Decidability of Cancellative Extension of Monoidal T-norm Based Logic

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Abstract

It is known that the monoidal t-norm based logic (MTL) and many of its schematic extensions are decidable. The usual way how to prove decidability of some schematic extension of MTL is to show that the corresponding class of algebras of truth values has the finite model property (FMP) or the finite embeddability property (FEP). However this method does not work for the extensions whose corresponding classes of algebras have only trivial finite members. Typical examples of such extensions are the product logic and the cancellative extension of MTL (ΠMTL) because the only finite algebras belonging to the corresponding varieties are finite Boolean algebras. The product logic is known to be decidable because of its connection with ordered Abelian groups. However the decidability of ΠMTL was not known. This paper solves this problem.

1 Introduction

In the last two decades many complexity results on propositional many-valued logics have appeared. This research in our context was started by Mundici in [17] where it is shown that the set of tautologies of Łukasiewicz logic (Ł) is coNP-complete. The same was proved also for Gödel logic (G) and the product logic (Π) in [2]. The coNP-completeness of tautologies of Hájek's basic fuzzy logic (BL) was shown in [3]. Finally in [13], the same result was also obtained for any particular t-algebra (i.e. BL-algebra on the real interval $[0, 1]$ with the monoidal operation given by a continuous t-norm). An overview of these results can be found in [1].

Although for the logics above BL we have the above mentioned results, the situation in the monoidal t-norm based logic (MTL) and its schematic extensions is still unclear. So far there are only results on decidability. The usual way how to prove decidability of some schematic extension of MTL is to show that the corresponding class of algebras of truth values has the finite model property (FMP) or the finite embeddability property (FEP). MTL, SMTL, and IMTL are decidable since they have the FEP (for the proof see [6]). However this method does not work for the extensions whose corresponding classes of algebras have only trivial finite members. Typical examples of such extensions are Π and the cancellative extension of MTL (ΠMTL) because the only finite algebras belonging to the corresponding varieties are finite Boolean algebras. Π is known to be decidable because of its connection with ordered Abelian groups. However the decidability of ΠMTL was not known. In this paper we solve this problem.

The paper is organized as follows; Section 2 recalls basic definitions and main results needed in the sequel. In Section 3 we improve the completeness theorem in the sense that ΠMTL is complete w.r.t. the class of ΠMTL-chains arising from finitely generated submonoids of finite lexicographic products of reals. Section 4 then shows that we can consider instead of lexicographic products of reals lexicographic products of integers. Thus we obtain a completeness w.r.t. a countable class of algebras. Finally, using this result we prove that ΠMTL is decidable in Section 5.

2 Preliminaries

We start with basic definitions and results on MTL and its schematic extensions which we will need in the sequel. The language of MTL consists of a countable set of propositional variables, a strong conjunction &, a minimum conjunction $\wedge$, an implication $\Rightarrow$, and a truth constant $\bot$. Derived connectives are defined as follows:

$$\begin{array}{ccl} \varphi \vee \psi & \text{is} & ((\varphi \Rightarrow \psi) \Rightarrow \psi) \wedge ((\psi \Rightarrow \varphi) \Rightarrow \varphi)\,, \\ \neg\varphi & \text{is} & \varphi \Rightarrow \bot\,, \\ \varphi \equiv \psi & \text{is} & (\varphi \Rightarrow \psi)\&(\psi \Rightarrow \varphi)\,, \\ \top & \text{is} & \neg\bot\,. \end{array}$$

MTL has a Hilbert style calculus introduced in [7]. The following are the axioms of MTL:

$$\begin{array}{ll} \text{(A1)} & (\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))\,, \\ \text{(A2)} & \varphi\&\psi \Rightarrow \varphi\,, \\ \text{(A3)} & \varphi\&\psi \Rightarrow \psi\&\varphi\,, \\ \text{(A4)} & (\varphi \wedge \psi) \Rightarrow \varphi\,, \\ \text{(A5)} & (\varphi \wedge \psi) \Rightarrow (\psi \wedge \varphi)\,, \\ \text{(A6)} & (\varphi\&(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \wedge \psi)\,, \\ \text{(A7a)} & (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\varphi\&\psi \Rightarrow \chi)\,, \\ \text{(A7b)} & (\varphi\&\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))\,, \\ \text{(A8)} & ((\varphi \Rightarrow \psi) \Rightarrow \chi) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \chi) \Rightarrow \chi)\,, \\ \text{(A9)} & \bot \Rightarrow \varphi\,. \end{array}$$

The deduction rule of MTL is modus ponens. The notion of a proof is defined in the usual way. Let T be a theory over MTL and φ be a formula. Then we write $T \vdash \varphi$ if φ is provable from T in the system MTL.

MTL has several important axiomatic extensions. Some of them were mentioned in the introduction. BL is the extension of MTL by the following axiom schema:

$$\text{(Div)} \quad (\varphi \wedge \psi) \Rightarrow (\varphi\&(\varphi \Rightarrow \psi))\,.$$

Łukasiewicz logic can be obtained by adding the law of involution to BL, i.e.

$$\text{(Inv)} \quad \neg\neg\varphi \Rightarrow \varphi\,.$$

Finally, the product logic is the extension of BL by the following schema:

$$\text{(C)} \quad \neg\psi \vee ((\psi \Rightarrow \varphi\&\psi) \Rightarrow \varphi)\,.$$

Originally the product logic was defined as the extension of BL by the axiom schemata (Π1) and (Π2), see [10]. Our definition comes from [18] where it was shown that both definitions are equivalent.

If we add (Inv) to MTL, we obtain the involutive monoidal t-norm based logic (IMTL) which is strictly weaker than Łukasiewicz logic. If we extend MTL by (C), we get the cancellative extension of MTL (ΠMTL) which is strictly weaker than product logic. Similarly as in the case of product logic, the original definition of ΠMTL in [12] used the schemata (Π1) and (Π2) instead of (C), for details see [18].

The semantical part of MTL is based on the notion of an MTL-algebra which is a special kind of residuated lattice (for details on residuated lattices see [16]). A commutative residuated lattice $\mathbf{A} = (A, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is an algebraic structure, where $(A, *, \mathbf{1})$ is a commutative monoid, $(A, \wedge, \vee)$

is a lattice, and $(*, \to)$ forms a residuated pair, i.e. $x * y \le z$ iff $x \le y \to z$. When we refer to a commutative residuated lattice, we will omit the word commutative since we will deal here only with the commutative case. The symbol x^n stands for $x * \cdots * x$ (n times). In the absence of parenthesis, $*$ is performed first, followed by $\to$, and finally $\vee$ and $\wedge$. The operation $\to$ is called residuum. It follows from the definition that $\to$ is antitone in the first argument and monotone in the second one. Moreover, the residuum is fully determined by $*$ and the order. It can be expressed as follows:

$$x \to y = \max\{z \in A \mid x * z \le y\}\,.$$

We also have the following:

Lemma 2.1 *Let* $\mathbf{A} = (A, *_A, \to_A, \wedge_A, \vee_A, \mathbf{1}_A)$ *and* $\mathbf{B} = (B, *_B, \to_B, \wedge_B, \vee_B, \mathbf{1}_B)$ *be residuated lattices such that their* $\to$*-free reducts are isomorphic. Then* $\mathbf{A}$ *and* $\mathbf{B}$ *are isomorphic as residuated lattices as well.*

PROOF: Let $h : A \to B$ be the isomorphism between the $\to$-free reducts of $\mathbf{A}$ and $\mathbf{B}$. We have to show that $h(x \to_A y) = h(x) \to_B h(y)$. By residuation we have $x *_A (x \to_A y) \le_A y$. Since h is a lattice and monoidal homomorphism, we get

$$h(x *_A (x \to_A y)) = h(x) *_B h(x \to_A y) \le_B h(y)\,.$$

Thus again by residuation we obtain $h(x \to_A y) \le_B h(x) \to_B h(y)$.

On the other hand, we have $h(x) *_B (h(x) \to_B h(y)) \le_B h(y)$. Thus we get

$$h^{-1}(h(x) *_B (h(x) \to_B h(y))) = x *_A h^{-1}(h(x) \to_B h(y)) \le_A y\,.$$

Finally by residuation we get $h^{-1}(h(x) \to_B h(y)) \le_A x \to_A y$ which is equivalent to $h(x) \to_B h(y) \le_B h(x \to_A y)$. □

A residuated lattice $\mathbf{A}$ is said to be integral if $\mathbf{1}$ is the top element of A. In this case we have $x \le y$ iff $x \to y = \mathbf{1}$. If a residuated lattice possesses a bottom element $\mathbf{0}$, then we have $\mathbf{0} * x = \mathbf{0}$. A residuated lattice $\mathbf{A}$ is said to be cancellative if for all $x, y, z \in L$, $x * z = y * z$ implies $x = y$. A totally ordered residuated lattice is called a residuated chain. A residuated lattice, which is isomorphic to a subdirect product of residuated chains, is said to be representable.

Definition 2.2 An MTL-algebra is a structure $\mathbf{L} = (L, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1})$ where the following conditions are satisfied:

1. $(L, *, \to, \wedge, \vee, \mathbf{1})$ is an integral residuated lattice,
2. $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice,
3. the identity $(x \to y) \vee (y \to x) \approx \mathbf{1}$ is valid in $\mathbf{L}$.

A totally ordered MTL-algebra is called an MTL-chain.

A mapping e from formulas to L is called an $\mathbf{L}$-evaluation if $e(\top) = \mathbf{1}$, $e(\bot) = \mathbf{0}$, and $e(\varphi \circ \psi) = e(\varphi) \circ e(\psi)$ for $\circ \in \{\&, \Rightarrow, \wedge\}$. We will write $T \models_{\mathbf{L}} \varphi$ if for each $\mathbf{L}$-evaluation e such that $e(T) \subseteq \{\mathbf{1}\}$ we have $e(\varphi) = \mathbf{1}$. Throughout the text we will use without mentioning also the alternative signature of an MTL-algebra where the lattice operations $\wedge$ and $\vee$ are substituted by the corresponding order $\le$.

MTL is an algebraizable logic in the sense of Blok and Pigozzi [4] and the variety of MTL-algebras is its equivalent algebraic semantics. This means that there is one-to-one correspondence between schematic extensions of MTL and subvarieties of MTL-algebras. If Σ is a set of formulas and L is the logic obtained by adding to MTL the formulas from Σ as axiom schemata, then the equivalent algebraic semantics of L is the subvariety of $\mathbb{MTL}$ axiomatized by the equations $\{\varphi \approx 1 : \varphi \in \Sigma\}$. Since this paper deals mainly with ΠMTL, we recall also the precise definition of a ΠMTL-algebra.

Definition 2.3 A ΠMTL-algebra $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is an MTL-algebra satisfying the identity $\neg y \vee ((y \rightarrow x * y) \rightarrow x) \approx \mathbf{1}$, where $\neg y = y \rightarrow \mathbf{0}$. A totally ordered ΠMTL-algebra is called a ΠMTL-*chain*.

ΠMTL is complete w.r.t. its algebraic semantics. Moreover, since each ΠMTL-algebra is isomorphic to a subdirect product of ΠMTL-chains, we have even completeness w.r.t. the class of totally ordered ΠMTL-algebras.

Theorem 2.4 *Let T be a theory over* ΠMTL *and φ be a formula. Then $T \vdash \varphi$ if and only if $T \models_{\mathbf{L}} \varphi$ for each* ΠMTL-*chain* $\mathbf{L}$.

Let $\mathbb{V}$ denote the variety of all ΠMTL-algebras and $E_{\mathbf{L}}$ be the set of all $\mathbf{L}$-evaluations. The main goal of this paper is to show that ΠMTL is decidable, i.e. the following set

$$TAUT = \{\varphi \mid (\forall \mathbf{L} \in \mathbb{V})(\forall e \in E_{\mathbf{L}})(e(\varphi) = \mathbf{1})\}$$

is decidable. Clearly $TAUT \in \Sigma_1$ by Theorem 2.4. Thus it is sufficient to show that also its complement is in Σ_1. Observe also that since ΠMTL is an algebraizable logic, the decidability of $TAUT$ is equivalent to the decidability of the equational theory of ΠMTL-algebras.

As we mentioned in the introduction the usual method how to prove the decidability of an algebraizable logic is to show that the corresponding algebraic semantics possesses the finite embeddability property (FEP). Let us recall what it means.

Definition 2.5 Let $\mathbf{A} = (A, \langle f_i : i \in I \rangle)$ be an algebra and let $B \subseteq A$ be an non-empty set. The partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ with domain B is the partial algebra $(B, \langle f_i : i \in I \rangle)$, where for every $i \in I$, f_i n-ary, $b_1, \ldots, b_n \in B$,

$$f_i^{\mathbf{B}}(b_1, \ldots, b_n) = \begin{cases} f_i^{\mathbf{A}}(b_1, \ldots, b_n) & \text{if } f_i^{\mathbf{A}}(b_1, \ldots, b_n) \in B, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

By an embedding of a partial algebra $\mathbf{B}$ into an algebra $\mathbf{A}$ we mean one-to-one mapping $h : B \rightarrow A$ that preserves all existing operations, i.e., if for some operation symbol f_i and $b_1, \ldots, b_k \in B$ we have $f_i^{\mathbf{B}}(b_1, \ldots, b_k)$ is defined, then $h(f_i^{\mathbf{B}}(b_1, \ldots, b_k)) = f_i^{\mathbf{B}}(h(b_1), \ldots, h(b_k))$.

Given a class $\mathbb{K}$ of algebras, $\mathbb{K}_{fin}$ will denote the class of its finite members.

Definition 2.6 A class $\mathbb{K}$ of algebras has the finite embeddability property (FEP, for short) if, and only if, every finite partial subalgebra of some member of $\mathbb{K}$ can be embedded in some algebra of $\mathbb{K}_{fin}$.

The FEP is interesting for its connection to other well-known properties.

Definition 2.7 A class $\mathbb{K}$ of algebras of the same type has the strong finite model property (SFMP, for short) if, and only if, every quasiequation that fails to hold in every algebra of $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.

Definition 2.8 A class $\mathbb{K}$ of algebras of the same type has the finite model property (FMP, for short) if, and only if, every equation that fails to hold in every algebra of $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.

A variety has the FMP if, and only if, it is generated by its finite members and a quasivariety has the SFMP if, and only if, it is generated (as a quasivariety) by its finite members. In [5] it is proved that for classes of algebras of finite type closed under finite products (hence, in particular, for varieties of MTL-algebras) the FEP and the SFMP are equivalent. This means that if some variety has the FEP then its decidability already follows.

However in our case we cannot proceed in this way because the variety of ΠMTL-algebras does not possess the FEP. In [12] it was shown the following result.

Lemma 2.9 *An* MTL-*chain* $\mathbf{L}$ *is a* ΠMTL-*chain if and only if for any* $x, y, z \in L$, $z \neq \mathbf{0}$, *we have* $x * z = y * z$ *implies* $x = y$.

As a corollary of this lemma we obtain that the only nontrivial finite ΠMTL-chain is the two element Boolean algebra. Thus the class of all finite ΠMTL-algebras is the class of all finite Boolean algebras. Consequently, we need to proceed in a different way. In fact, we will prove that it is enough to show that there is a countable class of ΠMTL-algebras such that each finite partial subalgebra of a ΠMTL-chain is embeddable into a member of this class.

Before we continue, we will show that we need not consider the bottom element during the construction of the embedding. Due to Lemma 2.9 it can be shown that there is a connection between cancellative residuated chains and ΠMTL-chains.

Lemma 2.10 *Let* $\mathbf{L} = (L, *, \rightarrow, \leq, \mathbf{0}, \mathbf{1})$ *be a* ΠMTL-*chain and* $\mathbf{L}' = L - \{\mathbf{0}\}$. *Then the subreduct* $\mathbf{L}' = (L', *, \rightarrow, \leq, \mathbf{1})$ *is an integral cancellative residuated chain.*

Also the other direction is possible. If we have an integral cancellative residuated chain $\mathbf{L} = (L, *, \rightarrow, \leq, \mathbf{1})$, we can extend it to a ΠMTL-chain by adding a bottom element $\mathbf{0}$. Let $L_0 = L \cup \{\mathbf{0}\}$ be the new universe. The order $\leq'$ is an extension of $\leq$ in such a way that $\mathbf{0} \leq' x$ for all $x \in L$. The operations are defined as follows:

$$x *' y = \begin{cases} x * y & x, y \in L\,, \\ \mathbf{0} & x = \mathbf{0} \text{ or } y = \mathbf{0}\,, \end{cases}$$

$$x \rightarrow' y = \begin{cases} x \rightarrow y & x, y \in L\,, \\ \mathbf{1} & x = \mathbf{0}\,, \\ \mathbf{0} & y = \mathbf{0} \text{ and } x > \mathbf{0}\,. \end{cases}$$

Lemma 2.11 *Let* $\mathbf{L} = (L, *, \rightarrow, \leq, \mathbf{1})$ *be an integral cancellative residuated chain. Then the structure* $\mathbf{L}_0 = (L_0, *', \rightarrow', \leq', \mathbf{0}, \mathbf{1})$ *is a* ΠMTL-*chain.*

Lemma 2.12 *Let* $\mathbf{A}$ *and* $\mathbf{L}$ *be integral cancellative residuated chains and* $\mathbf{B}$ *be a partial subalgebra of* $\mathbf{A}$. *Further, let* $\mathbf{A}_0$ *and* $\mathbf{L}_0$ *be the* ΠMTL-*chains constructed as in Lemma 2.11. Suppose that there is an embedding* $h' : B \rightarrow L$. *Then there is an embedding* $h : B_0 \rightarrow L_0$ *of the partial subalgebra* $\mathbf{B}_0$ *into* $\mathbf{L}_0$ *where* $B_0 = B \cup \{\mathbf{0}\}$.

PROOF: We will extend h' to h as follows:

$$h(x) = \begin{cases} h'(x) & \text{if } x \neq \mathbf{0}\,, \\ \mathbf{0} & \text{otherwise}\,. \end{cases}$$

We have to show that h is an embedding. Obviously it is one-to-one. Let $x \in B_0$. Then

$$h(\mathbf{0}) * h(x) = \mathbf{0} * h(x) = \mathbf{0} = h(\mathbf{0}) = h(\mathbf{0} * x)\,.$$

Similarly for $\wedge$ and $\vee$. If $\mathbf{0} \rightarrow x = \mathbf{1} \in B_0$ we have

$$h(\mathbf{0}) \rightarrow h(x) = \mathbf{0} \rightarrow h(x) = \mathbf{1} = h(\mathbf{1}) = h(\mathbf{0} \rightarrow x)\,.$$

Finally, if $x > \mathbf{0}$ then $h(x) > \mathbf{0}$ and we have

$$h(x) \rightarrow h(\mathbf{0}) = h(x) \rightarrow \mathbf{0} = \mathbf{0} = h(\mathbf{0}) = h(x \rightarrow \mathbf{0})\,.$$

Hence h is an embedding of the partial subalgebra $\mathbf{B}_0$ into $\mathbf{L}_0$. □

At the end of this section we also recall the notion of an Archimedean class and some of its properties. Let us recall that a totally ordered commutative monoid $\mathbf{A} = (A, *, \leq, \mathbf{1})$ is a structure where $(A,*,\mathbf{1})$ is a commutative monoid, $(A,\leq)$ is a chain, and $x\leq y$ implies $x*z\leq y*z$ for any $x,y,z\in A$. Again the word commutative will be omitted since we will deal only with the commutative monoids. Similarly as in the case of a residuated lattice, a totally ordered monoid is said to be integral if the neutral element $\mathbf{1}$ is the top element as well.

Definition 2.13 Let $\mathbf{A}=(A,*,\leq,\mathbf{1})$ be an integral totally ordered monoid, $a,b\in A$, and $\sim$ be an equivalence on A defined as follows:

$$a\sim b \text{ iff there exists an } n\in\mathbb{N} \text{ such that } a^n\leq b\leq a \text{ or } b^n\leq a\leq b.$$

Then for any $a\in A$ the equivalence class $[a]_\sim$ is called an Archimedean class.

We list several easy results about Archimedean classes.

Lemma 2.14 *Let $\mathbf{A}=(A,*,\leq,\mathbf{1})$ be an integral totally ordered monoid and $a,b\in A$. Then the following holds:*

1. *$[a]_\sim$ is closed under $*$.*
2. *$[a]_\sim$ is convex, i.e. for all $x,y\in[a]_\sim$, $x\leq z\leq y$ implies $z\in[a]_\sim$.*
3. *$[a*b]_\sim=[\min\{a,b\}]_\sim$.*

Note that by Lemma 2.14(2) Archimedean classes of the monoid $\mathbf{A}$ can be totally ordered by letting $[a]_\sim<[b]_\sim$ iff $a<b$ and $[a]_\sim\cap[b]_\sim=\emptyset$. Further, if $\mathbf{A}$ is finitely generated then the set of Archimedean classes of $\mathbf{A}$ is finite by Lemma 2.14(3).

Now we will discuss a certain class of totally ordered monoids which will be used at the sequel. Let $\mathbf{R}^n_{\mathrm{lex}}$ denote the lexicographic product of n-copies of the additive group of reals $(\mathbb{R},+,-,\leq,0)$, i.e., the universe of $\mathbf{R}^n_{\mathrm{lex}}$ is $\mathbb{R}^n$, the group operation is defined component-wise, and the order is lexicographic. The neutral element $\langle 0,\dots,0\rangle$ of $\mathbf{R}^n_{\mathrm{lex}}$ will be denoted by $\overline{0}$ for short. By $\mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$ we will denote the negative cone of $\mathbf{R}^n_{\mathrm{lex}}$. Observe that $\mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$ forms an integral cancellative residuated chain if we define $x\to y=y-x$ for $x>y$ and $\overline{0}$ otherwise. Moreover, $\mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$ has $n+1$ Archimedean classes.

We are especially interested in the finitely generated submonoids of $\mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$. Let $\mathbf{A}$ be such a submonoid. Then $\mathbf{A}$ has finitely many Archimedean classes as we already mentioned. Let $\mathcal{C}_A=\{C_1,\dots,C_k\}$ be the chain of all Archimedean classes of $\mathbf{A}$. Note that $k\leq n+1$ and $C_k=\{\overline{0}\}$. Moreover, for each C_r, $r<k$, there is $j\in\{1,\dots,n\}$ such that $\langle x_1,\dots,x_n\rangle\in C_r$ iff $x_j<0$ and $x_i=0$ for all $i<j$. Let t be the following function:

$$t(r)=\begin{cases} n+1 & \text{if } r=k\,,\\ \min\{j\mid x_j<0\} \text{ for some } \langle x_1,\dots,x_n\rangle\in C_r \text{ otherwise}\,. \end{cases} \tag{1}$$

If we define an equivalence on $\mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$ by $\langle x_1,\dots,x_n\rangle\sim_j\langle y_1,\dots,y_n\rangle$ iff $x_i=y_i$ for all $1\leq i<j$, then for $r<k$ we have

$$C_r=\{x\in A\mid x\sim_{t(r)}\overline{0}\}-\{x\in A\mid x\sim_{t(r+1)}\overline{0}\}\,.$$

Example 2.15 Consider the following subset of $\mathrm{N}(\mathbf{R}^4_{\mathrm{lex}})$:

$$X=\{\langle -1,1,1,1\rangle,\langle 0,0,-2,2\rangle,\langle 0,0,0,-3\rangle\}\,.$$

Let $\mathbf{A}$ be the submonoid of $\mathbf{R}^4_{\mathrm{lex}}$ generated by X with the order inherited from $\mathbf{R}^4_{\mathrm{lex}}$. Since all elements in X are less than $\overline{0}$, $\mathbf{A}$ is integral. The following are the Archimedean classes of $\mathbf{A}$:

$$\begin{aligned} C_1 &= \{\langle x_1,x_2,x_3,x_4\rangle\in A\mid x_1<0\}\,,\\ C_2 &= \{\langle x_1,x_2,x_3,x_4\rangle\in A\mid x_1=x_2=0,\ x_3<0\}\,,\\ C_3 &= \{\langle x_1,x_2,x_3,x_4\rangle\in A\mid x_1=x_2=x_3=0,\ x_4<0\}\,,\\ C_4 &= \{\overline{0}\}\,. \end{aligned}$$

The function defined by Equation 1 looks in our particular case as follows:

$$t(1)=1\,,\quad t(2)=3\,,\quad t(3)=4\,,\quad t(4)=5\,.$$

3 A smaller generating class for ΠMTL-algebras arising from lexicographic products of reals

In this section we will improve Theorem 2.4 by showing that there is a smaller class generating the variety of ΠMTL-algebras. Let $\mathbf{A}$ be an integral cancellative residuated chain. By $\hat{\mathbf{A}}$ we will denote the $\rightarrow$-free reduct of $\mathbf{A}$. Given $X \subseteq A$, $\mathbf{M}(X)$ will denote the subalgebra of $\hat{\mathbf{A}}$ generated by X. Note that $\mathbf{M}(X)$ is in fact submonoid of $\mathbf{A}$ generated by X with the order inherited from $\mathbf{A}$. The following lemma shows that $\mathbf{M}(X)$ can be sometimes extended to an integral cancellative residuated chain (for the proof see [14, Lemma 3.1] or [15, Lemma 5]).

Lemma 3.1 *Let $\mathbf{B}$ be an integral finitely generated totally ordered monoid. Then $\mathbf{B}$ is inversely well ordered (i.w.o.), i.e., each nonempty subset of B has a maximum.*

Thus by the latter lemma $\mathbf{M}(X)$ is i.w.o. and a residuum in $\mathbf{M}(X)$ exists. It can be computed as follows:

$$x \rightarrow y = \max\{z \in M(X) \mid x * z \leq y\}\,.$$

Let us denote the resulting integral cancellative residuated chain by $\mathbf{M}(X)^{\rightarrow}$. The extension of $\mathbf{M}(X)^{\rightarrow}$ to a ΠMTL-chain by Lemma 2.11 will be denoted by $\mathbf{M}(X)_0^{\rightarrow}$.

We are going to improve Theorem 2.4 by showing that ΠMTL is complete w.r.t. those ΠMTL-chains which arise from finitely generated submonoids of $\mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$. More precisely we are claiming the following statement.

Theorem 3.2 *Let T be a finite theory over* ΠMTL *and φ be a formula. Then $T \vdash \varphi$ iff $T \models_{\mathbf{M}(X)_0^{\rightarrow}} \varphi$ for each finite subset $X \subseteq \mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$ and each $n \in \mathbb{N}$.*

Proof of Theorem 3.2

Let T be a finite theory over ΠMTL and φ be a formula such that $T \not\vdash \varphi$. Then there is a ΠMTL-chain $\mathbf{L} = (L, *_L, \rightarrow_L, \leq_L, \mathbf{0}_L, \mathbf{1}_L)$ such that $T \not\models_{\mathbf{L}} \varphi$ by Theorem 2.4. Let e be the $\mathbf{L}$-evaluation such that $e(\varphi) < \mathbf{1}_L$ and $e(T) = \{\mathbf{1}_L\}$. Let $X = \{e(\psi) \mid \psi \text{ is a subformula of } \tau \in T \cup \varphi\} \cup \{\mathbf{1}_L\}$. Thus $\mathbf{X}$ is a finite partial subalgebra of $\mathbf{L}$. We are going to construct an embedding of the partial subalgebra $\mathbf{X}$ into a suitable ΠMTL-chain. By Lemma 2.12 we can assume that $\mathbf{0}_L \notin X$. Firstly we will show the following results.

Lemma 3.3 *Let $\mathbf{M}(X)^{\rightarrow} = (M(X), *, \rightarrow, \leq, \mathbf{1})$ be the integral cancellative residuated chain arising from the submonoid of $\mathbf{L}$ generated by X. The identity mapping $id : X \rightarrow M(X)^{\rightarrow}$ is an embedding of the partial subalgebra $\mathbf{X}$ into $\mathbf{M}(X)^{\rightarrow}$.*

PROOF: Note that $*$, $\leq$ are just restrictions of $*_L$ and $\leq_L$ respectively. Moreover $\mathbf{1} = \mathbf{1}_L$. The mapping id is obviously one-to-one and order-preserving. Let $x, y, z \in X$. Suppose that $z = x *_L y$. Then $id(x) * id(y) = x * y = x *_L y = z = id(z)$. Now assume that $z = x \rightarrow_L y = \max\{z' \in L \mid x *_L z' \leq_L y\}$. Then

$$id(x) \rightarrow id(y) = \max\{z' \in M(X) \mid x *_L z' \leq_L y\} \leq_L x \rightarrow_L y = z = id(z)\,,$$

because $\{z' \in M(X) \mid x *_L z' \leq_L y\} \subseteq \{z' \in L \mid x *_L z' \leq_L y\}$. On the other hand since $z \in M(X)$ and $x * z \leq y$, we have

$$id(z) = z \leq \max\{z' \in M(X) \mid x * z' \leq y\} = id(x) \rightarrow id(y)\,.$$

Thus $id(z) = id(x) \rightarrow id(y)$. □

Now we finish the proof of Theorem 3.2. Each cancellative totally ordered monoid can be embedded into a totally ordered group by introducing fractions (see e.g. [8, Page 161, Corollary 5]). Thus $\mathbf{M}(X)$ can be embedded into a totally ordered Abelian group $\mathbf{G}$. Since $\mathbf{M}(X)$

is finitely generated, $\mathbf{G}$ is finitely generated as well. Further it is well-known that each totally ordered Abelian group can be embedded into a full Hahn group (see [8, 9]). Since $\mathbf{G}$ is finitely generated, this Hahn group is just $\mathbf{R}^n_{\text{lex}}$ for some $n \in \mathbb{N}$. Thus we can embed $\mathbf{M}(X)$ into $\mathbf{R}^n_{\text{lex}}$ by an order-preserving embedding h. Since $\mathbf{M}(X)$ is integral, we have $h(M(X)) \subseteq \mathrm{N}(\mathbf{R}^n_{\text{lex}})$. Clearly $\mathbf{M}(X) \cong h(\mathbf{M}(X)) = \mathbf{M}(h(X))$. Due to Lemma 2.1 we in fact get $\mathbf{M}(X)^{\rightarrow} \cong \mathbf{M}(h(X))^{\rightarrow}$. Thus $h \circ id : X \to M(h(X))$ is an embedding of the partial subalgebra $\mathbf{X}$ into $\mathbf{M}(h(X))$ and the theorem follows. □

Remark 3.4 Let T be a finite theory over ΠMTL and φ be a formula. It follows from the proof of Theorem 3.2 that if $T \not\vdash \varphi$ then there exists a finite subset $X \subseteq \mathrm{N}(\mathbf{R}^n_{\text{lex}})$, $\bar{0} \in X$, and $\mathbf{M}(X)^{\rightarrow}_{\bar{0}}$-evaluation e such that $e(T) \subseteq \{\bar{0}\}$ and $e(\varphi) < \bar{0}$ (note that $\bar{0}$ is the top element in $\mathbf{M}(X)^{\rightarrow}_{\bar{0}}$). Moreover, let $S = \{\psi \mid \psi \text{ is a subformula of } \tau \in T \cup \{\varphi\}\}$. Then $e(S) \subseteq X \cup \{\mathbf{0}\}$ where $\mathbf{0}$ is the bottom element in $\mathbf{M}(X)^{\rightarrow}_{\bar{0}}$ (see Lemma 2.11).

Corollary 3.5 *The class of all* $\mathbf{M}(X)^{\rightarrow}_{\bar{0}}$ *for each finite subset* $X \subseteq \mathrm{N}(\mathbf{R}^n_{\text{lex}})$ *generates (as a quasivariety) the variety of* ΠMTL*-algebras.*

Corollary 3.6 *The class of all* $\mathbf{M}(X)^{\rightarrow}$ *for each finite subset* $X \subseteq \mathrm{N}(\mathbf{R}^n_{\text{lex}})$ *generates (as a quasivariety) the variety of commutative integral cancellative representable residuated lattices.*

4 A generating class for ΠMTL-algebras arising from lexicographic products of integers

In this section we will further reduce the generating set for ΠMTL-algebras. We in fact show that instead of lexicographic products of reals, we can consider lexicographic products of integers. Let $\mathbf{Z}^n_{\text{lex}}$ denote the lexicographic product of n-copies of the additive group of integers $(\mathbb{Z}, +, -, \leq, 0)$. By $\mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ we will denote the negative cone of $\mathbf{Z}^n_{\text{lex}}$. Again $\mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ forms an integral cancellative residuated chain.

We will start with a crucial lemma which enables us to approximate reals by integers. If $B \subseteq V$ is a basis of a vector space V and $x \in V$ then we write $x = \langle q_1, \ldots, q_n \rangle_B$ for expressing the fact that the vector x has coordinates $q_1, \ldots, q_n$ in the basis B. By $\mathbb{R}^-$ we will denote the set of non-positive reals.

Lemma 4.1 *Let* X *be a finite subset of* $\mathbb{R}^-$ *and* $a \in M(X)$*. Then there is a monoid homomorphism* $h : M(X) \to \mathbb{Z}$ *such that* h *is one-to-one and order-preserving for all* $x, y \in [a, 0] \cap M(X)$*.*

PROOF: Let us assume that $X \neq \{0\}$. For $X = \{0\}$ the statement is trivial. Firstly, observe that the set $[a, 0] \cap M(X)$ is finite. Indeed, since the addition in $\mathbb{R}^-$ is Archimedean (i.e., for each $x, y \in \mathbb{R}^-$ such that $y < x < 0$ there is $k \in \mathbb{N}$ such that $kx < y$), there is $k_i \in \mathbb{N}$ for each $x_i \in X$ such that $k_i x_i < a$. Since each element $y \in M(X)$ can be expressed as $y = \sum l_i x_i$ for some $l_i \in \mathbb{N}$ and $x_i \in X$, we get that only finitely many elements from $M(X)$ can be greater or equal to a.

Now recall that $\mathbb{R}$ forms a vector space over the field of rational numbers $\mathbb{Q}$. Thus there is a maximal linearly independent subset $B \subseteq X$ which is a basis of the subspace generated by X. Let d be its dimension, i.e. $d = |B|$. Then any element $x \in M(X)$ can be uniquely expressed as $x = \sum_{i=1}^{d} q_i b_i$ where $b_i \in B$, $q_i \in \mathbb{Q}$. Let $C = \{\langle q_1, \ldots, q_d \rangle \mid x = \langle q_1, \ldots, q_d \rangle_B, x \in X\}$ and $s \in \mathbb{N}$ be the smallest natural number such that $sC \subseteq \mathbb{Z}^d$. There is such s since C is finite.

Let $m = \min\{|x - y| \mid x \neq y,\ x, y \in [a, 0] \cap M(X)\}$. Such a minimum exists since $[a, 0] \cap M(X)$ is finite. Let $\varepsilon = m/(2u)$ where

$$u = \max\left\{ \sum_{i=1}^{d} |q_i| \;\middle|\; x = \langle q_1, \ldots, q_d \rangle_B, x \in [a, 0] \cap M(X) \right\}.$$

Let $x \in \mathbb{R}$. Let $[x]$ denote a function such that $[x]$ is some of the nearest integers to x for every x. Clearly $[x] = x + \delta$ for some $|\delta| < 1$. Then for every $n \in \mathbb{N}$ we have

$$\big|[nx]/n - x\big| = \big|\delta/n\big| < 1/n\,.$$

Let us choose $n_0 \in \mathbb{N}$ such that one has $\big|[n_0x]/n_0 - x\big| < 1/n_0 \leq \varepsilon$ for all $x \in B$.

Now we define the mapping $h : M(X) \to \mathbb{Z}$. If $b \in B$ then $h(b) = s[n_0b]$. If $x \in M(X)$ then x can be uniquely expressed as $x = \sum q_ib_i$ where $b_i \in B$ and $q_i \in \mathbb{Q}$. Thus we can extend h to a monoid homomorphism by $h(x) = \sum q_ih(b_i)$ for every $x \in M(X)$. We will show that $h(x) \in \mathbb{Z}$. Since $x \in M(X)$, we have $x = \sum_{i=1}^n k_ix_i$ for some $k_i \in \mathbb{N}$ and $x_i \in X$. For each $x_i \in X$, we have $x_i = \langle q_{i1}, \ldots, q_{id}\rangle_B$. By the choice of s, the numbers sq_{ij} are integers for every $x_i \in X$. Hence, $h(x_i) \in \mathbb{Z}$ for every $x_i \in X$ and, consequently also for every $x \in M(X)$.

We have to show that h is one-to-one and order-preserving on $[a, 0] \cap M(X)$. Suppose that $x, y \in [a, 0] \cap M(X)$ such that $x < y$. Then $x = \sum q_ib_i$ and $y = \sum t_ib_i$ for some $q_i, t_i \in \mathbb{Q}$. We need to prove that $h(x) < h(y)$ which is equivalent to $h(x)/sn_0 < h(y)/sn_0$. For every i we have

$$\big|h(b_i)/sn_0 - b_i\big| < 1/n_0 \leq \varepsilon\,.$$

Hence,

$$\big|h(x)/sn_0 - x\big| = \Big|\sum q_i\big(h(b_i)/sn_0 - b_i\big)\Big| < \Big|\sum q_i\Big|\varepsilon \leq u\varepsilon\,.$$

Similarly, $\big|h(y)/sn_0 - y\big| < u\varepsilon$. Consequently,

$$h(y)/sn_0 - h(x)/sn_0 > y - x - 2u\varepsilon = y - x - m \geq 0\,.$$

□

Lemma 4.2 *Let X be a finite subset of $\mathbb{R}^-$ and $h : M(X) \to \mathbb{Z}$ be the monoid homomorphism from Lemma 4.1 such that h is one-to-one and order-preserving for all $x, y \in [2m, 0] \cap M(X)$, where $m = \min(X)$. If $x, y \in M(X)$ such that $x < y$ and $y \geq m$, then $h(x) < h(y)$.*

PROOF: Let us assume that $m < 0$ otherwise it is trivial. Firstly, if $x \geq 2m$ then $h(x) < h(y)$ since h is one-to-one and order-preserving on $[2m, 0] \cap M(X)$. Thus assume that $x < 2m$. Since $x \in M(X)$, we have $x = \sum_{i=1}^n k_ix_i$ for some $k_i \in \mathbb{N}$ and $x_i \in X$. Let

$$S = \{z < 2m \mid z = \sum_{i=1}^n a_ix_i,\ a_i \in \mathbb{N},\ a_i \leq k_i\}\,.$$

Since $M(X)$ is i.w.o., S has a maximum. Let $\max S = \sum l_ix_i$. We can assume w.l.o.g. that $l_1 \geq 1$ and $x_1 < 0$. Let $u = (l_1 - 1)x_1 + \sum_{i=2}^n l_ix_i$. As $u > \max S$, we get $u \geq 2m$. Further, since $x_1 \geq m$, we obtain $u < m$. Indeed, suppose that $u \geq m$. Then $\max S = u + x_1 \geq m + x_1 \geq 2m$ (a contradiction). Moreover, we get $h(x) \leq h(u)$ since

$$h(x) = \sum_{i=1}^n k_ih(x_i) \leq \sum_{i=1}^n l_ih(x_i) \leq (l_1 - 1)h(x_1) + \sum_{i=2}^n l_ih(x_i) = h(u)\,.$$

As $2m \leq u < m \leq y$, we have $h(u) < h(y)$. Thus $h(x) \leq h(u) < h(y)$. □

Now thanks to Theorem 3.2 we know that if $T \not\vdash \varphi$ for some finite theory T, then there is a finite subset X of $\mathrm{N}(\mathbf{R}^n_{\mathrm{lex}})$ such that $T \not\models_{\mathbf{M}(X)_0^{\rightarrow}} \varphi$. We want to find a finite subset Y of $\mathrm{N}(\mathbf{Z}^n_{\mathrm{lex}})$ such that $T \not\models_{\mathbf{M}(Y)_0^{\rightarrow}} \varphi$. For this purpose we will use the mapping h from Lemma 4.1. However we cannot use it directly since it maps only nonpositive reals to integers. More precisely, let $X_i = \pi_i(X)$ be the set of the i-th components of elements from X and $x \in X$. We denote the i-th component $\pi_i(x)$ of x by x_i. We want to define $h(x) = \langle h_1(x_1), \ldots, h_n(x_n)\rangle$ where $h_i : M(X_i) \to \mathbb{Z}$ are the mappings from Lemma 4.1. Nevertheless some of x_i may be positive even if x is from the negative cone of $\mathbf{R}^n_{\mathrm{lex}}$. Thus we have to firstly prove that it is possible to consider that all components of the elements in X are less or equal to 0.

Lemma 4.3 *Let X be a finite subset of $\mathrm{N}(\mathbf{R}^n_{\text{lex}})$ and $C \neq \{\overline{0}\}$ be an Archimedean class of $\mathbf{M}(X)$. Then there is an order-preserving automorphism g of $\mathbf{R}^n_{\text{lex}}$ such that all components of $g(x)$ are bounded by 0 for all $x \in C \cap X$. Moreover, let D be an Archimedean class. If $D > C$ then $g(D) = D$.*

PROOF: Let $x \in C$ and j be the smallest natural number such that $x_j < 0$. Observe that $y \in C$ implies $y_j < 0$ and $y_i = 0$ for all $i < j$. Let $k = \min\{x_i/x_j \mid \langle 0, \ldots, x_j, \ldots, x_n\rangle \in C \cap X, j < i \leq n\}$. We define the mapping $g : \mathbf{R}^n_{\text{lex}} \to \mathbf{R}^n_{\text{lex}}$ as follows:

$$g(x_1, \ldots, x_n) = \langle x_1, \ldots, x_j, x_{j+1} - kx_j, x_{j+2} - kx_j, \ldots, x_n - kx_j\rangle\,.$$

We will show that g is a group homomorphism. Clearly we have $g(\overline{0}) = \overline{0}$. Let $\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle \in \mathbf{R}^n_{\text{lex}}$. Then

$$\begin{aligned}
&g(x_1 + y_1, \ldots, x_n + y_n) = \\
&\quad = \langle x_1 + y_1, \ldots, x_j + y_j, x_{j+1} + y_{j+1} - k(x_j + y_j), \ldots, x_n + y_n - k(x_j + y_j)\rangle \\
&\quad = \langle x_1 + y_1, \ldots, x_j + y_j, x_{j+1} - kx_j + y_{j+1} - ky_j, \ldots, x_n - kx_j + y_n - ky_j\rangle \\
&\quad = g(x_1, \ldots, x_n) + g(y_1, \ldots, y_n)\,.
\end{aligned}$$

Let $x, y \in \mathbf{R}^n_{\text{lex}}$ such that $x < y$. Then there is the smallest $i \leq n$ such that $x_i < y_i$ and $x_s = y_s$ for all $s < i$. If $i \leq j$ then $g(x) < g(y)$ since i-th component is not modified by g. If $i > j$, then $\pi_i(g(x)) = x_i - kx_j < y_i - kx_j = y_i - ky_j = \pi_i(g(y))$. Thus g is an order-preserving group automorphism.

Now we will show that all components of elements from $g(C \cap X)$ are less or equal to 0. Let $x \in C \cap X$. Then $x_j < 0$ and $x_i = 0$ for all $i < j$. Clearly, $\pi_i(g(x)) = x_i = 0$ for $i < j$ and $\pi_j(g(x)) = x_j < 0$. Let $i > j$. Then $\pi_i(g(x)) = x_i - kx_j$. Since $k \leq x_i/x_j$ and $x_j < 0$, we have $kx_j \geq x_i$. Thus $\pi_i(g(x)) = x_i - kx_j \leq 0$.

Finally, $g(x) = x$ for any $x \in D$ since the j-th component of x equals 0. □

Lemma 4.4 *Let X be a finite subset of $\mathrm{N}(\mathbf{R}^n_{\text{lex}})$. Then there is a finite subset $Y \subseteq \mathrm{N}(\mathbf{R}^n_{\text{lex}})$ such that $\mathbf{M}(X)^{\to} \cong \mathbf{M}(Y)^{\to}$ and all components of any $x \in M(Y)$ are bounded by 0.*

PROOF: Let $\{C_1, \ldots, C_k\}$ be the chain of Archimedean classes of $M(X)$. Clearly $k \leq n+1$ and $C_k = \{\overline{0}\}$. We will construct the set Y iteratively. Let $X_{k-1} = X$ and $X_{i-1} = g_i(X_i)$ where $g_i : \mathbf{R}^n_{\text{lex}} \to \mathbf{R}^n_{\text{lex}}$ is the order-preserving automorphism from Lemma 4.3 shifting elements from $C_i \cap X$ for $1 \leq i < k$. Let $Y = X_0$ and $g = g_1 \circ \cdots \circ g_{k-1}$. Clearly g is an order-preserving automorphism of $\mathbf{R}^n_{\text{lex}}$. Since g_i keeps the already modified Archimedean classes the same, we get that all components of any $y \in Y$ are bounded from above by 0. Consequently, also all components of $x \in M(Y)$ are less or equal to 0 since $x = \sum k_i y_i$ for some $k_i \in \mathbb{N}$ and $y_i \in Y$. Finally, as g is obviously an order-preserving monoid isomorphism between $\mathbf{M}(X)$ and $\mathbf{M}(Y)$, we obtain $\mathbf{M}(X)^{\to} \cong \mathbf{M}(Y)^{\to}$ by Lemma 2.1. □

Theorem 4.5 *Let X be a finite subset of $\mathrm{N}(\mathbf{R}^n_{\text{lex}})$. Then there is a mapping $f : X \to \mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ such that f is an embedding of a partial subalgebra $\mathbf{X}$ of $\mathbf{M}(X)^{\to}$ into $\mathbf{M}(f(X))^{\to}$, i.e. f is one-to-one, order-preserving and satisfies the following conditions for all $x, y, z \in X$:*

1. *if $z = x + y$ then $f(z) = f(x) + f(y)$,*
2. *if $z = x \to_R y$ then $f(z) = f(x) \to_Z f(y)$,*

where $\to_R$ is the residuum in $\mathbf{M}(X)^{\to}$ and $\to_Z$ is the residuum in $\mathbf{M}(f(X))^{\to}$.

PROOF: By Lemma 4.4 we can assume that all $x \in X$ are bounded by $\overline{0}$. Let $X_i = \pi_i(X)$ be the set of the i-th components of elements from X. Let $m_i = \min(X_i)$. By Lemma 4.1 there are monoid homomorphisms $h_i : M(X_i) \to \mathbb{Z}$ such that h_i are one-to-one and order-preserving on $[2m_i, 0] \cap M(X_i)$.

Let $x \in M(X)$ and $x = \langle x_1, \ldots, x_n \rangle$. We define $h(x) = \langle h_1(x_1), \ldots, h_n(x_n) \rangle$. Then $h : M(X) \to \mathbf{Z}^n_{\text{lex}}$ is a monoid homomorphism since h_i are monoid homomorphisms. Further, let $x, y \in M(X)$ such that $x < y$. Then there is the smallest j such that $x_j < y_j$ and $x_i = y_i$ for all $i < j$. If $x_j, y_j \geq 2m_j$ then $h(x) < h(y)$ since $h_j(x_j) < h_j(y_j)$ and $h_i(x_i) = h_i(y_i)$ for all $i < j$. Let $f = h|X$. Since for any $x \in X$ we have $x_i \geq 2m_i$, f is order-preserving and one-to-one.

Now we will show that f satisfies the conditions 1, 2. The first condition is obvious since f is a restriction of h which is a monoid homomorphism. Let $x, y, z \in X$ and $z = x \to_R y$. Then $x + z \leq y$. If $x + z = y$ then we get

$$f(x) + f(z) = h(x) + h(z) = h(x + z) = h(y) = f(y)\,.$$

Thus $f(z) = f(y) - f(x) = f(x) \to_Z f(y)$ since $f(y) - f(x)$ is the maximal possible element of $\mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ such that $f(x) + (f(y) - f(x)) \leq f(y)$.

Suppose that $x + z < y$. Then there is $j \leq n$ such that $x_j + z_j < y_j$ and $x_i + z_i = y_i$ for all $i < j$. Since $x_j + z_j \geq 2m_j$, we have

$$f(y) = h(y) > h(x + z) = h(x) + h(z) = f(x) + f(z)\,,$$

i.e. $f(z) \leq f(x) \to_Z f(y)$. Let $w = f(x) \to_Z f(y) \in M(f(X))$. Then $w = \sum k_i f(x_i)$ for some $k_i \in \mathbb{N}$ and $x_i \in X$. We get

$$h(y) = f(y) \geq f(x) + w = f(x) + \sum k_i f(x_i) = h(x + \sum k_i x_i)\,. \qquad (*)$$

Suppose that $x + \sum k_i x_i > y$, i.e. there is j such that $\pi_j(x + \sum k_i x_i) > \pi_j(y)$ and $\pi_s(x + \sum k_i x_i) = \pi_s(y)$ for all $s < j$. Then $h(x + \sum k_i x_i) > h(y)$ since $\pi_j(y) \geq 2m_j$ which contradicts $(*)$. Hence $x + \sum k_i x_i \leq y$. Consequently $\sum k_i x_i \leq x \to_R y = z$. Finally, applying of h again leads to $w = h(\sum k_i x_i) \leq h(z) = f(z)$ by Lemma 4.2. Indeed, if $\sum k_i x_i = z$ then it is obvious. If $\sum k_i x_i < z$ then there is the smallest j such that $\pi_j(\sum k_i x_i) < \pi_j(z)$. Since $\pi_j(z) \geq m_j$, the conclusion follows by Lemma 4.2. □

Thanks to Theorem 4.5 we get the following result.

Theorem 4.6 *Let T be a finite theory over* ΠMTL *and φ be a formula. Then $T \vdash \varphi$ iff $T \models_{\mathbf{M}(X)^{\to}_{0}} \varphi$ for each finite subset $X \subseteq \mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ and each $n \in \mathbb{N}$.*

PROOF: One implication already follows from Theorem 2.4. Suppose that $T \nvdash \varphi$. Then there is a finite subset $X \subseteq \mathrm{N}(\mathbf{R}^n_{\text{lex}})$ and an $\mathbf{M}(X)^{\to}_{0}$-evaluation e such that $e(T) \subseteq \{\overline{0}\}$ and $e(\varphi) < \overline{0}$ by Theorem 3.2. Moreover, by Remark 3.4 $e(\psi) \in X \cup \{\mathbf{0}\}$ ($\mathbf{0}$ is the bottom element of $\mathbf{M}(X)^{\to}_{0}$) for each subformula ψ of any formula from $T \cup \{\varphi\}$. Finally, it follows from Theorem 4.5 that there is a finite subset $Y \subseteq \mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ such that the partial subalgebra $\mathbf{X}$ can be embedded into $\mathbf{M}(Y)^{\to}$. By Lemma 2.12 we can extend this embedding for the case when some of the subformulas of formulas from $T \cup \{\varphi\}$ are evaluated by $\mathbf{0}$. Thus the proof is done. □

Corollary 4.7 *The class of all $\mathbf{M}(X)^{\to}_{0}$ for each finite subset $X \subseteq \mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ generates (as a quasivariety) the variety of* ΠMTL*-algebras.*

Corollary 4.8 *The class of all $\mathbf{M}(X)^{\to}$ for each finite subset $X \subseteq \mathrm{N}(\mathbf{Z}^n_{\text{lex}})$ generates (as a quasivariety) the variety of commutative integral cancellative representable residuated lattices.*

5 Decidability

In Theorem 4.6 we have shown that the variety of ΠMTL-algebras is generated as a quasivariety by the class of ΠMTL-chains arising from finitely generated submonoids of $\mathrm{N}(\mathbf{Z}^n_{\mathrm{lex}})$. This class is clearly countable and it is not difficult to find its enumeration. Thus it is possible to construct a Turing machine which accepts nonprovable formulas. It goes through all algebras from this class and computes whether a given formula φ is valid or not. The only thing which we have to show is that there is an algorithm which computes the residuum in $M(X)^{\rightarrow}_{\overline{0}}$ for some finite subset $X \subseteq \mathrm{N}(\mathbf{Z}^n_{\mathrm{lex}})$. Note that if we are able to compute the residuum in $\mathbf{M}(X)^{\rightarrow}$ then we are able to compute it also in $\mathbf{M}(X)^{\rightarrow}_{\overline{0}}$ by Lemma 2.11. Thus we will describe an algorithm only for the residuum in $\mathbf{M}(X)^{\rightarrow}$.

Algorithm 5.1 The algorithm consists of one function named Residuum which calls recursively itself. The function is defined as follows:

FUNCTION Residuum(d, r)

1. If $\overline{0} \leq d$ then Return$(\overline{0})$.
2. Find the set $H = \{b \mid b = \sum a_i x_i,\ a_i \in \mathbb{N},\ x_i \in C_r \cap X,\ b \sim_{t(r+1)} d\}$, where the equivalence $\sim_j$ was defined at the end of Section 2 and the function t was defined by Equation (1).
3. If $H = \emptyset$ then $z = \max\{z' \mid z' = \sum a_i x_i,\ a_i \in \mathbb{N},\ x_i \in C_r \cap X,\ z' \leq d\}$ and Return(z).
4. If $H = \{b_1, \ldots, b_s\}$ then for each $b_i \in H$ call $p_i = b_i +$ Residuum$(d - b_i, r+1)$ where the difference $d - b_i$ is computed in $\mathbf{Z}^n_{\mathrm{lex}}$.
5. Set $z = \max\{p_1, \ldots, p_s\}$. Return$(z)$.

Let $x, y \in M(X)$. The command Residuum$(y - x, 1)$ starts the algorithm computing $x \rightarrow y$, where $y - x$ is a difference in $\mathbf{Z}^n_{\mathrm{lex}}$. The second input r tells the algorithm in which Archimedean class it starts to search for the residuum.

Now we will prove that Algorithm 5.1 really computes the residuum $x \rightarrow y$. Since $z = \max\{z' \in M(X) \mid x + z' \leq y\} = \max\{z' \in M(X) \mid z' \leq y - x\}$, it is sufficient if we prove the following lemma.

Proposition 5.2 *Algorithm 5.1 finds the best approximation of $d = y - x$ from below by elements from $M(X)$.*

PROOF: We will prove it by induction on the number of Archimedean classes of $\mathbf{M}(X)$. If $\mathbf{M}(X)$ has only one Archimedean class C_1, then $M(X) = C_1 = \{\overline{0}\}$. Consequently $d = \overline{0}$ and the first step of the algorithm returns correctly $\overline{0}$.

Thus assume that Algorithm 5.1 computes a residuum correctly for any $\mathbf{M}(Y)$ consisting of m Archimedean classes where Y is a finite subset of $\mathrm{N}(\mathbf{Z}^n_{\mathrm{lex}})$. Let $C_1 < \cdots < C_{m+1} = \{\overline{0}\}$ be the chain of Archimedean classes of $\mathbf{M}(X)$. If $d \geq \overline{0}$ then the algorithm correctly returns $\overline{0}$. Suppose that $d < \overline{0}$. We will denote by z the desired approximation of d. Since $z \in M(X)$, we have $z = \sum k_i x_i$ for some $k_i \in \mathbb{N}$ and $x_i \in X$. Observe that we can split this sum according to Archimedean classes, i.e., $z = c_1 + \cdots + c_m$, where $c_j = \sum k_{ij} g_{ij}$ for some $k_{ij} \in \mathbb{N}$ and $g_{ij} \in C_j \cap X$. The algorithm finds the possible values of $c_1, \ldots, c_m$.

At the beginning we search for the possible values of c_1. If we want to find the best approximation of d, we have to try whether there is an element b whose initial segment of the components corresponding to the first Archimedean class is equal to the same initial segment of d, i.e., $b \sim_{t(2)} d$. Then b is a possible value for c_1. If there is no such an element then $z = c_1 = \max\{z' \mid z' = \sum a_i x_i,\ a_i \in \mathbb{N},\ x_i \in C_1 \cap X,\ z' \leq d\}$ and $c_2 = \cdots = c_m = \overline{0}$. This is the content of the second and the third step of the algorithm. In particular, the second step finds the set H of all elements b such that $b \sim_{t(2)} d$ and the third step computes z in case that $H = \emptyset$. We should show that the both steps can be computed.

Lemma 5.3 *Steps 2 and 3 can be computed.*

PROOF: Since the elements from H are finite sums of elements from $C_1 \cap X$, we can find for each $x_i \in C_1 \cap X$ a natural number k_i such that $k_i x_i < d$. Let k be the maximal k_i. Then it is sufficient to go through all sums whose length is less or equal to k because all longer sums are surely strictly less then d. Thus there is only finitely many possibilities where we have to search for the elements of H. Also in case when $H = \emptyset$ we can find $\max\{z' \mid z' = \sum a_i x_i,\ a_i \in \mathbb{N},\ x_i \in C_1 \cap X,\ z' \leq d\}$ among these finitely many possibilities. $\square$

Now we have to discuss what happens if H is finite, say $H = \{b_1, \ldots, b_s\}$. Note that if we know the value of c_1 then we can reformulate the problem as follows: find the maximal element w in $M(X)$ such that $w \leq d - c_1$, i.e.,

$$w = \max\{z' \in M(X) \mid z' \leq d - c_1\}\,.$$

Then the best approximation $z = c_1 + w$. Indeed, we have obviously $c_1 + w \leq d$. Thus $z \geq c_1 + w$. Suppose that $z > c_1 + w$. Recall that $z = c_1 + c_2 + \cdots + c_m$. It follows that $w < c_2 + \cdots + c_m$. Since $c_2 + \cdots + c_m \in M(X)$, we get $c_2 + \cdots + c_m \leq w$ (a contradiction).

As we know that c_1 is one of the elements in $H = \{b_1, \ldots, b_s\}$, we can find for each b_i the element $w_i = \max\{z' \in M(X) \mid z' \leq d - b_i\}$. Then it is sufficient to find the maximum among all $b_i + w_i$, $1 \leq i \leq s$. This is content of the fourth and the fifth step of the algorithm. The only thing to be proved is the computability of all w_i. Since for each b_i we have $b_i \sim_{t(2)} d$, we obtain $d - b_i \sim_{t(2)} \overline{0}$. But this means that w_i must belong to a greater Archimedean class than C_1. Observe that $M(X) - C_1$ is a submonoid of $M(X)$ which has only m Archimedean classes by Lemma 2.14. Thus by the induction assumption we can find w_i since it belongs to $M(X) - C_1$ and we are done. $\square$

Theorem 5.4 *The set of tautologies TAUT of* ΠMTL *is decidable. The equational theory of* ΠMTL-*algebras is decidable as well.*

Since Theorem 4.6 works also for finite theories and ΠMTL is an algebraizable logic, we obtain also the following theorem.

Theorem 5.5 *The finite consequence relation of* ΠMTL *is decidable. The quasi-equational theory of* ΠMTL-*algebras is decidable.*

The intended semantics for fuzzy logics is usually the real unit interval $[0, 1]$ endowed with interpretations of logical connectives. Such semantics is called standard. The interpretation of the conjuction in this case is a t-norm[1]. Thus we are usually interested in complexity results formulated for this standard semantics. In [14] we proved that ΠMTL enjoys the standard completeness theorem. It says that ΠMTL is complete w.r.t. the standard semantics, i.e., the class of ΠMTL-chains whose lattice reduct is the real interval $[0, 1]$ endowed with the usual order and the monoidal operation is a cancellative left-continuous t-norm. Let us call a formula φ t-tautology if φ is valid in each standard ΠMTL-chain. Then due to Theorem 5.4 and the standard completeness theorem we obtain the following corollary.

Corollary 5.6 *The problem of recognizing t-tautologies is decidable.*

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[1] A t-norm is a binary operation on $[0, 1]$ which is commutative, associative, monotone, and 1 is its neutral element.

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