Finite Embeddability Property for Residuated Lattices via Regular Languages

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Abstract Let V be a variety of residuated lattices axiomatized by a set of identites in the language $\{\vee, \cdot, 1\}$. We characterize when V has the finite embeddability property via regularity of a certain collection of languages. Several applications of this characterization are presented.

1 Introduction

A class of algebras K has the finite embeddability property (FEP) if every finite subset of any of its members has an isomorphic copy in a finite member of K. Finitely axiomatizable classes of algebras with this property have decidable universal theories. Due to this reason there are several decidability proofs in the literature which rely on this property. Likely one of the first proofs using the FEP (without an explicit definition of this property) was done in 1940's by McKinsey. He proved in [20] that the modal logics S2 and S4 are decidable by showing that the corresponding algebraic semantics has the FEP. A few years later McKinsey and Tarski showed that the variety of closure algebras (i.e., Boolean algebras with a closure operator) has the FEP and concluded that its universal theory is decidable [21]. Later they used this result in order to show that the variety of Heyting algebras has decidable universal theory [22]. In 1950's Henkin mentioned in [11] that the variety of abelian groups has the FEP (he attributed the result to John Tate). Afterward he introduced (likely for the first time) explicitly the notion of FEP under the name *the finite imbedding property* (see [12]). In late 1960's Evans introduced in [8] the same notion again under the name *finite embeddability property*. He also gave a characterization of this property for varieties of algebras. Namely, a variety has the FEP iff its finitely presented algebras are residually finite.

In this chapter we deal with the FEP for classes of residuated lattices. Residuated lattices form an algebraic semantics for substructural logics. Roughly speaking, they play the same role as Boolean algebras in classical logic and Heyting algebras in intuitionistic logic (see [10]). Although the above-mentioned McKinsey and Tarski's result on Heyting algebras can be viewed as a result dealing with special residuated lattices, the first explicit results on the FEP for classes of residuated lattices were proved by Blok and van Alten. In [1] they proved that the variety of integral commutative residuated lattices has the FEP. They also gave a characterization of the FEP for classes closed under finite direct products. Namely, such a class K has the FEP iff its quasi-equational theory has the finite model property (i.e., every quasi-identity which fails in K fails in a finite member of K). Later they generalized this result to various classes of residuated groupoids (see [2]). Among others results on the FEP let us mention [25] where it is shown that the variety of commutative residuated lattices satisfying $x^m \leq x^n$ for $m \neq n$ has the FEP. The paper [9] contains a generalization of the result from [1]. Namely, [9] proves that every variety of integral residuated lattices axiomatized by identities in the language $\{\vee, \cdot, 1\}$ has the FEP. Last but not least there is a recent result

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by Cardona and Galatos [3] where the FEP was established for varieties satisfying $x^m \leq x^n$ and having a weaker form of commutativity.

On the other hand, there are also negative results. Usually non-integral varieties do not enjoy the FEP. For instance, the variety of all residuated lattices does not enjoy the FEP for the simple reason since its universal theory is undecidable. In fact, even word problem is undecidable for the variety of all residuated lattices (see [16]). The same holds for the variety of commutative residuated lattices as follows from results on linear logic in [18]. Further, it follows from the result in [1] that every class of residuated lattices containing the abelian lattice-ordered group on integers with addition and the usual linear order cannot have the FEP (see also [10]).

The above-mentioned list of positive and negative results on the FEP for classes of residuated lattices is not for sure exhaustive. Nevertheless it might suggest that we are closer to delimit the border between varieties of residuated lattices having the FEP and varieties without this property. This chapter is an attempt in this direction. For any variety V of residuated lattices axiomatized by a set of identities in the language $\{\vee, \cdot, 1\}$ it provides sufficient and necessary conditions for V to have the FEP. Another interesting fact worth to stress consists in a tight connection between the FEP and the theory of regular languages. Namely, we are going to characterize the FEP for V via regularity of certain finite collections of languages. Due to this many of currently known results can be easily proved using various regularity conditions from the formal language theory as is shown in the last section of this chapter. Although the chapter is rather a compilation of already known facts, the connection with the theory of regular languages and our characterization of the FEP seem to be new.

The presented material is heavily based on the results from [4] and [9]. Both these papers use residuated frames as a tool for proving their results. Although the content of this chapter could be fully presented in terms of residuated frames, we decided to work rather with collections of languages. The reason for this is that we believe that in this way the connection with formal languages is more transparent. We should also point out that the characterization of FEP provided in this chapter might be done for any quasi-variety of FL-algebras (i.e., pointed residuated lattices) axiomatized by a set of analytic structural quasi-identities. In order to be less technical, we restrict ourselves only to varieties of residuated lattices axiomatized by a set of identities in the signature $\{\vee, \cdot, 1\}$.

2 Finite embeddability property

We recall the definition of the FEP together with a couple of auxiliary definitions. Then we prove that the FEP for a class of algebras K is equivalent to the finite model property for its universal theory.

Let $\mathbf{A} = \langle A, \langle f_i^{\mathbf{A}} \mid i \in K \rangle\rangle$ be an algebra and $B \subseteq A$. Then $\mathbf{B} = \langle B, \langle f_i^{\mathbf{B}} \mid i \in K \rangle\rangle$ is a *partial subalgebra* of $\mathbf{A}$ where for every n-ary operation f_i, $i \in K$, we define

$$f_i^{\mathbf{B}}(a_1,\dots,a_n) = \begin{cases} f_i^{\mathbf{A}}(a_1,\dots,a_n) & \text{if } f_i^{\mathbf{A}}(a_1,\dots,a_n) \in B, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Given an algebra $\mathbf{C}$ of the same type as $\mathbf{A}$ and a one-to-one map $g\colon B \to C$, we call g an *embedding* of $\mathbf{B}$ into $\mathbf{C}$ if g preserves all existing operations, i.e., for every n-ary operation f_i, $i \in K$, and $a_1,\dots,a_n \in B$ we have

$$g(f_i^{\mathbf{B}}(a_1,\dots,a_n)) = f_i^{\mathbf{C}}(g(a_1),\dots,g(a_n)),$$

whenever $f_i^{\mathbf{B}}(a_1,\dots,a_n)$ is defined. Finally, we say that a partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ is *embeddable* into $\mathbf{C}$ if there is an embedding $g\colon B \to C$.

Definition 1. Let K be a class of algebras of the same type. Then K is said to have the *finite embeddability property* (FEP) if every finite partial subalgebra $\mathbf{B}$ of any member $\mathbf{A} \in \mathsf{K}$ is embeddable into a finite member $\mathbf{C} \in \mathsf{K}$.

Let K be a class of algebras of the same type. Consider its universal theory $\mathrm{Th}_\forall(\mathsf{K})$, i.e., the set of all universal sentences valid in K. We say that $\mathrm{Th}_\forall(\mathsf{K})$ has the *finite model property* if every universal sentence Φ which fails to hold in K fails in a finite member of K. The importance of the FEP stems from the following theorem.

Theorem 1. *A class of algebras* K *of the same type has the FEP iff* $\mathrm{Th}_\forall(\mathsf{K})$ *has the finite model property.*

Proof. ($\Rightarrow$): Suppose that K has the FEP and a universal formula $\forall x_1 \dots x_n \Phi(x_1,\dots,x_n)$ fails in K. Note that Φ is a Boolean combination of atomic formulas of the form $t(x_1,\dots,x_n) = s(x_1,\dots,x_n)$ for some terms t,s. Since $\forall x_1 \dots x_n \Phi(x_1,\dots,x_n)$ fails in K, there are $\mathbf{A} \in \mathsf{K}$ and elements $a_1,\dots,a_n \in A$ such that $\mathbf{A} \not\models \Phi(a_1,\dots,a_n)$. Consider a set T of all subterms occurring in atomic subformulas of Φ. We define a subset $B \subseteq A$ as follows

$$B = \{t^{\mathbf{A}}(a_1,\dots,a_n) \in A \mid t(x_1,\dots,x_n) \in T\}.$$

By the FEP $\mathbf{B}$ can be embedded into a finite member $\mathbf{C} \in \mathsf{K}$ via an embedding g. Then we claim that $\mathbf{C} \not\models \Phi(g(a_1),\dots,g(a_n))$ which implies that $\forall x_1 \dots x_n \Phi(x_1,\dots,x_n)$ fails in $\mathbf{C}$. To see this observe that for any atomic subformula $t = s$ occurring in Φ we have

$$\begin{aligned} t^{\mathbf{A}}(a_1,\dots,a_n) = s^{\mathbf{A}}(a_1,\dots,a_n) \quad &\text{iff} \quad g(t^{\mathbf{A}}(a_1,\dots,a_n)) = g(s^{\mathbf{A}}(a_1,\dots,a_n)) \\ &\text{iff} \quad t^{\mathbf{C}}(g(a_1),\dots,g(a_n))) = s^{\mathbf{C}}(g(a_1),\dots,g(a_n)), \end{aligned}$$

because g is an embedding, i.e., it is injective and preserves existing operations.

($\Leftarrow$): Let $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A}$. Suppose that $B = \{b_1,\dots,b_n\}$, where $b_i \neq b_j$ for $1 \leq i < j \leq n$. Consider a formula $\Psi(x_1,\dots,x_n)$ which is a conjunction of all atomic formulas of the form

$$f(x_{i_1},\dots,x_{i_k}) = x_{i_j},$$

where

$$f^{\mathbf{B}}(b_{i_1},\dots,b_{i_k}) = b_{i_j}$$

for an operation f in the signature of $\mathbf{B}$. The following sentence

$$\forall x_1,\dots,x_n \left(\Psi(x_1,\dots,x_n) \Rightarrow \bigvee_{1\leq i<j\leq n} (x_i = x_j) \right)$$

fails in K because $\mathbf{A} \models \Psi(b_1,\dots,b_n)$ and $\mathbf{A} \not\models \bigvee_{1\leq i<j\leq n}(b_i = b_j)$. By the finite model property there is a finite member $\mathbf{C} \in \mathsf{K}$ and elements $c_1,\dots,c_n \in C$ such that $\mathbf{C} \models \Psi(c_1,\dots,c_n)$ and $\mathbf{C} \models \bigwedge_{1\leq i<j\leq n}(c_i \neq c_j)$. Thus $b_i \mapsto c_i$ for $1 \leq i \leq n$ defines an embedding of $\mathbf{B}$ into $\mathbf{C}$. $\square$

Corollary 1. *Let* K *be a finitely axiomatized class of algebras of the same type. If* K *has the FEP then* $\mathrm{Th}_\forall(\mathsf{K})$ *is decidable.*

3 Regular languages and syntactic congruences

Since the notion of regular language will play an important role in our characterization of the FEP, we recall several well-known facts on regular languages.

Throughout the rest of this chapter B always denotes a finite set which is called *alphabet*. The set of all finite words (i.e., finite sequences) over B is denoted B^*. Recall that B^* together with empty word ε and concatenation forms the B-generated free monoid which we denote $\mathbf{B}^* = \langle B^*, \cdot, \varepsilon\rangle$. A subset $L \subseteq B^*$ is called a *language*. Given languages $L, K \subseteq B^*$, one can define their concatenation as follows

$$LK = \{uv \in B^* \mid u \in L,\ v \in K\}. \tag{1}$$

This operation is associative so we can define powers of a language L by letting $L^0 = \{\varepsilon\}$ and $L^{n+1} = LL^n$. The union of all L^n is known as *Kleene-star* of the language L, i.e.,

$$L^* = \bigcup_{n \in \mathbb{N}} L^n. \tag{2}$$

Definition 2. The collection of *regular languages* over B is defined as the smallest collection containing all finite languages $L \subseteq B^*$ and closed under concatenation, union and Kleene-star.

It follows from the above definition that $\{b\}$ is a regular language for every $b \in B$. Other regular languages can be constructed by means of concatenation, union and Kleene-star. For instance,

$$(\{a\}(\{a\} \cup \{b\})^*) \cup \{b\}^*$$

is a regular language. In what follows we will abuse notation in the usual way and omit symbols $\{,\}$ in the above expression. In addition, we assume that Kleene-star binds stronger than concatenation which in turn binds stronger than union. Thus the above expression is simplified as follows

$$a(a \cup b)^* \cup b^*.$$

Note that b might stand in different contexts for an element from B and also for the singleton language $\{b\}$. Similarly ε denotes the empty word and at the same time the language $\{\varepsilon\}$.

Let $L \subseteq B^*$ be a language and $\sim$ a monoid congruence on $\mathbf{B}^*$. The language L is said to be *saturated by* $\sim$ if L is a union of congruence classes, i.e., $a \in L$ and $a \sim_L b$ implies $b \in L$. We call L *recognizable* if there is a monoid congruence $\sim$ on $\mathbf{B}^*$ saturating L and having finite index (i.e., the quotient $\mathbf{B}^*/{\sim}$ is finite). Equivalently, L is recognizable if there is a finite monoid $\mathbf{M}$, a monoid homomorphism $h\colon B^* \to M$ and $S \subseteq M$ such that $h^{-1}[S] = L$.

Recognizable languages are precisely languages accepted by finite automata (see e.g. [19]). Thus the following result can be viewed as a variant of famous Kleene's theorem [17].

Theorem 2. *Let $L \subseteq B^*$. Then L is regular iff it is recognizable.*

Next we recall the notion of syntactic congruence. This notion plays a crucial role in the algebraic theory of regular languages (see [23]). Let $L \subseteq B^*$ be a *language*. The *syntactic congruence* of the language L is defined as follows:

$$x \sim_L y \quad \text{iff} \quad \forall u,v \in B^*(uxv \in L \Longleftrightarrow uyv \in L). \tag{3}$$

The syntactic congruence is also known in Abstract Algebraic Logic under the name *Leibniz congruence* (see [5]). The syntactic congruence can be characterized as follows (see e.g. [19] or [5]).

Theorem 3. *Let $L \subseteq B^*$. The syntactic congruence $\sim_L$ is the largest monoid congruence (with respect to the inclusion) which saturates L.*

Note that it follows from Theorem 3 that $L \subseteq B^*$ is recognizable iff $\mathbf{B}^*/{\sim_L}$ is finite. Thus using Theorem 2, we obtain the following characterization of regular languages.

Theorem 4. *A language $L \subseteq B^*$ is regular iff $\mathbf{B}^*/{\sim_L}$ is finite.*

In order to understand the syntactic congruence of a language $L \subseteq B^*$ better, we introduce the following languages for every $u,v \in B^*$:

$$u\backslash L/v = \{x \in B^* \mid uxv \in L\}. \tag{4}$$

Then the syntactic congruence $\sim_L$ can be defined as follows:

$$x \sim_L y \quad \text{iff} \quad \forall u,v \in B^*(x \in u\backslash L/v \Longleftrightarrow y \in u\backslash L/v). \tag{5}$$

Consequently, $x,y \in B^*$ are equivalent iff they belong to the same languages $u\backslash L/v$. Thus congruence classes are determined by subsets of the collection

$$\mathcal{B} = \{u\backslash L/v \mid u,v \in B^*\}.$$

More precisely, if we define for $x \in B^*$ a set $E_x = \{\langle u,v\rangle \in B^* \times B^* \mid x \in u\backslash L/v\}$, we can describe the congruence class $[x]_{\sim_L}$ as follows:

$$[x]_{\sim_L} = \left(\bigcap_{\langle u,v\rangle \in E_x} u\backslash L/v\right) - \left(\bigcup_{\langle u,v\rangle \notin E_x} u\backslash L/v\right). \tag{6}$$

In the above formula we adopt the convention $\bigcap \emptyset = B^*$. Now we can reformulate Theorem 4 in the following way.

Theorem 5. *Let $L \subseteq B^*$ be a language. Then L is regular iff the collection $\mathcal{B} = \{u\backslash L/v \mid u,v \in B^*\}$ is finite.*

Proof. ($\Rightarrow$): Since L is regular, its syntactic congruence $\sim_L$ is of finite index by Theorem 4. Consequently, there are only finitely many languages which are saturated by $\sim_L$ (precisely 2^n where n is the cardinality of $\mathbf{B}^*/{\sim_L}$). Thus it suffices to show that every $u\backslash L/v$ is saturated by $\sim_L$. Let $x \in u\backslash L/v$ and $x \sim_L y$. We have $uxv \in L$. It follows from the definition (3) of $\sim_L$ that $uyv \in L$. Hence $y \in u\backslash L/v$.

($\Leftarrow$): Assume that $|\mathcal{B}| = n$. Since the congruence classes of $\sim_L$ are determined by subsets of $\mathcal{B}$ there are at most 2^n congruence classes. Thus L is regular by Theorem 4. □

Example 1. Let $B = \{a,b\}$. We compute the syntactic monoid of language $L = a(a\cup b)^*b = aB^*b$. First, we have to compute languages $\mathcal{B} = \{u\backslash L/v \mid u,v \in B^*\}$. By Theorem 5 we know that there are only finitely many different languages of this form. Namely, we have six pairwise different languages

$$\begin{aligned}
\varepsilon\backslash L/\varepsilon &= L \\
a\backslash L/\varepsilon &= B^*b \\
\varepsilon\backslash L/b &= aB^* \\
b\backslash B^*b/\varepsilon &= B^*b\cup\varepsilon \\
\varepsilon\backslash B^*b/b &= B^* \\
\varepsilon\backslash aB^*/a &= aB^*\cup\varepsilon.
\end{aligned}$$

Figure 1 shows how the above languages are ordered by inclusion.

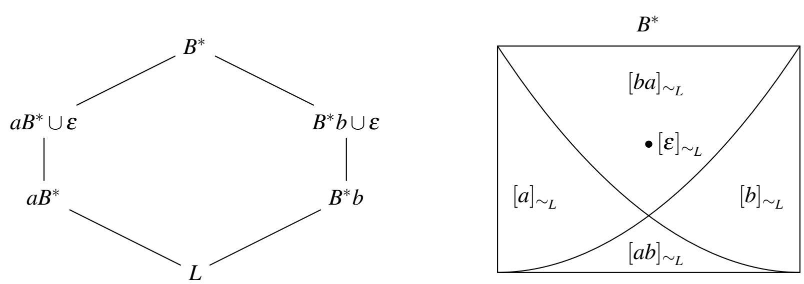


Fig. 1 (Left) The collection $\mathcal{B} = \{u\backslash L/v \mid u,v \in B^*\}$ ordered by inclusion. (Right) Congruence classes of $\sim_L$. The parabolas separate respectively aB^* and B^*b from the rest of B^*.

Now we can describe congruence classes of $\sim_L$ corresponding to various subsets of $\mathcal{B}$. Namely, using the formula (6), we obtain the following congruence classes (see Figure 1):

$$\begin{aligned}
[ab]_{\sim_L} &= \bigcap \mathcal{B} = L \\
[a]_{\sim_L} &= aB^* - L \\
[b]_{\sim_L} &= B^*b - L \\
[\varepsilon]_{\sim_L} &= (aB^*\cup\varepsilon)\cap(B^*b\cup\varepsilon) = \{\varepsilon\} \\
[ba]_{\sim_L} &= B^* - (aB^*\cup B^*b\cup\varepsilon).
\end{aligned}$$

4 Residuated lattices induced by a collection of languages

In this section we recall all necessary notions and facts about residuated lattices. Moreover, we describe a construction which produces a residuated lattice out of a finite collection of languages. Then we prove that this residuated lattice is finite iff the collection contains regular languages.

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid and for all $a, b, c \in A$ we have

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a\backslash c \quad \text{iff} \quad a \leq c/b. \tag{7}$$

We will omit some parentheses in residuated-lattice terms. Namely, in absence of parentheses we assume that the multiplication $\cdot$ is performed first, followed by divisions $\backslash$, $/$ and finally by the lattice connectives $\wedge$, $\vee$. Given residuated-lattice terms t, s, we refer to the inequality $s \leq t$ as identity because it is equivalent to $s \vee t = t$. A residuated lattice $\mathbf{A}$ is said to be *complete* if its lattice reduct is a complete lattice. For details on residuated lattices see [10].

It follows from the above condition that the multiplication is monotone in both arguments. The divisions $\backslash$, $/$ are monotone in nominator and antitone in denominator. In fact, we have even the following distribution laws:

- Multiplication distributes over any existing join, i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subseteq A$, then so does $\bigvee_{x\in X, y\in Y}(x \cdot y)$, and
$$(\bigvee X) \cdot (\bigvee Y) = \bigvee_{x\in X, y\in Y} (x \cdot y). \tag{8}$$
- Divisions preserve all existing meets in the numerator and convert all existing joins in the denominator to meets, i.e., if $\bigvee X$ and $\bigwedge Y$ exist for $X, Y \subseteq A$, then for any $z \in A$ the following equalities hold (in particular the right-hand sides exist):
$$z\backslash(\bigwedge Y) = \bigwedge_{y\in Y} (z\backslash y), \qquad (\bigwedge Y)/z = \bigwedge_{y\in Y} (y/z), \tag{9}$$
$$(\bigvee X)\backslash z = \bigwedge_{x\in X} (x\backslash z), \qquad z/(\bigvee X) = \bigwedge_{x\in X} (z/x). \tag{10}$$

Lemma 1. *The following holds in every residuated lattice:*

1. $a \cdot (a\backslash b) \leq b$ *and* $(b/a) \cdot a \leq b$,
2. $(a \cdot b)\backslash c = b\backslash(a\backslash c)$ *and* $c/(b \cdot a) = (c/a)/b$,
3. $(a\backslash c)/b = a\backslash(c/b)$ *so we can write* $a\backslash c/b$ *without parentheses,*
4. $1\backslash a = a = a/1$.

Let B be a set. It is known (see e.g. [10]) that the powerset of free monoid $\mathbf{B}^*$ forms a complete residuated lattice $\mathcal{P}(\mathbf{B}^*) = \langle B^*, \cap, \cup, \cdot, \backslash, /, \varepsilon\rangle$, where for $U, V, W \subseteq B^*$ the multiplication UV is just the concatenation of languages (1) and

$$U\backslash W = \{v \in B^* \mid Uv \subseteq W\},$$
$$W/V = \{u \in B^* \mid uV \subseteq W\}.$$

Note that in the above definition ε stands for the language $\{\varepsilon\}$. Similarly Uv and uV denote respectively languages $U\{v\}$ and $\{u\}V$. Let $u, v \in B^*$ and $L \subseteq B^*$. We will use a similar convention also for divisions, namely the division of the form $\{u\}\backslash L/\{v\}$ is denoted shortly $u\backslash L/v$. Note that the set $u\backslash L/v$ is precisely the set defined in (4). Further, observe that the free monoid $\mathbf{B}^*$ embeds (as a monoid) into $\mathcal{P}(\mathbf{B}^*)$ via $x \mapsto \{x\}$.

We have seen that the collection of all languages forms a complete residuated lattice. Also other collections form a complete residuated lattice. In fact, every collection $\mathcal{L}$ of languages over B^* induces a collection which forms a complete residuated lattice $\mathcal{L}(\mathbf{B}^*)$. Let $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ be a collection of languages over B^* indexed by the set B. Although the most of following considerations works for every collection of languages, we restrict ourselves only to collections indexed by the elements of B. This is precisely what we need in the sequel.

Before we describe the construction of $\mathcal{L}(\mathbf{B}^*)$, we recall the notion of a closure operator. A self-map $\gamma\colon \mathcal{P}(B^*) \to \mathcal{P}(B^*)$ is called *closure operator* if for every $X, Y \subseteq B^*$ it satisfies the following conditions (see e.g. [6]):

- $\gamma(\gamma(X)) = \gamma(X)$,
- $X \subseteq \gamma(X)$,
- $X \subseteq Y$ implies $\gamma(X) \subseteq \gamma(Y)$.

A language $X \subseteq B^*$ is said to be γ-closed if $\gamma(X) = X$. The collection of all γ-closed languages is denoted $\mathcal{P}(B^*)_\gamma$. A subset $\mathcal{B} \subseteq \mathcal{P}(B^*)_\gamma$ is referred to as *basis* for γ if every γ-closed language is a (possibly infinite) intersection of languages from $\mathcal{B}$. Suppose that $\mathcal{B}$ is a basis for γ. Then for each language $X \subseteq B^*$ we have

$$\gamma(X) = \bigcap\{L \subseteq \mathcal{B} \mid X \subseteq L\}. \tag{11}$$

Observe also that for every $X, Y \subseteq B^*$ we have

$$\gamma(X) \subseteq \gamma(Y) \quad \text{iff} \quad \forall L \in \mathcal{B}(Y \subseteq L \Longrightarrow X \subseteq L). \tag{12}$$

Hence the kernel of γ can be described as follows:

$$\gamma(X) = \gamma(Y) \quad \text{iff} \quad \forall L \in \mathcal{B}(X \subseteq L \Longleftrightarrow Y \subseteq L). \tag{13}$$

Now we are ready to give the promised construction of the residuated lattice $\mathcal{L}(\mathbf{B}^*)$ from the collection $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$. The construction has two steps.

1. First, we close $\mathcal{L}$ under divisions by elements of B^*. Namely, we define
$$\mathcal{B} = \{u\backslash L_b/v \mid u, v \in B^*,\ b \in B\}.$$
2. Second, we define a closure operator $\gamma\colon \mathcal{P}(B^*) \to \mathcal{P}(B^*)$ such that $\mathcal{B}$ is a basis for γ, i.e., we define
$$\gamma(X) = \bigcap\{L \in \mathcal{B} \mid X \subseteq L\}.$$

Then the collection of γ-closed languages $\mathcal{P}(B^*)_\gamma$ forms a residuated lattice
$$\mathcal{L}(\mathbf{B}^*) = \langle \mathcal{P}(B^*)_\gamma, \cap, \cup_\gamma, \circ_\gamma, \backslash, /, \gamma(\varepsilon)\rangle, \tag{14}$$
where
$$X \cup_\gamma Y = \gamma(X \cup Y),$$
$$X \circ_\gamma Y = \gamma(XY).$$

A proof that $\mathcal{L}(\mathbf{B}^*)$ is a residuated lattice can be found in [9][1]. We also provide an explanation why $\mathcal{L}(\mathbf{B}^*)$ is a residuated lattice in order to stress some connection with the language theory. It is well known that the collection of γ-closed sets forms a complete lattice where meets are intersections and joins are closures of unions (see e.g. [6]). Further, for every $X \subseteq B^*$ and $u\backslash L_b/v \in \mathcal{B}$ we have by Lemma 1 and the distribution laws (8), (10) the following equality:

$$X\backslash(u\backslash L_b/v) = uX\backslash L_b/v = (\bigcup_{x\in X} ux)\backslash L_b/v = \bigcap_{x\in X}(ux\backslash L_b/v).$$

Thus dividing a language $L \in \mathcal{B}$ from the basis by any language X on the left produces a γ-closed language. Analogous claim holds for division on the right. Consequently, γ-closed sets are closed under divisions

[1] In fact, $\mathcal{L}(\mathbf{B}^*)$ is nothing else than the dual algebra $\mathbf{W}^+$ of unital residuated frame $\mathbf{W} = \langle B^*, B^* \times B^* \times B, N, \cdot, \backslash\!\!\backslash, /\!\!/, \{\varepsilon\}\rangle$ where $x\,N\,\langle u,v,b\rangle$ iff $uxv \in L_b$ and $x \backslash\!\!\backslash \langle u,v,b\rangle = \langle ux,v,b\rangle$, $\langle u,v,b\rangle /\!\!/ x = \langle u,xv,b\rangle$; for details see [9].

because by (9) we have for every $X,Y \subseteq B^*$:

$$X\backslash\gamma(Y) = X\backslash\bigcap\{L \in \mathcal{B} \mid Y \subseteq L\} = \bigcap\{X\backslash L \in \mathcal{P}(B^*)_\gamma \mid L \in \mathcal{B},\ Y \subseteq L\}$$

and likewise for $\gamma(Y)/X$. Now it is easy to see that (7) holds for $\circ_\gamma, \backslash, /$, i.e., for γ-closed languages $X,Y,Z \subseteq B^*$ we have

$$X \circ_\gamma Y = \gamma(XY) \subseteq Z \quad \text{iff} \quad XY \subseteq Z \quad \text{iff} \quad Y \subseteq X\backslash Z \quad \text{iff} \quad X \subseteq Z/Y.$$

It remains to show that $\circ_\gamma$ is a monoid operation. To see this note that by (13) the kernel $\ker(\gamma)$ of the self-map γ can be described as follows:

$$\gamma(X) = \gamma(Y) \quad \text{iff} \quad \forall b \in B\, \forall u,v \in B^* (X \subseteq u\backslash L_b/v \Longleftrightarrow Y \subseteq u\backslash L_b/v). \tag{15}$$

We claim that $\ker(\gamma)$ is a monoid congruence on $\mathcal{P}(\mathbf{B}^*)$. Let $X,Y,Z \subseteq B^*$ such that $\gamma(Y) = \gamma(Z)$. Then for every $u,v \in B^*$ and $b \in B$ we have

$$\begin{aligned} XY \subseteq u\backslash L_b/v \quad &\text{iff} \quad Y \subseteq uX\backslash L_b/v = \bigcap_{x\in X}(ux\backslash L_b/v) \\ &\text{iff} \quad Z \subseteq uX\backslash L_b/v \\ &\text{iff} \quad XZ \subseteq u\backslash L_b/v. \end{aligned}$$

Thus $\gamma(XY) = \gamma(XZ)$. Analogously one can show that $\gamma(YX) = \gamma(ZX)$. Consequently, the kernel γ is a monoid congruence and $\langle \mathcal{P}(B^*)_\gamma, \circ_\gamma, \gamma(\varepsilon)\rangle$ is a monoid.

Compare (15) also with the definition of syntactic congruence (5). If X,Y are singletons (i.e., $X = \{x\}$ and $Y = \{y\}$), then $\gamma(X) = \gamma(Y)$ iff $x \sim_{L_b} y$ for all $b \in B$. Thus $\ker(\gamma)$ can be seen as a lifting of a monoid congruence $\bigcap_{b\in B} \sim_{L_b}$ from $\mathbf{B}^*$ to $\mathcal{P}(\mathbf{B}^*)$.

One can also show that $\ker(\gamma)$ is a semilattice congruence with respect to join. Namely, $\gamma(Y) = \gamma(Z)$ implies $\gamma(X \cup Y) = \gamma(X \cup Z)$. Indeed,

$$\begin{aligned} X \cup Y \subseteq u\backslash L_b/v \quad &\text{iff} \quad X,Y \subseteq u\backslash L_b/v \\ &\text{iff} \quad X,Z \subseteq u\backslash L_b/v \\ &\text{iff} \quad X \cup Z \subseteq u\backslash L_b/v. \end{aligned}$$

Thus γ can be viewed as $\{\vee,\cdot,1\}$-homomorphism from $\mathcal{P}(\mathbf{B}^*)$ to $\mathcal{L}(\mathbf{B}^*)$.

Example 2. Let $B = \{a,b\}$ and $\mathcal{L} = \{L_a, L_b\}$, where $L_a = ab^*$ and $L_b = ba^*$. We compute the residuated lattice $\mathcal{L}(\mathbf{B}^*)$. First, we construct the basis

$$\mathcal{B} = \{u\backslash L_a/v \mid u,v \in B^*\} \cup \{u\backslash L_b/v \mid u,v \in B^*\}$$

for the closure operator γ. Since L_a and L_b are regular languages, the basis has to be finite by Theorem 5. The basis $\mathcal{B}$ consists of the following languages:

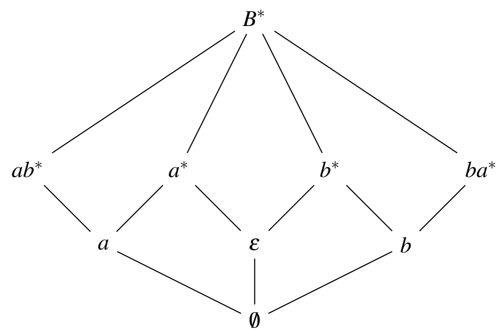
$$\begin{aligned} &\varepsilon\backslash L_a/\varepsilon = L_a && \varepsilon\backslash L_b/\varepsilon = L_b \\ &a\backslash L_a/\varepsilon = b^* && b\backslash L_b/\varepsilon = a^* \\ &\varepsilon\backslash L_a/a = \varepsilon. \end{aligned}$$

Then the γ-closed languages are those which are given by intersections of the basis elements. Namely, the lattice reduct of $\mathcal{L}(\mathbf{B}^*)$ is depicted in Figure 2. Note that B^* can be viewed as the empty intersection.

The multiplication in $\mathcal{L}(\mathbf{B}^*)$ is computed as follows. Given two languages L_1, L_2 from $\mathcal{L}(B^*)$, we concatenate them and then find the least language in $\mathcal{L}(\mathbf{B}^*)$ containing L_1L_2. For instance to compute $b \circ_\gamma ab^*$ note that the only γ-closed language containing bab^* is B^*. Thus $b \circ_\gamma ab^* = B^*$. On the other hand, we have $ab^*b = ab^*$ so $ab^* \circ_\gamma b = ab^*$.

In Example 2 we have seen that the resulting residuated lattice $\mathcal{L}(\mathbf{B}^*)$ is finite. This is not accidental because we have started with a finite collection $\mathcal{L}$ of regular languages.

Fig. 2 The lattice reduct of $\mathcal{L}(\mathbf{B}^*)$.



Theorem 6. *Let B be an alphabet and $\mathcal{L} = \{L_b \mid b \in B\}$ a finite collection of languages. Then $\mathcal{L}(\mathbf{B}^*)$ is finite iff all members of $\mathcal{L}$ are regular.*

Proof. ($\Rightarrow$): Let $L_b \in \mathcal{L}$. We have to show that L_b is regular. Since $\mathcal{L}(\mathbf{B}^*)$ is finite, the collection

$$\{u\backslash L_b/v \mid u,v \in B^*\} \subseteq \mathcal{P}(B^*)_\gamma$$

has to be finite as well. Thus L_b is regular by Theorem 5.

($\Leftarrow$): It suffices to show that the basis $\mathcal{B} = \{u\backslash L_b/v \mid u,v \in B^*,\ b \in B\}$ is finite since then the collection of γ-closed languages generated by $\mathcal{B}$ contains only finite intersections and so it has to be finite. We have

$$\mathcal{B} = \bigcup_{b\in B} \{u\backslash L_b/v \mid u,v \in B^*\}.$$

By Theorem 5 each $\{u\backslash L_b/v \mid u,v \in B^*\}$ is finite. Thus $\mathcal{B}$ is finite and we are done. $\square$

5 Analytic identities and corresponding rules

From the previous section we know how to recognize whether a collection $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ induces a finite residuated lattice $\mathcal{L}(\mathbf{B}^*)$ or not. Next we would like to know which identities are valid in $\mathcal{L}(\mathbf{B}^*)$. More precisely, we will show that the residuated lattice $\mathcal{L}(\mathbf{B}^*)$ satisfies an identity iff the languages in $\mathcal{L}$ are closed under certain rules corresponding to the identity. We restrict our attention only to identities of a special form. In principle, the method used in this paper would work for all $\mathcal{N}_2$-identities which are preserved by Dedekind-McNeille completion (for details see [4]). In order to keep this paper simpler, we restrict ourselves even more and consider only identities built up from variables using the operations in $\{\vee,\cdot,1\}$.

Let $\{x_1,\dots,x_m\}$ be a set of distinct variables. An identity is said to be *analytic*[2] if it is of the form

$$x_1 \dots x_m \leq t_1 \vee \dots \vee t_n, \qquad (e)$$

where the terms $t_1,\dots,t_n$ are products of some variables from $\{x_1,\dots,x_m\}$. For instance, $x_1x_2 \leq 1 \vee x_1^2x_2$ is an analytic identity. The following lemma is proved in [9].

Lemma 2. *Every identity in the signature $\{\vee,\cdot,1\}$ is equivalent to a set of analytic identities.*

Due to Lemma 2, we can deal only with analytic identities because every variety of residuated lattices axiomatized by a set of identities in the signature $\{\vee,\cdot,1\}$ can be also axiomatized by a set of analytic identities.

[2] Identities of the form are called *simple* in [9]. We call them analytic because they are equivalent to analytic structural quasi-identities; for details see [4].

Given an analytic identity (e), we have to clarify when (e) holds in $\mathcal{L}(\mathbf{B}^*)$. We start with a general fact. Let $\mathbf{A}$ be a complete residuated lattice which is generated as a complete join-semilattice by a set E, i.e., for every $a \in A$ there is $E_a \subseteq E$ such that $a = \bigvee_{e \in E_a} e$. Consider any term $t(x_1, \ldots, x_m)$ in the signature $\{\vee, \cdot, 1\}$. Then by the distribution law (8) we have for all $a_1, \ldots, a_m \in A$ the following equality:

$$t^{\mathbf{A}}(a_1, \ldots, a_m) = t^{\mathbf{A}}\left(\bigvee_{e_1 \in E_{a_1}} e_1, \ldots, \bigvee_{e_m \in E_{a_m}} e_m\right) = \bigvee_{\substack{e_i \in E_{a_i} \\ 1 \leq i \leq m}} t^{\mathbf{A}}(e_1, \ldots, e_m). \tag{16}$$

Now consider an identity $s \leq t$ for some terms t, s built up from variables $x_1, \ldots, x_m$ using operations in $\{\vee, \cdot, 1\}$. In order to check that $s \leq t$ holds in $\mathbf{A}$, we claim that it suffices to focus only on interpretations into the set E. In other words, we claim that $\mathbf{A} \models s \leq t$ iff

$$\forall e_1, \ldots, e_m \in E(s^{\mathbf{A}}(e_1, \ldots, e_m) \leq t^{\mathbf{A}}(e_1, \ldots, e_m)). \tag{17}$$

Clearly, $\mathbf{A} \models s \leq t$ implies (17) because (17) just restricts possible interpretations for variables $x_1, \ldots, x_m$. Conversely, assume that (17) holds. Then using (16) we have for arbitrary $a_1, \ldots, a_m \in A$ the following:

$$s^{\mathbf{A}}(a_1, \ldots, a_m) = \bigvee_{\substack{e_i \in E_{a_i} \\ 1 \leq i \leq m}} s^{\mathbf{A}}(e_1, \ldots, e_m) \leq \bigvee_{\substack{e_i \in E_{a_i} \\ 1 \leq i \leq m}} t^{\mathbf{A}}(e_1, \ldots, e_m) = t^{\mathbf{A}}(a_1, \ldots, a_m). \tag{18}$$

The above observation simplify our task because $\mathcal{L}(\mathbf{B}^*)$ is generated as a complete join-semilattice by $\{\gamma\{w\} \mid w \in B^*\}$. Indeed, for every γ-closed language $W \subseteq B^*$ we have

$$W = \bigcup_{w \in W} \{w\} \subseteq \bigcup_{w \in W} \gamma\{w\} \subseteq W.$$

Next we discuss how various terms occurring in (e) are interpreted in the free monoid $\mathbf{B}^*$ and residuated lattices $\mathcal{P}(\mathbf{B}^*)$, $\mathcal{L}(\mathbf{B}^*)$. We have seen in the previous section that these algebras are related as follows:

$$\mathbf{B}^* \xrightarrow{\ \{\_\}\ } \mathcal{P}(\mathbf{B}^*) \xrightarrow{\ \gamma\ } \mathcal{L}(\mathbf{B}^*)$$

where $\{\_\}$ is a monoid embedding and γ is a surjective $\{\vee, \cdot, 1\}$-homomorphism.

Lemma 3. *Let $s \leq t$ be an analytic identity (e), i.e., $s = x_1 \ldots x_m$ and $t = t_1 \vee \ldots \vee t_n$. Then the following hold for all $w_1, \ldots, w_m \in B^*$ and $1 \leq i \leq n$:*

$$s^{\mathcal{L}(\mathbf{B}^*)}(\gamma\{w_1\}, \ldots, \gamma\{w_m\}) = \gamma\{w_1 \ldots w_m\} \tag{19}$$
$$t_i^{\mathcal{P}(\mathbf{B}^*)}(\{w_1\}, \ldots, \{w_m\}) = \{t^{\mathbf{B}^*}(w_1, \ldots, w_m)\} \tag{20}$$
$$t^{\mathcal{L}(\mathbf{B}^*)}(\gamma\{w_1\}, \ldots, \gamma\{w_m\}) = \gamma\{t_1^{\mathbf{B}^*}(w_1, \ldots, w_m), \ldots, t_n^{\mathbf{B}^*}(w_1, \ldots, w_m)\}. \tag{21}$$

Proof. The equality (19) follows since γ is a $\{\vee, \cdot, 1\}$-homomorphism. Namely, we have

$$s^{\mathcal{L}(\mathbf{B}^*)}(\gamma\{w_1\}, \ldots, \gamma\{w_m\}) = \gamma(s^{\mathcal{P}(\mathbf{B}^*)}(\{w_1\}, \ldots, \{w_m\})) = \gamma\{w_1 \ldots w_m\}.$$

To prove (20) we use the fact that $\{\_\}$ is a monoid homomorphism. The equality (21) follows from (20) and the fact that γ is a $\{\vee, \cdot, 1\}$-homomorphism. Namely, we have

$$\begin{aligned} t^{\mathcal{L}(\mathbf{B}^*)}(\gamma\{w_1\}, \ldots, \{w_m\}) &= \gamma(t^{\mathcal{P}(\mathbf{B}^*)}(\{w_1\}, \ldots, \{w_m\})) \\ &= \gamma(t_1^{\mathcal{P}(\mathbf{B}^*)}(\{w_1\}, \ldots, \{w_m\}) \cup \ldots \cup t_n^{\mathcal{P}(\mathbf{B}^*)}(\{w_1\}, \ldots, \{w_m\})) \\ &= \gamma\{t_1^{\mathbf{B}^*}(w_1, \ldots, w_m), \ldots, t_n^{\mathbf{B}^*}(w_1, \ldots, w_m)\}. \end{aligned}$$

$\square$

Using Lemma 3, it follows that (e) holds $\mathcal{L}(\mathbf{B}^*)$ iff for all words $w_1,\ldots,w_m \in B^*$ we have

$$\gamma\{w_1\ldots w_m\} \subseteq \gamma\{t_1^{\mathbf{B}^*}(w_1,\ldots,w_m),\ldots,t_n^{\mathbf{B}^*}(w_1,\ldots,w_m)\}.$$

Since $\mathcal{B} = \{u\backslash L_b/v \mid u,v \in B^*,\ b \in B\}$ is a basis for γ, the above inclusion is equivalent to the validity of following implication for all $u,v,w_1,\ldots,w_m \in B^*$ and $b \in B$:

$$t_1^{\mathbf{B}^*}(w_1,\ldots,w_m),\ldots,t_n^{\mathbf{B}^*}(w_1,\ldots,w_m) \in u\backslash L_b/v \implies w_1\ldots w_m \in u\backslash L_b/v. \tag{22}$$

Thus we have proved that $\mathcal{L}(\mathbf{B}^*)$ satisfies an analytic identity iff the implication (22) holds. This explains the next definition.

Definition 3. Let (e) be an analytic identity. A language $L \subseteq B^*$ is said to be closed under the rule corresponding to (e) if the following implication holds for all $u,v,w_1,\ldots,w_m \in B^*$:

$$t_1^{\mathbf{B}^*}(w_1,\ldots,w_m),\ldots,t_n^{\mathbf{B}^*}(w_1,\ldots,w_m) \in u\backslash L/v \implies w_1\ldots w_m \in u\backslash L/v. \tag{R_e}$$

Example 3. Consider an identity $x^2 \leq x$ which is equivalent to an analytic identity $x_1x_2 \leq x_1 \vee x_2$. We say that a language $L \subseteq B^*$ satisfies the corresponding rule if the following implication holds for all $u,v,w_1,w_2 \in B^*$:

$$w_1,w_2 \in u\backslash L/v \implies w_1w_2 \in u\backslash L/v. \tag{23}$$

For instance, the language $L = aB^*b$ from Example 1 satisfies the above condition. To see this note that (23) expresses the fact that all the languages in

$$\{u\backslash L/v \mid u,v \in B^*\} = \{L, aB^*, B^*b, aB^* \cup \varepsilon, B^*b \cup \varepsilon, B^*\}$$

are subsemigroups of $\mathbf{B}^*$ which is easy to check.

To sum up what we have proved above, we can formulate the main result of this section. It was originally proved in [4].

Theorem 7. *Let $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ be a collection of languages and (e) an analytic identity. Then $\mathcal{L}(\mathbf{B}^*)$ satisfies (e) iff every language $L_b \in \mathcal{L}$ is closed under the rule (R_e).*

Example 4. In Example 2 we started with languages $L_a = ab^*$ and $L_b = ba^*$. Both of them are closed under the (contraction) rule

$$ww \in u\backslash L/v \implies w \in u\backslash L/v.$$

Equivalently, this implication can be written as follows:

$$uwwv \in L \implies uwv \in L.$$

Therefore the resulting residuated lattice $\mathcal{L}(\mathbf{B}^*)$ from Example 2 satisfies the identity $x \leq x^2$. On the other hand, L_a and L_b are not closed under the (weakening) rule

$$\varepsilon \in u\backslash L/v \implies w \in u\backslash L/v$$

or equivalently

$$uv \in L \implies uwv \in L.$$

Consequently, the residuated lattice $\mathcal{L}(\mathbf{B}^*)$ from Example 2 does not satisfy the identity $x \leq 1$.

6 Finite embeddability property

Now we are ready to prove the promised characterization of FEP via regular languages. We start with a crucial definition of a Gentzen collection of languages. Similarly as surjective homomorphisms from an algebra **A** correspond to congruences on **A**, embeddings of partial subalgebras correspond to separating Gentzen collections of languages.

Let **A** be a residuated lattice and **B** a finite partial subalgebra of **A**. Assume that there is an embedding g of **B** into a residuated lattice **C**. Since $\mathbf{B}^*$ is the free B-generated monoid, there is a unique monoid homomorphism $g^*\colon B^* \to C$ such that $g^*(b) = g(b)$ for all $b \in B$, i.e., the diagram in Figure 3 commutes.

$$\begin{tikzcd} \mathbf{B} \arrow[r, "g"] \arrow[d, "\iota"'] & \mathbf{C} \\ \mathbf{B}^* \arrow[ur, "g^*"'] & \end{tikzcd}$$

Fig. 3 The extension $g^*\colon B^* \to C$ of $g\colon B \to C$ where $\iota\colon B \to B^*$ is the inclusion map.

Every $b \in B$ defines a language over B^* as follows:

$$L_b = \{x \in B^* \mid g^*(x) \leq g(b)\}.$$

One can ask what properties languages L_b's have if we know that g is an embedding of partial subalgebra **B** into a residuated lattice **C**. In this situation it can be shown that the collection $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ is closed under the rules from Figure 4 for any $a,b,c \in B$ and $u,v,x \in B^*$. These rules are analogous to the rules from Gentzen's sequent calculus so we call them *Gentzen rules*. Every rule might have several premises (the upper part of the rule) and has a conclusion (the lower part). A rule can be applied only if the element from **B** in its conclusion is defined. For instance, the rule

$$(\backslash\mathrm{L})\ \frac{x \in L_a \qquad ubv \in L_c}{ux(a\backslash b)v \in L_c}$$

applies only if $a\backslash b \in B$.

Definition 4. Let **A** be a residuated lattice and **B** a finite partial subalgebra of **A**. A collection of languages $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ is called *Gentzen*[3] if it is closed under the rules in Figure 4 for all $u,v,x \in B^*$ and $a,b,c \in B$. In addition, the system $\mathcal{L}$ is said to be *separating* if every language $L_b \in \mathcal{L}$ does not contain any $a \in B$ such that $a \not\leq b$.

Now we show that for every embedding of a partial subalgebra there is a separating Gentzen collection of languages. In what follows, the multiplication in the partial subalgebra **B** is denoted by $\cdot$ and the multiplication in the free monoid $\mathbf{B}^*$ by concatenation.

Lemma 4. *Let* **A** *be a residuated lattice and* **B** *a finite partial subalgebra of* **A***. Assume that there is an embedding g of* **B** *into a residuated lattice* **C***. Then the collection $\mathcal{L} = \{L_b \mid b \in B\}$, where*

$$L_b = \{x \in B^* \mid g^*(x) \leq g(b)\},$$

is separating and Gentzen. In addition, if **C** *satisfies an analytic identity (e) then every L_b is closed under the corresponding rule (R_e).*

Proof. It is easy to see that $\mathcal{L}$ is separating since g is an order-embedding. Indeed, since g preserves existing operations, we have

$$a \leq b \quad \text{iff} \quad a \vee b = b \quad \text{iff} \quad g(a) \vee g(b) = g(b) \quad \text{iff} \quad g(a) \leq g(b).$$

[3] Gentzen collections are closely related to Gentzen residuated frames defined in [9]. More precisely, a Gentzen collection $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ is nothing else than the basis of the nucleus induced by the unital Gentzen residuated frame $\langle \mathbf{W}, \mathbf{B} \rangle$, where $\langle B^*, B^* \times B^* \times B, N, \cdot, \backslash\!\backslash, /\!/, \{\varepsilon\} \rangle$, $x \backslash\!\backslash \langle u,v,b \rangle = \langle ux,v,b \rangle$, $\langle u,v,b \rangle /\!/ x = \langle u,xv,b \rangle$ and $x\, N\, \langle u,v,b \rangle$ iff $uxv \in L_b$; for details see [9].

Fig. 4 Gentzen rules for a collection $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$; $a,b,c \in B$ and $u,v,x \in B^*$.

$$(\mathrm{Id})\ \frac{}{a \in L_a} \qquad (\mathrm{Cut})\ \frac{x \in L_a \qquad uav \in L_b}{uxv \in L_b}$$

$$(\cdot\mathrm{L})\ \frac{uabv \in L_c}{u(a\cdot b)v \in L_c} \qquad (\cdot\mathrm{R})\ \frac{x \in L_a \qquad y \in L_b}{xy \in L_{a\cdot b}}$$

$$(\backslash\mathrm{L})\ \frac{x \in L_a \qquad ubv \in L_c}{ux(a\backslash b)v \in L_c} \qquad (\backslash\mathrm{R})\ \frac{ax \in L_b}{x \in L_{a\backslash b}}$$

$$(/\mathrm{L})\ \frac{x \in L_a \qquad ubv \in L_c}{u(b/a)xv \in L_c} \qquad (/\mathrm{R})\ \frac{xa \in L_b}{x \in L_{b/a}}$$

$$(\vee\mathrm{L})\ \frac{uav \in L_c \qquad ubv \in L_c}{u(a\vee b)v \in L_c} \qquad (\vee\mathrm{R})\ \frac{x \in L_{a_i}}{x \in L_{a_1\vee a_2}} \text{ for } i=1,2$$

$$(\wedge\mathrm{L})\ \frac{ua_iv \in L_c}{u(a_1\wedge a_2)v \in L_c} \text{ for } i=1,2 \qquad (\wedge\mathrm{R})\ \frac{x \in L_a \qquad x \in L_b}{x \in L_{a\wedge b}}$$

$$(1\mathrm{L})\ \frac{uv \in L_a}{u1v \in L_a} \qquad (1\mathrm{R})\ \frac{}{\varepsilon \in L_1}$$

It is also clear that $b \in L_b$ for every $b \in B$.

We check that $\mathcal{L}$ is closed under the rule (Cut). Suppose that $x \in L_a$ and $uav \in L_b$. Then

$$g^*(x) \leq g(a) \quad \text{and} \quad g^*(uav) = g^*(u)g(a)g^*(v) \leq g(b).$$

Consequently, $g^*(uxv) \leq g(b)$ by monotonicity of multiplication in $\mathbf{C}$. Thus $uxv \in L_b$.

Next consider the rule (\L). Suppose that $a\backslash b \in B$ and $x \in L_a$, $ubv \in L_c$. Then $g^*(x) \leq g(a)$ and $g^*(ubv) \leq g(c)$. Since g preserves operations in $\mathbf{B}$, we have $g(a)g(a\backslash b) = g(a)(g(a)\backslash g(b)) \leq g(b)$ by Lemma 1. Consequently, we obtain $ux(a\backslash b)v \in L_c$ because we have

$$g^*(u)g^*(x)g(a\backslash b)g^*(v) \leq g^*(u)g(a)g(a\backslash b)g^*(v) \leq g^*(u)g(b)g^*(v) \leq g(c).$$

Now we show that $\mathcal{L}$ is closed under (\R). Assume that $a\backslash b \in B$ and $ax \in L_b$, i.e., $g(a)g^*(x) \leq g(b)$. Thus $g^*(x) \leq g(a)\backslash g(b) = g(a\backslash b)$, i.e. $x \in L_{a\backslash b}$.

Consider the rule (·L). Suppose that $a\cdot b \in B$. If $uabv \in L_c$ then $g^*(u)g(a)g(b)g^*(v) \leq g(c)$ by definition of L_c. Since g is an embedding, we have $g(a\cdot b) = g(a)g(b)$. Thus $u(a\cdot b)v \in L_c$. To check the rule (·R) we assume that $x \in L_a$ and $y \in L_b$. Thus $g^*(x) \leq g(a)$ and $g^*(y) \leq g(b)$. Consequently, $g^*(xy) = g^*(x)\cdot g^*(y) \leq g(a)\cdot g(b) = g(a\cdot b)$ and so $xy \in L_{a\cdot b}$. The proof for the remaining rules is similar.

Finally, we prove the additional part of the lemma. Suppose that $\mathbf{C}$ satisfies an analytic identity

$$x_1 \ldots x_m \leq t_1 \vee \ldots \vee t_n. \tag{e}$$

Consider its corresponding rule for the language L_b:

$$t_1^{\mathbf{B}^*}(w_1,\ldots,w_m),\ldots,t_n^{\mathbf{B}^*}(w_1,\ldots,w_m) \in u\backslash L_b/v \implies w_1\ldots w_m \in u\backslash L_b/v. \tag{R_e}$$

Assume that the premises are satisfied for some $u,v,w_1,\ldots,w_m \in B^*$ and $b \in B$. Thus for $1 \leq i \leq n$ we have

$$g^*(u)\cdot t_i^{\mathbf{C}}(g^*(w_1),\ldots,g^*(w_m))\cdot g^*(v) = g^*(ut_i^{\mathbf{B}^*}(w_1,\ldots,w_m)v) \leq g(b),$$

where the first equality holds because g^* is a monoid homomorphism from $\mathbf{B}^*$ to $\mathbf{C}$. Using the fact that (e) holds in $\mathbf{C}$, we obtain

$$\begin{aligned} g^*(uw_1\dots w_m v) &= g^*(u)\cdot g^*(w_1)\cdot\ldots\cdot g^*(w_m)\cdot g^*(v)\\ &\le g^*(u)\cdot(t_1^{\mathbf{C}}(g^*(w_1),\dots,g^*(w_m))\vee\ldots\vee t_n^{\mathbf{C}}(g^*(w_1),\dots,g^*(w_m)))\cdot g^*(v)\\ &\le g(b). \end{aligned}$$

Consequently, $w_1\dots w_m \in u\backslash L_b/v$ and so we have shown that L_b is closed under (R_e). □

Now we will prove the converse of Lemma 4. Namely, for every separating Gentzen collection of languages $\mathcal{L}=\{L_b\subseteq B^*\mid b\in B\}$ we show that there is an embedding g from the partial subalgebra $\mathbf{B}$ into the residuated lattice $\mathcal{L}(\mathbf{B}^*)$. The following lemma was originally proved in [9].

Lemma 5. *Let* $\mathbf{A}$ *be a residuated lattice and* $\mathbf{B}$ *a finite partial subalgebra of* $\mathbf{A}$*. Suppose that we have a separating Gentzen collection* $\mathcal{L}=\{L_b\subseteq B^*\mid b\in B\}$*. Then* $\mathbf{B}$ *can be embedded into* $\mathcal{L}(\mathbf{B}^*)$*.*

Proof. We claim that the map $g\colon B\to\mathcal{P}(B^*)_\gamma$ given by $g(b)=L_b$ is an embedding of $\mathbf{B}$ into $\mathcal{L}(\mathbf{B}^*)$. It is clearly injective since $\mathcal{L}$ is separating. Indeed, if $a\neq b$ then either $a\not\le b$ or $b\not\le a$. Without any loss of generality assume that $a\not\le b$. Then $a\notin L_b$ and $a\in L_a$ by (Id). Thus $L_a\neq L_b$. It remains to check that g preserves all existing products, joins, meets, divisions and multiplicative unit 1 in $\mathbf{B}$.

Suppose that $1\in B$. We have to check that $L_1=\gamma(\varepsilon)$. First, $\gamma(\varepsilon)\subseteq L_1$ by (1R). Conversely, it suffices to show that $\varepsilon\in u\backslash L_b/v \implies L_1\subseteq u\backslash L_b/v$ for all languages $u\backslash L_b/v$ from the basis $\mathcal{B}$. Assume that $\varepsilon\in u\backslash L_b/v$, i.e., $uv\in L_b$. By (1L) we obtain $1\in u\backslash L_b/v$. Thus it follows that $L_1\subseteq u\backslash L_b/v$ by the rule (Cut).

Now assume that $a,b,a\backslash b\in B$. We have to show $L_{a\backslash b}=L_a\backslash L_b$. By the definition of $\mathcal{L}(\mathbf{B}^*)$ we have

$$L_a\backslash L_b=\{x\in B^*\mid L_a x\subseteq L_b\}.$$

Suppose that $L_a x\subseteq L_b$ for $x\in B^*$. By the rule (Id) we have $ax\in L_b$. Thus $x\in L_{a\backslash b}$ by ($\backslash$R). Conversely, assume that $x\in L_{a\backslash b}$. We have to show that $L_a x\subseteq L_b$, i.e., $yx\in L_b$ for all $x\in L_a$. Using Gentzen rules, we obtain the following derivation:

$$\text{(Cut)}\ \dfrac{x\in L_{a\backslash b}\qquad (\backslash\text{L})\ \dfrac{y\in L_a\qquad (\text{Id})\ \dfrac{}{b\in L_b}}{y(a\backslash b)\in L_b}}{yx\in L_b}$$

Next assume that $a,b,a\cdot b\in B$. We have to show $L_{a\cdot b}=L_a\circ_\gamma L_b$. By the definition of operations in $\mathcal{L}(\mathbf{B}^*)$ we have

$$L_a\circ_\gamma L_b=\gamma(L_aL_b).$$

Clearly, $L_aL_b\subseteq L_{a\cdot b}$ by ($\cdot$R). Thus $\gamma(L_a\cdot L_b)\subseteq L_{a\cdot b}$. Conversely, suppose that $L_aL_b\subseteq u\backslash L_c/v$. Then $uabv\in L_c$ by (Id). Hence $u(a\cdot b)v\in L_c$ by ($\cdot$L). Thus $L_{a\cdot b}\subseteq u\backslash L_c/v$ by (Cut).

Finally, we check that g preserves existing joins. The proof for meets is analogous. Assume that $a,b,a\vee b\in B$. We have to show

$$L_{a\vee b}=L_a\cup_\gamma L_b=\gamma(L_a\cup L_b).$$

We have $L_a,L_b\subseteq L_{a\vee b}$ by ($\vee$R). Thus $\gamma(L_a\cup L_b)\subseteq L_{a\vee b}$. Conversely, suppose that $L_a,L_b\subseteq u\backslash L_c/v$. Then $a\vee b\in u\backslash L_c/v$ by ($\vee$L). Thus $L_{a\vee b}\subseteq u\backslash L_c/v$ by (Cut). □

Now we combine all the above results in order to obtain the main result of this chapter.

Theorem 8. *Let* V *be a variety of residuated lattices axiomatized by a set* E *of analytic identities. Then the following are equivalent:*

1. V *has the FEP.*
2. *For every algebra* $\mathbf{A}\in\mathsf{V}$ *and a finite partial subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *there is a separating Gentzen collection* $\mathcal{L}=\{L_b\subseteq B^*\mid b\in B\}$ *of regular languages closed under the rules from* $\{R_e\mid e\in E\}$*, i.e., the rules corresponding to the analytic identities in* E*.*

Proof. (1⇒2): Let $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A} \in \mathsf{V}$. By the FEP there is a finite residuated lattice $\mathbf{C}$ and an embedding $g\colon B \to C$. Consider the unique monoid homomorphism $g^*\colon B^* \to C$. By Lemma 4 the collection of languages $L_b = \{x \in B^* \mid g^*(x) \leq g(b)\}$ for $b \in B$ is Gentzen, separating and every L_b is closed under the rules R_e. Since $\mathbf{C}$ is finite, L_b is regular because it is saturated by $\ker(g^*)$.

(2⇒1): By Lemma 5 the partial subalgebra $\mathbf{B}$ is embeddable into the residuated lattice $\mathcal{L}_B(\mathbf{B}^*)$. Moreover, this residuated lattice is finite by Theorem 6 since the languages are regular. Finally, $\mathcal{L}_B(\mathbf{B}^*)$ belongs to V by Theorem 7. □

The following theorem which might be useful for disproving the FEP for a class of algebras follows from the above proof of (1⇒2) just by disregarding parts concerning the rules R_e.

Theorem 9. *Let K be a class of algebras of the same type. If K has the FEP then for every algebra $\mathbf{A} \in \mathsf{K}$ and a finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ there is a separating Gentzen collection $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ of regular languages.*

7 Applications

In this section we present several application of the main result. In particular, we cover several known results from literature concerning the FEP for classes of residuated lattices. We also give a new one. Finally, we illustrate how to use Theorem 9 in order to show that a class do not have the FEP.

If we want to show that a variety V of residuated lattices axiomatized by a set E of analytic identities has the FEP, we have to find for every partial subalgebra $\mathbf{B}$ of a residuated lattice $\mathbf{A} \in \mathsf{V}$ a separating Gentzen collection $\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}$ of regular languages. First, note that the identity map is an embedding of $\mathbf{B}$ into $\mathbf{A}$. In particular, the diagram in Figure 5 commutes. Thus the collection

$$\mathcal{L} = \{L_b \subseteq B^* \mid b \in B\}, \text{ where } L_b = \{x \in B^* \mid id^*(x) \leq b\}, \tag{24}$$

is a separating Gentzen collection closed under the rules R_e for $e \in E$ by Lemma 4. By Theorem 8 the only remaining step to prove the FEP for V is to show that the languages L_b are regular. In what follows we will apply several results from the theory of regular languages in order to prove for various varieties of residuated lattices that the collection $\mathcal{L}$ consists of regular languages.

$$\begin{tikzcd} \mathbf{B} \arrow[r, "id"] \arrow[d, "\iota"'] & \mathbf{A} \\ \mathbf{B}^* \arrow[ur, "id^*"'] & \end{tikzcd}$$

Fig. 5 The extension $id^*\colon B^* \to A$ of $id\colon B \to A$ where $\iota\colon B \to B^*$ is the inclusion map.

7.1 *Integral residuated lattices*

A residuated lattice $\mathbf{A}$ is said to be integral if the identity $x \leq 1$ holds in $\mathbf{A}$. It was proved in [1, 2] that the variety of integral residuated lattices IRL has the FEP. Later this result was generalized to any subvariety of IRL axiomatized by analytic identities [9]. We prove this result using the above machinery. We first recall a useful characterization of regular languages from [7]. Before that recall that a *quasi-order* $\preceq \subseteq B^* \times B^*$ is a reflexive and transitive binary relation. If it is in addition antisymmetric then $\preceq$ is an *order*. The quasi-order

$\preceq$ is called *well* if there is neither infinite antichain nor infinite descending chain (see e.g. [19]). Finally, $\preceq$ is said to be *compatible* if $x \preceq y$ implies $uxv \preceq uyv$ for all $u,v,x,y \in B^*$.

Theorem 10 (Generalized Myhill Theorem). *A language $L \subseteq B^*$ is regular iff there is a compatible well-quasi-order on $\mathbf{B}^*$ with respect to which L is upward closed (i.e. $x \in L$ & $x \preceq y \implies y \in L$).*

Now we can simply apply well-known Higman's lemma [13] in order to show that subvarieties of IRL axiomatized by analytic identities has the FEP. Recall the divisibility order on B^*. It is defined as follows:

$$x \preceq y \quad \text{iff} \quad x = a_1 \ldots a_n \text{ and } y = u_0 a_1 u_1 a_2 u_2 \ldots u_{n-1} a_n u_n \text{ for some } a_1, \ldots, a_n \in B \text{ and } u_0, \ldots, u_n \in B^*.$$

It is easy to see that this order is compatible.

Lemma 6 (Higman Lemma). *The divisibility order $\preceq$ on $\mathbf{B}^*$ is well.*

Let $\mathcal{L}$ be the separating Gentzen collection defined by (24) for a finite partial subalgebra $\mathbf{B}$ of an integral residuated lattice $\mathbf{A}$. Since $\mathbf{A}$ satisfies $x \leq 1$, every language $L_b = \{x \in B^* \mid id^*(x) \leq b\}$ is closed under the rule $uv \in L_b \implies uwv \in L_b$ by Lemma 4. Consequently, each L_b is upward closed with respect to the divisibility order $\preceq$. Thus by Theorem 10 each language L_b is regular and we obtain the following theorem originally proved in [9].

Theorem 11. *Every V variety of integral residuated lattices axiomatized by a set of analytic identities has the FEP.*

7.2 Knotted residuated lattices

A residuated lattice is said to be *commutative* if its monoid reduct is commutative, i.e., $x \cdot y = y \cdot x$. The variety of commutative residuated lattices is denoted CRL. Let RL^n_m denote the variety of residuated lattices satisfying $x^m \leq x^n$. In [25] it was proved that the variety $\mathsf{CRL} \cap \mathsf{RL}^n_m$ for any $m \neq n$ has the FEP. We will generalize this result a little bit. We need not deal with the cases for $m = 0$ or $n = 0$. It is easy to show that RL^n_0 is the trivial variety and $\mathsf{RL}^0_m = \mathsf{IRL}$. Thus we will assume $m, n \neq 0$ in the rest of this section.

We recall another characterization of regular languages from [19, Chapter 6]. A language $L \subseteq B^*$ is called *permutable* if there exist $k > 1$ such that for any sequence of words $w_1, \ldots, w_k \in B^*$ there exists a nontrivial permutation $\sigma \in S_k$ such that for all $u, v \in B^*$ we have

$$uw_1 \ldots w_k v \in L \Longleftrightarrow uw_{\sigma(1)} \ldots w_{\sigma(k)} v \in L. \tag{25}$$

The language L is called *quasi-periodic* if for all $w \in B^*$ there exist integers $0 < n < m$ such that for all $u, v \in B^*$ one has

$$uw^n v \in L \implies uw^m v \in L. \tag{26}$$

A language L is called *co-quasi-periodic* if the complement of L is quasi-periodic. Observe that co-quasi-periodic languages satisfy the converse implication of (26).

Theorem 12 ([19]). *A language $L \subseteq B^*$ is regular iff it is permutable and quasi-periodic or co-quasi-periodic.*

Let $\mathbf{A} \in \mathsf{CRL} \cap \mathsf{RL}^n_m$ for $m \neq n$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$. Consider the collection $\mathcal{L}$ defined by (24). Since $\mathbf{A}$ is commutative, every language $L_b \in \mathcal{L}$ is closed under the rule

$$uw_1 w_2 v \in L_b \implies uw_2 w_1 v \in L_b$$

by Lemma 4. Thus L_b is obviously permutable. Moreover, it is quasi-periodic or co-quasi-periodic. It does not follow immediately from Lemma 4 because the identity $x^m \leq x^n$ is not analytic. Nevertheless, it is easy to see that L_b is closed under the rule $uw^n v \in L_b \implies uw^m v \in L_b$. Indeed, suppose that $uw^n v \in L_b$. Then we have

$$id^*(uw^m v) = id^*(u) \cdot (id^*(w))^m \cdot id^*(v) \leq id^*(u) \cdot (id^*(w))^n \cdot id^*(v) = id^*(uw^n v) \leq b.$$

Hence L_b is quasi-periodic if $n < m$ and co-quasi-periodic if $m < n$. Consequently, L_b is regular by Theorem 12 and we obtain the following theorem.

Theorem 13. *Let $m \neq n$ and let $\mathsf{V} \subseteq \mathsf{CRL} \cap \mathsf{RL}_m^n$ be a subvariety axiomatized by a set of analytic identities. Then V has the FEP.*

In the above case, the commutativity immediately implies permutability of the languages L_b. Nevertheless, permutability is a weaker notion so that it is implied by weaker forms of commutativity. Consider the subvariety of RL_m^n for $m \neq n$ axiomatized by $xyx = x^2y$ which might be seen as a weaker form of commutativity. It was proved in [3] that this variety has the FEP. In fact, they proved the same result also for other analogous forms of weak commutativity. We will use our results in order to show that any subvariety $\mathsf{V} \subseteq \mathsf{RL}_m^n$ for $m \neq n$ axiomatized by $xyx = x^2y$ and any set of analytic identities has the FEP.

Let $\mathbf{A} \in \mathsf{V}$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$. Consider the collection $\mathcal{L}$ defined by (24). We know from the above considerations that each language L_b is quasi-periodic of co-quasi-periodic because $\mathbf{A} \models x^m \leq x^n$. We will prove that L_b is also permutable. Since $xyx = x^2y$ holds in $\mathbf{A}$, one can show by analogous reasoning as above that L_b is closed under the following rule:

$$uw_1w_2w_1v \in L_b \iff uw_1^2w_2v \in L_b. \tag{27}$$

Let $x \in B^*$. We define the *content* $\mathsf{c}(x)$ of x as the set of all letters occurring in x. For example, if $B = \{a,b,c\}$ and $x = ccaaac$ then $\mathsf{c}(x) = \{a,c\}$.

Lemma 7. *Let $u,v,x,y,z \in B^*$. If $\mathsf{c}(z) \subseteq \mathsf{c}(x)$ then $uxyzv \in L_b \iff uxzyv \in L_b$.*

Proof. By induction on the length $|z|$. For $z = \varepsilon$ there is nothing to prove. Assume that $z = z'a$ for some $a \in B$ and $z' \in B^*$. By induction hypothesis we have $uxyz'av \in L_b \iff uxz'yav \in L_b$. It follows from $\mathsf{c}(z) \subseteq \mathsf{c}(x)$ that $x = x_1ax_2$ for some $x_1, x_2 \in B^*$. Thus by (27) we have

$$ux_1ax_2z'yav \in L_b \iff ux_1a^2x_2z'yv \in L_b \iff ux_1ax_2z'ayv \in L_b.$$

□

Clearly there are only $2^{|B|}$ many different contents. Consider $uw_1 \dots w_k v \in L_b$ for $k = 2^{|B|} + 2$. Since $k = 2^{|B|} + 2$, there must be $1 \leq i < j < k$ such that $\mathsf{c}(w_i) = \mathsf{c}(w_j)$ by the pigeonhole principle. If $j \geq i+1$ then

$$uw_1 \dots w_k v \in L_b \iff uw_1 \dots w_i w_j w_{i+1} \dots w_{j-1} w_{j+1} \dots w_k v \in L_b$$

by Lemma 7. If $j = i+1$ then

$$uw_1 \dots w_k v \in L_b \iff uw_1 \dots w_i w_{i+2} \dots w_k w_{i+1} v \in L_b$$

again by Lemma 7. Therefore the language L_b is permutable. Consequently, L_b is regular by Theorem 12 and we obtain the following theorem.

Theorem 14. *Let $\mathsf{V} \subseteq \mathsf{RL}_m^n$ for $m \neq n$ be a subvariety axiomatized by $xyx = x^2y$ and a set of analytic identities. Then V has the FEP.*

Without any form of commutativity most of the varieties RL_m^n do not have the FEP because they have undecidable word problem (see [15]). Nevertheless for some combinations of m and n, they could have the FEP. Apart from the above-mentioned cases $m = 0$ or $n = 0$, the only cases which are not covered by [15] and therefore still might have the FEP are varieties satisfying $x^m \leq x$ for $m > 1$. We show here that for $m = 2$ this is the case.

Consider any subvariety $\mathsf{V} \subseteq \mathsf{RL}_2^1$ axiomatized by a set of analytic identities. Let $\mathbf{A} \in \mathsf{V}$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$. Recall that the identity $x^2 \leq x$ is equivalent to the analytic identity $x_1x_2 \leq x_1 \vee x_2$ (see Example 3). Since $\mathbf{A} \models x_1x_2 \leq x_1 \vee x_2$, the collection $\mathcal{L}$ defined by (24) is closed under the rule

$$uw_1v, uw_2v \in L_b \implies uw_1w_2v \in L_b.$$

The following theorem was proved in [14].

Theorem 15. *Let $L \subseteq B^*$ be a language closed under the following rule:*

$$uw_1v, uw_2v \in L \implies uw_1w_2v \in L.$$

Then L is regular.

Consequently, all languages $L_b \in \mathcal{L}$ are regular. Thus the next theorem follows immediately.

Theorem 16. *Let $\mathsf{V} \subseteq \mathsf{RL}^1_2$ by a subvariety axiomatized by a set of analytic identities. Then V has the FEP.*

7.3 Disproving the FEP

Our characterization can be used also for disproving the FEP for a class of algebras. In particular, Theorem 9 might be of use. It is known that any variety of residuated lattices containing the abelian ℓ-group on integers with addition and the usual linear order cannot have the FEP (see [1, 10]). We prove this fact via Theorem 9.

Let V be a variety of residuated lattices containing the above-mentioned abelian ℓ-group $\mathbf{G}$. We will use multiplicative notation, i.e., $\mathbf{G} = \langle G, \wedge, \vee, \cdot, \div, 1\rangle$, where $G = \{2^n \in \mathbb{R} \mid n \in \mathbb{Z}\}$, $\wedge, \vee$ are interpreted respectively by min and max, $\cdot$ is the usual multiplication of reals and $\div$ the usual division. If V would have the FEP then for the set $B = \{2^{-1}, 1, 2\}$ there would be a separating Gentzen collection $\mathcal{L}$ of regular languages. Let $a = 2$, $b = 2^{-1}$. Thus $\mathcal{L} = \{L_a, L_b, L_1\}$. We will show that $L_b \cap a^*b^* = \{a^m b^n \mid m, n \in \mathbb{N}$ and $m < n\}$. This language is not regular (see e.g. [24]) which is a contradiction because a^*b^* is regular and regular languages are closed under finite intersections.

To show that $a^m b^n \in L_b$ for $m < n$, note that $a^m b^n = a^m b^m b^k$ for some $k \geq 1$. Using (Id) and ($\cdot$R), it follows that $ab \in L_{a\cdot b}$. Since $a \cdot b = 1$, we have $L_{a\cdot b} = L_1$. Thus $ab \in L_1$. Consequently, one can show that $a^m b^m \in L_1$ by induction on m. For $m = 0$ we have $1 \in L_1$ by (Id). For $m > 0$ we have $a^{m-1}b^{m-1} \in L_1$ by induction hypothesis. By (1L) it follows that $a^{m-1}1b^{m-1} \in L_1$. Thus $a^m b^m \in L_1$ by (Cut). Now it remains to show that $a^m b^m b^k \in L_b$ by induction on k. If $k = 1$ then $a^m b^m b \in L_b$ by (Id) and ($\cdot$R). Suppose that $k > 1$. Since $b \vee 1 = 1$, we have $b \in L_{b\vee 1} = L_1$ by ($\vee$R). Thus $x \in L_b$ implies $xb \in L_{b\cdot 1} = L_b$ by ($\cdot$R). Consequently, $a^m b^m b^k \in L_b$ for all $k \geq 1$.

Suppose now that $a^m b^n \in L_b$ and $m \geq n$. Since $b \in L_1$, we have $a^m b^m \in L_b$ by repetitive applications of ($\cdot$R). Consequently, $\varepsilon \in L_b$ by repetitive applications of ($\cdot$L), (Cut) and $\varepsilon \in L_1 = L_{a\cdot b}$. Thus $1 \in L_b$ by (1L) which contradicts the fact that $\mathcal{L}$ is separating because $1 \not\leq b$.

Theorem 17. *Let V be a variety of residuated lattices containing the abelian ℓ-group $\mathbf{G}$. Then V does not enjoy the FEP.*

Acknowledgements The author was supported by the project P202/12/G061 of the Czech Science Foundation and by the long-term strategic development financing of the Institute of Computer Science (RVO:67985807).

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