)$ is the sub-$\ell$-monoid of $\mathbf{V}(\Gamma)$ generated by $G \cup G_0$.

Observe that $G_0$ forms a submonoid of $\mathbf{V}(\Gamma)$ isomorphic to $\mathbf{R}^-$. The submonoid of $\mathbf{C}_G$ generated by $G$ will be denoted by $\mathbf{Sg}(G)$. Note that $\mathbf{Sg}(G)$ is i.w.o. by Theorem 6.5 and is at most countable because $\mathbf{Sg}(G)$ is an at most countably generated algebra (see [6, Corollary 3.6, Page 32]). Let us define the following relation on $C_G$:

$$f\ \theta_0\ g \text{ iff } \ -|f-g| \in G_0\,.$$

It can be easily seen that $\theta_0$ is an equivalence. We show that each equivalence class w.r.t. $\theta_0$ has a maximum. Let $f = \sum_{i=1}^{n} g_i$ for some $g_i \in (G \cup G_0) \setminus \{0\}$. If one of $g_i$ is from $G_0$ then $f$ cannot be maximum. Without any loss of generality suppose that $g_n \in G_0$. Then $f\ \theta_0\ f'$ for $f' = \sum_{i=1}^{n-1} g_i$ and moreover $f' > f$. Thus if $[f]_{\theta_0}$ has a maximum then this maximum belongs to $\mathbf{Sg}(G)$. Since $\mathbf{Sg}(G)$ is i.w.o., $\max{[f]_{\theta_0}} = \max{([f]_{\theta_0} \cap \mathbf{Sg}(G))}$. Let us denote this maximum by $f_m$. Each equivalence class $[f]_{\theta_0}$ is convex. Let $f < g < h$ such that $f\ \theta_0\ h$. Then $f - h < g - h < 0$ and $f - h \in G_0$. Since all functions between 0 and $f - h$ belong to $G_0$, $g - h$ also belongs to $G_0$. Thus $g\ \theta_0\ h$. Further, it is obvious that each element $g$ of $[f]_{\theta_0}$ can be decomposed as $g = f_m + z$ for some $z \in G_0$. Indeed, since $-|g - f_m| = g - f_m \in G_0$, we can set $z = g - f_m$.

Now, we show that $\mathbf{C}_G$ is residuated. It is sufficient to prove that a maximum of $M = \{h \in C_G \mid f + h \leq g\}$ exists for each $f, g \in C_G$. We denote the maximum of $M \cap \mathbf{Sg}(G)$ by $k$. If $k = \max M$ we are done. Thus assume that $k \neq \max M$. Let $h \in M$ such that $h > k$. According to the latter paragraph, we can write $h = h_m + z$ for some $h_m \in \mathbf{Sg}(G)$ and $z \in G_0$. Then $f + h_m + z \leq g < f + h_m$. Hence $g\ \theta_0\ f + h_m$, i.e. $s = g - (f + h_m) \in G_0$. Consequently, $f + h_m + s = g$ and $h_m + s = \max M$.

Let us define a structure $\mathbf{C}_G^{\rightarrow} = (C_G, +, \rightarrow, \leq, 0)$ where $(C_G, +, \leq, 0)$ is the ordered submonoid of $\mathbf{V}(\Gamma)$ and the operation $\rightarrow$ is defined as follows:

$$f \rightarrow g = \max\{h \in C_G \mid f + h \leq g\}\,.$$

**Lemma 7.2.** *The structure* $\mathbf{C}_G^{\rightarrow} = (C_G, +, \rightarrow, \leq, 0)$ *is an integral cancellative CRC. Moreover,* $\mathbf{C}_G^{\rightarrow}$ *is subdirectly irreducible.*

*Proof.* Since $\mathbf{C}_G$ is a submonoid of the group $\mathbf{V}(\Gamma)$, $\mathbf{C}_G$ is obviously cancellative. The definition of $\rightarrow$ ensures that $(+, \rightarrow)$ is a residuated pair. Moreover, since $G \cup G_0 \subseteq V(\Gamma)^-$, $\mathbf{C}_G^{\rightarrow}$ is integral. The irreducibility of $\mathbf{C}_G^{\rightarrow}$ follows from the fact that $G_0$ forms a minimum nontrivial filter and $\theta_0$ is its corresponding congruence. □

The final step in the construction is to prove that the ΠMTL-chain $\mathbf{2} \oplus \mathbf{C}_G^{\rightarrow}$ is isomorphic to a standard ΠMTL-chain. For this it suffices to show that $C_G$ is order-isomorphic to $(0, 1]$. It is known from set theory (see e.g. [15, Theorem 5.7, Page 89]) that a chain $S$ is order-isomorphic to $(0, 1)$ iff it satisfies the following conditions:

(1) $S$ is complete (each non-empty subset has a supremum and an infimum),
(2) $S$ has no minimum and no maximum,
(3) $S$ contains a countable dense subset in it.

Clearly $C_G \setminus \{0\}$ has no maximum because $G_0 \setminus \{0\}$ has no maximum. Since $\mathbf{C}_G$ is a cancellative $o$-monoid, it has no minimum. Let us denote the set of maxima of the equivalence classes w.r.t. $\theta_0$ by $M$. Since $M$ is a subset of $\mathbf{Sg}(G)$, we get that $M$ is at most countable and i.w.o. Let $D = \{g \in G_0 \mid g(\gamma_0) \in \mathbb{Q}\}$. The set $D$ is clearly countable and dense in $G_0$. Consequently, the set

$$M + D = \{f + g \mid f \in M,\ g \in D\}$$

is obviously countable and dense in $C_G$ because each element from $C_G$ can be expressed as $f + g$ for some $f \in M$ and $g \in G_0$. Finally, let $\emptyset \neq S \subseteq C_G$. As $M$ is i.w.o., we can define $m = \max\{f \in M \mid f\ \theta_0\ g \text{ for some } g \in S\}$. Then $\bigvee S = \bigvee([m]_{\theta_0} \cap S)$ since $[m]_{\theta_0} = \{m\} + G_0$ is complete. Thus $C_G \setminus \{0\}$ is order-isomorphic to $(0, 1)$ and $C_G$ to $(0, 1]$.

**Theorem 7.3.** *The integral cancellative CRC $\mathbf{C}_G^{\rightarrow} = (C_G, +, \rightarrow, \leq, 0)$ is isomorphic to a subdirectly irreducible $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$ and the* $\Pi$MTL*-chain $\mathbf{2} \oplus \mathbf{C}_G^{\rightarrow}$ is isomorphic to a standard subdirectly irreducible* $\Pi$MTL*-chain.*

## 8. Structural theorem

We are ready to prove that each subdirectly irreducible $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$ is isomorphic to some $\mathbf{C}_G^{\rightarrow}$. According to Definition 7.1 we have to define the following three sets:

(1) Let $\Gamma(\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*})$ be the chain of all values of the fraction group $\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*}$. So by Theorem 5.9 there is an $\ell$-monoidal embedding $\Phi\colon (\mathbf{0},\mathbf{1}]_* \to \mathbf{V}(\Gamma(\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*}))$.
(2) Let $G = \Phi(E_*)$ where $E_*$ is the set of product irreducible elements from Lemma 4.5.
(3) $G_0 = \{f \in V(\Gamma(\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*})) \mid f(\gamma_0) \leq 0,\ f{\restriction}(\gamma_0, \rightarrow) = 0\}$.

Further, we have to show that (1) $\Gamma(\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*})$ is at most countable and contains a minimum, (2) $G$ is at most countable and i.w.o.

(2) The set $G = \Phi(E_*)$ is at most countable and i.w.o. by Lemma 4.5.

(1) The $o$-group $\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*}$ is subdirectly irreducible by Lemma 5.2. Thus there is a minimum nontrivial convex subalgebra $V^{\gamma_0} \in \mathcal{CS}_{\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*}}$. Since $V^{\gamma_0}$ is principal, its predecessor $V_{\gamma_0}$ is the minimum of $\Gamma(\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*})$. We will denote this minimum shortly by $\gamma_0$.

**Lemma 8.1.** *The set of values $\Gamma(\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*})$ is at most countable.*

*Proof.* Let $\mathcal{N}$ be the chain of all principal convex subalgebras in $\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*}$. Since the set of values $\Gamma(\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*})$ is equipotent (via the mapping $V_g \mapsto V^g$) to the set $\mathcal{N} \setminus \{1\}$, it suffices to prove that $\mathcal{N}$ is at most countable. Let $F_\Delta$ be the nontrivial minimum filter in $(\mathbf{0},\mathbf{1}]_*$. By Theorem 3.8 we know that the set $M = \{m_x^{F_\Delta} \mid x \in L\}$ is i.w.o. Since $M \subseteq [0, 1]$, $M$ must be at most countable. By Theorem 4.3 each element $x \in (0, 1]$ can be expressed as $x = m_x^{F_\Delta} * s$ for some $s \in F_\Delta$. Thus each element in $\mathbf{G}_{(\mathbf{0},\mathbf{1}]_*}$ is of the form $a/b = (m_a^{F_\Delta} * s)/(m_b^{F_\Delta} * r) = (m_a^{F_\Delta}/m_b^{F_\Delta}) * (s/r)$ for some $s/r \in \overline{F_\Delta}$.

Let us define a mapping $\lambda : M^2 \to \mathcal{N}$ by $\lambda(m_a^{F_\Delta}, m_b^{F_\Delta}) = V^h$ where $h = m_a^{F_\Delta}/m_b^{F_\Delta}$. We will show that $\lambda(M^2) = \mathcal{N} \setminus \{\overline{F_\Delta}\}$. Let $V^g \in \mathcal{N}$ such that $V_g \neq \overline{F_\Delta}$. If $g = 1$ then we have $V^g = \{1\} = \lambda(1, 1)$. Suppose that $g \neq 1$. Then $g$ cannot belong to $\overline{F_\Delta}$. Thus $g = (m_a^{F_\Delta}/m_b^{F_\Delta}) * (s/r)$ for some $s/r \in \overline{F_\Delta}$ and $m_a^{F_\Delta}/m_b^{F_\Delta} \notin \overline{F_\Delta}$. Let $h = m_a^{F_\Delta}/m_b^{F_\Delta}$. Since $V^g, V^h \supseteq \overline{F_\Delta}$, we have $g \in V^h$ and $h \in V^g$, i.e. $V^h = V^g$. Thus $\lambda(m_a^{F_\Delta}, m_b^{F_\Delta}) = V^g$. As $M^2$ is at most countable, its image $\lambda(M^2)$ is at most countable as well (see [15, Theorem 3.4, Page 74]). Hence $\mathcal{N} = \lambda(M^2) \cup \{\overline{F_\Delta}\}$ is at most countable. □

Thus $\mathbf{C}_G^{\rightarrow} \in \mathcal{C}an\mathcal{ICRC}$ by Lemma 7.2 and it remains to prove that $(\mathbf{0},\mathbf{1}]_*$ is isomorphic to $\mathbf{C}_G^{\rightarrow}$.

**Lemma 8.2.** *We have $\Phi(F_\Delta) = G_0$.*

*Proof.* By Theorem 4.4, $\mathbf{F}_\Delta$ is isomorphic to $\mathbf{R}^-$. Thus $\mathbf{F}_\Delta$ is complete and dense. Since $V^{\gamma_0} = \overline{F_\Delta} = F_\Delta \cup F_\Delta^{-1}$, we have $f{\restriction}(\gamma_0, \rightarrow) = 0$ for all $f \in \Phi(F_\Delta)$ by Lemma 5.10. Thus $\Phi(F_\Delta) \subseteq G_0$. Suppose that $\Phi(F_\Delta) \neq G_0$. Then there is $g \in G_0 \setminus \Phi(F_\Delta)$. Let $M = \{f \in \Phi(F_\Delta) \mid f \leq g\}$. Observe that $M \neq \emptyset$ since $G_0$ forms an Archimedean $o$-monoid. Indeed, let $x \in F_\Delta \setminus \{1\}$. Then $\Phi(x) < 0$ and there is $n \in \mathbb{N}$ such that $g \geq (\Phi(x))^n = \Phi(x^n)$. As $\mathbf{F}_\Delta$ is complete, we get $\bigvee M \in \Phi(F_\Delta)$ and

$\bigwedge(\Phi(F_\Delta) \setminus M) \in \Phi(F_\Delta)$. Thus $\bigvee M < g < \bigwedge(\Phi(F_\Delta) \setminus M)$ by our assumption. However, it is a contradiction with the fact that $\mathbf{F}_\Delta$ is dense. Hence $\Phi(F_\Delta) = G_0$. $\square$

**Theorem 8.3.** *The algebra $(\mathbf{0},\mathbf{1}]_*$ is isomorphic to $\mathbf{C}_G^{\rightarrow}$.*

*Proof.* We will show that $\Phi$ is an isomorphism between $(\mathbf{0},\mathbf{1}]_*$ and $\mathbf{C}_{G_*}^{\rightarrow}$. First, we prove $\Phi((0,1]) \subseteq C_G$. Let $x \in (0,1]$. Then $x = g_1 * \cdots * g_n * s$ for some $g_i \in E_*$ and $s \in F_\Delta$ by Theorem 4.6. Thus $\Phi(x) = \Phi(g_1 * \cdots * g_n * s) = \Phi(g_1) + \cdots + \Phi(g_n) + \Phi(s)$. Since $\Phi(g_i) \in G$ and by Lemma 8.2 $\Phi(s) \in G_0$, we have $\Phi(x) \in C_G$.

Now we prove that $\Phi$ is onto $C_G$. Let $h \in C_G$. Then $h = (\sum_{i=1}^n f_i) + g$ for some $f_i \in G$ and $g \in G_0$. By definition of $G$, there are elements $x_i \in (0,1]$ such that $\Phi(x_i) = f_i$. By Lemma 8.2 there is $s \in F_\Delta$ such that $\Phi(s) = g$. Let us take $x = x_1 * \cdots * x_n * s$. Since $\Phi$ is an embedding, we have $\Phi(x) = \Phi(x_1) + \cdots + \Phi(x_n) + \Phi(s) = (\sum_{i=1}^n f_i) + g = h$. Thus $\Phi$ is onto. Hence $\Phi$ is an isomorphism between the $\rightarrow$-free reduct of $(\mathbf{0},\mathbf{1}]_*$ and $\mathbf{C}_G$. Consequently, $(\mathbf{0},\mathbf{1}]_* \cong \mathbf{C}_G^{\rightarrow}$, as CRLs. $\square$

**Corollary 8.4.** *Each standard subdirectly irreducible* $\Pi$MTL*-chain* $\mathbf{L}$ *is isomorphic to* $\mathbf{2} \oplus \mathbf{C}_G^{\rightarrow}$ *for some at most countable i.w.o. subset $G \subseteq \Gamma$ where $\Gamma$ is an at most countable chain with a minimum.*

## 9. Subdirectly reducible integral cancellative CRCs on $(0,1]$

In this section we will focus on the structure of subdirectly reducible chains.

**Lemma 9.1.** *Let $\mathbf{C} \in \mathcal{CanICRC}$ be a complete subdirectly reducible CRC and $F \in \mathcal{F}_{\mathbf{C}}$ a nontrivial filter. Then $\mathbf{C}/F$ is subdirectly irreducible and* Con $\mathbf{C}/F$ *is well ordered.*

*Proof.* If $F = C$ then $\mathbf{C}/F$ is trivial and the statement clearly holds. Assume that $F \neq C$. Since each nontrivial filter $F \in \mathcal{F}_{\mathbf{C}}$ has a successor by Lemma 3.2, $\mathbf{C}/F$ is subdirectly irreducible. As $\mathbf{C}/F$ is complete by Lemma 3.6, Con $\mathbf{C}/F$ is well ordered by Theorem 3.7. $\square$

For a subdirectly irreducible $(\mathbf{0},\mathbf{1}]_* \in \mathcal{CanICRC}$ we have that Con $(\mathbf{0},\mathbf{1}]_*$ is well ordered by Theorem 3.7. For subdirectly reducible algebras we have the following result. Let $\omega^*$ be the inversely ordered set of natural numbers.

**Theorem 9.2.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{CanICRC}$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly reducible then* Con $(\mathbf{0},\mathbf{1}]_* \setminus \{\Delta\}$ *has the same order type as a subset of the lex. product $\omega^* \times \alpha$ where $\alpha$ is an at most countable ordinal. In particular,* Con $(\mathbf{0},\mathbf{1}]_*$ *is countable.*

*Proof.* Let $\theta_n$ be a decreasing sequence such that $\bigcap_{n\in\mathbb{N}} \theta_n = \Delta$. There is such a sequence. Indeed, let $\langle a_n \rangle_{n\in\mathbb{N}}$, $a_n \in (0,1]$, be an increasing sequence such that $\bigvee_{n\in\mathbb{N}} a_n = 1$. Then $\langle F^{a_n} \rangle_{n\in\mathbb{N}}$ is a decreasing sequence of filters whose limit is $\Delta$ because $(\mathbf{0},\mathbf{1}]_*$ is subdirectly reducible. Thus we can set $\theta_n = \theta_{F^{a_n}}$.

Let $I_0 = (\theta_0, \nabla]$ and $I_n = (\theta_n, \theta_{n-1}]$ for $n \geq 1$. Then Con $(\mathbf{0},\mathbf{1}]_* \setminus \{\Delta\} = \bigcup_{n\in\mathbb{N}} I_n$. Since Con $(\mathbf{0},\mathbf{1}]_*/\theta_n \cong [\theta_n, \nabla]$ is well ordered for all $n \in \mathbb{N}$ by Lemma 9.1, we get that $I_n \subseteq [\theta_n, \nabla]$ is well ordered. Moreover, Con $(\mathbf{0},\mathbf{1}]_*/\theta_n$ is at most countable by the same reasoning as in the proof of Lemma 4.1. Thus each $I_n$ is at most countable as well. Let $\alpha$ be the supremum of all ordinals order-isomorphic to the intervals $I_n$. Since all such intervals are at most countable and there are countable many of them, $\alpha$ is at most countable as well. Each $I_n$ can be order-embedded into $\alpha$ by some $\Phi_n\colon I_n \to \alpha$. If we define a mapping $\Phi\colon$ Con $(\mathbf{0},\mathbf{1}]_* \setminus \{\Delta\} \to \omega^* \times \alpha$ by $\Phi(\theta) = \langle n, \Phi_n(\theta) \rangle$ for $\theta \in I_n$, we get an order-preserving injection. $\square$

Now we prove that if $\mathbf{C} \in \mathcal{CanICRC}$ is not subdirectly irreducible then its fraction group $\mathbf{G_C}$ has the same property.

**Lemma 9.3.** *Let $\mathbf{C} \in \mathcal{CanICRC}$. If $\mathbf{C}$ is subdirectly reducible then*

$$\bigcap_{F\in\mathcal{F}_{\mathbf{C}},\ F\neq\{\mathbf{1}\}} \overline{F} = \Delta\,,$$

*where $\Delta$ is the minimum of* Con $\mathbf{G_C}$.

*Proof.* Suppose that there is a nontrivial convex subalgebra $V$ such that $V \subseteq \overline{F}$ for all $F \in \mathcal{F}_{\mathbf{C}}$, $F \neq \{\mathbf{1}\}$. Thus there is an element $a/b \in V$ such that $a < b$ and $a/b > x$ for all $x \in C \setminus \{\mathbf{1}\}$, i.e. $b * x < b * (a/b) = a$. Since $\mathbf{C}$ is subdirectly reducible it cannot possess a coatom $c$ otherwise $F^c$ would be the minimum nontrivial filter. Thus $\bigvee(C \setminus \{\mathbf{1}\}) = \mathbf{1}$. Consequently, we have for all $x \in C \setminus \{\mathbf{1}\}$

$$b = b * \bigvee(C \setminus \{\mathbf{1}\}) = \bigvee_{x \in C \setminus \{\mathbf{1}\}} (b * x) \leq a\,,$$

a contradiction with the fact that $a < b$. □

By Lemma 9.3 there is no minimum nontrivial congruence in Con $\mathbf{G_C}$. Thus $\mathbf{G_C}$ cannot be subdirectly irreducible.

**Theorem 9.4.** *Let $\mathbf{C} \in \mathcal{C}an\mathcal{ICRC}$. If $\mathbf{C}$ is subdirectly reducible then $\mathbf{G_C}$ is subdirectly reducible as well.*

If $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$ is subdirectly irreducible then it follows from Corollary 3.5 that the set

$$\{m_x^F \in (0,1] \mid x \in (0,1],\ F \in \mathcal{F}_{(\mathbf{0},\mathbf{1}]_*},\ F \neq \{1\}\} = \{m_x^{F_\Delta} \mid x \in (0,1]\}\,.$$

Thus this set is i.w.o. by Theorem 3.8. This is not the case if $(\mathbf{0},\mathbf{1}]_*$ is subdirectly reducible.

**Proposition 9.5.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$. If $(\mathbf{0},\mathbf{1}]_*$ is subdirectly reducible then the set*

$$M = \{m_x^F \in (0,1] \mid x \in (0,1],\ F \in \mathcal{F}_{(\mathbf{0},\mathbf{1}]_*},\ F \neq \{1\}\}$$

*is dense in* $(0,1]$.

*Proof.* Let $m_1, m_2 \in M$ and $m_1 > m_2$. Then $z = m_1 \to m_2$ belongs to the principal filter $F^z$. Let $F_z$ be the predecessor of $F^z$. Since $\mathbf{L}$ is subdirectly reducible, there exists a filter $F$ such that $F \subsetneq F_z$. Let $s$ be an element from $F_z \setminus F$ and $m = \max\,[m_1 * s]_F$. Then $m_1 \to m_1 * s = s \notin F$. Thus $m_1 \notin [m_1 * s]_F$ and consequently $m < m_1$. Since $s > z$, we have $m \geq m_1 * s > m_2$. Thus $m > m_2$ and the proof is done. □

Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$ be a subdirectly reducible CRC. It is a well-known fact from Universal Algebra that if we have an indexed family $\theta_i \in$ Con $(\mathbf{0},\mathbf{1}]_*$ such that $\bigcap_{i \in I} \theta_i = \Delta$, then the natural homomorphism

$$\nu : (\mathbf{0},\mathbf{1}]_* \to \prod_{i \in I} (\mathbf{0},\mathbf{1}]_*/\theta_i$$

defined by $\nu(x)(i) = [x]_{\theta_i}$ is a subdirect embedding (see [6, Lemma 8.2, Page 57]). We know from the proof of Theorem 9.2 that Con $(\mathbf{0},\mathbf{1}]_*$ contains a non-increasing sequence $\langle \theta_n \rangle_{n \in \mathbb{N}}$ such that $\bigcap_{n \in \mathbb{N}} \theta_n = \Delta$ and $\theta_n \neq \Delta$. In the following theorem we show that the structure of each factor $(\mathbf{0},\mathbf{1}]_*/\theta_n$ can be described by means of Theorem 8.3. Thus we obtain a characterization of the structure of $(\mathbf{0},\mathbf{1}]_*$ since $(\mathbf{0},\mathbf{1}]_*$ can be subdirectly embedded into the direct product $\prod_{n \in \mathbb{N}} (\mathbf{0},\mathbf{1}]_*/\theta_n$.

**Theorem 9.6.** *Let $(\mathbf{0},\mathbf{1}]_* \in \mathcal{C}an\mathcal{ICRC}$, $\theta \in$ Con $(\mathbf{0},\mathbf{1}]_*$, and $\theta \neq \Delta$. Then $(\mathbf{0},\mathbf{1}]_*/\theta$ is isomorphic to some $\mathbf{C}_G^{\rightarrow}/\theta_\Delta$ where $\theta_\Delta = \min(\text{Con } \mathbf{C}_G^{\rightarrow} \setminus \{\Delta\})$.*

*Proof.* Let $(\mathbf{0},\mathbf{1}]_\Pi$ be the integral cancellative CRC such that $\mathbf{2} \oplus (\mathbf{0},\mathbf{1}]_\Pi$ is isomorphic to the standard product algebra $[\mathbf{0},\mathbf{1}]_\Pi$. Then $\mathbf{A} = ((\mathbf{0},\mathbf{1}]_*/\theta) \overset{\rightarrow}{\times} (\mathbf{0},\mathbf{1}]_\Pi$ is an integral cancellative CRC by Proposition 1.16. It is easy to verify that the mapping $\Psi : (0,1]/\theta \to A$ defined by $\Psi(x) = \langle x, 1 \rangle$ is an embedding of $(\mathbf{0},\mathbf{1}]_*/\theta$ into $\mathbf{A}$.

Further, $A' = A \setminus \{\langle 1,1 \rangle\}$ is order-isomorphic to $(0,1)$ since $A'$ has no minimum and no maximum, $\{\langle a,x \rangle \in A' \mid x \in \mathbb{Q}\}$ is countable and dense in $A'$ (because $(0,1]/\theta$ is at most countable), and each non-empty subset of $A'$ has a supremum (because $(\mathbf{0},\mathbf{1}]_*/\theta$ is i.w.o. and $(0,1]$ is complete). Thus $\mathbf{A}$ is isomorphic to some cancellative integral CRC $(\mathbf{0},\mathbf{1}]_\odot = ((0,1], \odot, \to, \min, \max, 1)$. It is easy to see that $F_\Delta = \{\langle \mathbf{1}, r \rangle \in A \mid r \in (0,1]\}$ is the minimum nontrivial filter in $\mathbf{A}$ and $\mathbf{A}/F_\Delta \cong (\mathbf{0},\mathbf{1}]_*/\theta$. Hence $\mathbf{A}$ is subdirectly irreducible. Finally, $\mathbf{A}$ is isomorphic to some $\mathbf{C}_G^{\rightarrow}$ by Theorem 8.3. Hence $(\mathbf{0},\mathbf{1}]_*/\theta \cong \mathbf{A}/F_\Delta \cong \mathbf{C}_G^{\rightarrow}/\theta_\Delta$. □

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