Cancellative Residuated Lattices Arising on 2-generated Submonoids of Natural Numbers

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Abstract

It is known that there are only two cancellative atoms in the subvariety lattice of residuated lattices, namely the variety of Abelian ℓ-groups $\mathcal{CLG}$ generated by the additive ℓ-group of integers and the variety $\mathcal{CLG}^-$ generated by the negative cone of this ℓ-group. In this paper we consider all cancellative residuated chains arising on 2-generated submonoids of natural numbers and show that almost all of them generate a cover of $\mathcal{CLG}^-$. This proves that there are infinitely many covers above $\mathcal{CLG}^-$ which are commutative, integral, and representable.

1 Introduction

It was proved in [5] that there are only two atoms in the subvariety lattice of cancellative residuated lattices. The first one is the variety of Abelian ℓ-groups $\mathcal{CLG}$ which is known to be generated by the additive ℓ-group of integers $\mathbf{Z}$ (due to Weinberg [11]). The second atom $\mathcal{CLG}^-$ is the variety generated by the negative cone of $\mathbf{Z}$ (see [5]). However, if we go one step higher, the situation becomes very complicated, namely both the atoms already have uncountably many covers. The fact that there are uncountably many covers of $\mathcal{CLG}$ was proved in [7]. Then taking into account the results of [2], we get easily that $\mathcal{CLG}^-$ has uncountably many covers as well. We explain this in more details. For any variety $\mathcal{V}$ the subvariety lattice of $\mathcal{V}$ is denoted $\mathbf{\Lambda}(\mathcal{V})$. Let $\mathcal{K}$ be a subvariety of the variety of ℓ-groups $\mathcal{LG}$. The class of negative cones of members from $\mathcal{K}$ is denoted $\mathcal{K}^-$. It is proved in [2] that the mapping $\mathcal{K} \mapsto \mathcal{K}^-$ is a lattice isomorphism between $\mathbf{\Lambda}(\mathcal{LG})$ and $\mathbf{\Lambda}(\mathcal{LG}^-)$. Using this mapping on the uncountably many covers of $\mathcal{CLG}$ found in [7], one can produce uncountable many covers of $\mathcal{CLG}^-$. Let us stress here that the covers obtained in this way are non-commutative (i.e., they contain non-commutative algebras).

Considering the above-mentioned results, it is natural to ask what happens if we restrict our attention only to covers belonging to some well-known subvarieties of residuated lattices. In this paper we are going to prove that even if we restrict to commutative, integral, and representable covers, there are still infinitely many of them. More precisely, we consider all residuated chains arising on 2-generated submonoids of the dually ordered monoid of natural numbers. Then it is proved which of them exactly generate a cover of $\mathcal{CLG}^-$.

Notation 1.1 Let $\mathbf{A}$ be an algebra. Its congruence lattice is denoted $\mathrm{Con}(\mathbf{A})$. Given $x \in A$ and $\theta \in \mathrm{Con}(\mathbf{A})$, the congruence class of x with respect to θ is denoted x/θ. If I is a set and U an ultrafilter on I, then $\mathbf{A}^I/U$ denotes the corresponding ultrapower of $\mathbf{A}$ with respect to U. For any class of algebras $\mathcal{K}$ of the same type, $\mathcal{V}(\mathcal{K})$ is the variety generated by $\mathcal{K}$. We write $\mathcal{V}(\mathbf{A})$ instead of $\mathcal{V}(\{\mathbf{A}\})$. The sets of natural numbers, integers, and reals are denoted $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ respectively. The set of non-positive integers is denoted $\mathbb{Z}^-$. Given $a, b \in \mathbb{Z}$, the fact that a divides b is denoted $a|b$. Let $a, b \in \mathbb{Z}$ such that $a \leq b$. Then $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Given $a \in \mathbb{R}$, the greatest integer less than or equal to a is denoted $\lfloor a \rfloor$. Analogously, $\lceil a \rceil$ stands for the least integer greater than or equal to a. Let $a \in \mathbb{N} \setminus \{0\}$. Then every $b \in \mathbb{Z}$ can be expressed by integer division as $b = \lfloor b/a \rfloor a + r$ where and $r \in [0, a-1]$. The number r is the remainder on division of b by a and denoted $\rho_a(b)$.

Lemma 1.2 *The following are properties of the operations $\lfloor\ \rfloor, \lceil\ \rceil$:*

1. *Let $a \in \mathbb{R}$. Then $a - 1 < \lfloor a \rfloor \leq a$ and $a \leq \lceil a \rceil < a + 1$.*
2. *Let $a \in \mathbb{R}$ and $b \in \mathbb{Z}$. If $b > \lfloor a \rfloor$ then $b \geq \lceil a \rceil$. Dually, if $b < \lceil a \rceil$ then $b \leq \lfloor a \rfloor$.*
3. *Let $a, b, n \in \mathbb{N}$. If $n\rho_a(b) < a$, then $\lfloor nb/a \rfloor = n\lfloor b/a \rfloor$.*

PROOF: The first claim is obvious. In order to prove the second one, observe that $b > \lfloor a \rfloor$ implies $b \geq \lfloor a \rfloor + 1 \geq \lceil a \rceil$. Finally, if $n\rho_a(b) < a$ then $n\rho_a(b)/a < 1$. Consequently,

$$\lfloor nb/a \rfloor = \lfloor n(\lfloor b/a \rfloor a + \rho_a(b))/a \rfloor = \lfloor n\lfloor b/a \rfloor + n\rho_a(b)/a \rfloor = n\lfloor b/a \rfloor\,.$$

□

2 Preliminaries

A *commutative residuated lattice* (CRL) $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \to, 1\rangle$ is an algebraic structure, where $\langle L, \cdot, 1\rangle$ is a commutative monoid, $\langle L, \wedge, \vee\rangle$ is a lattice, and for all $x, y, z \in L$ we have

$$x \cdot y \leq z \text{ iff } x \leq y \to z\,.$$

It follows immediately from the definition that $a \to b$ is the greatest solution of the inequality $a \cdot x \leq b$ for x. It is well known that the class $\mathcal{CRL}$ of all CRLs forms a congruence distributive variety (see e.g. [6, 8]). In the absence of parentheses, $\cdot$ is performed first, followed by $\to$ and then $\wedge, \vee$.

The following theorem describes the congruences on CRLs. Recall that a subalgebra $\mathbf{B}$ of a CRL $\mathbf{L}$ is said to be convex if for all $x, y \in B$ and $z \in L$ we have $x \leq z \leq y$ implies $z \in B$.

Theorem 2.1 ([4]) *Let $\mathbf{L}$ be a CRL. Then $\mathrm{Con}(\mathbf{L})$ is isomorphic to the lattice of convex subalgebras of $\mathbf{L}$. This correspondence is given by the following mutually inverse maps:*

$$\theta \mapsto 1/\theta\,,$$
$$\mathbf{S} \mapsto \theta_{\mathbf{S}} = \{\langle x, y\rangle \in L^2 \mid (x \to y) \wedge 1 \in S \text{ and } (y \to x) \wedge 1 \in S\}\,.$$

Let us recall several subclasses of CRLs. A CRL $\mathbf{L}$ is said to be *cancellative* if its monoidal reduct is cancellative, i.e., for all $x, y, z \in L$ we have $x \cdot z = y \cdot z$ implies $x = y$. It was shown in [2] that the class $\mathcal{CanCRL}$ of cancellative CRLs forms a variety defined by $y \to y \cdot x \approx x$. A CRL $\mathbf{L}$ is called *integral* (ICRL) if 1 is the top element of L. A totally ordered ICRL is referred to as *integral commutative residuated chain* (ICRC). An ICRL is called *representable* if it is a subdirect product of ICRCs. A representable ICRL $\mathbf{L}$ satisfying the identity $x \wedge y \approx x \cdot (x \to y)$ is called *divisible*. This identity is equivalent to the property:

for all $x, y \in L$ such that $x \le y$, there is $z \in L$ (namely $y \to x$) such that $x = y \cdot z$.

Let $\mathbf{G} = \langle G, \wedge, \vee, +, \to, 0\rangle$ be an Abelian ℓ-group viewed as a CRL, i.e., $x \to y = y - x$. The negative cone of $\mathbf{G}$ forms a divisible ICRL $\mathbf{G}^- = \langle G^-, \wedge, \vee, +, \to, 0\rangle$, where $x \to y = (y - x) \wedge 0$ (see e.g. [2]). The Abelian ℓ-group of integers is denoted $\mathbf{Z}$.

Theorem 2.2 ([2, 3]) *The variety $\mathcal{CLG}^-$ consists of all negative cones of Abelian ℓ-groups and is generated by $\mathbf{Z}^-$.*

Let $\mathbf{L}$ be a cancellative ICRC. It is well known that divisibility of $\mathbf{L}$ implies that $\mathbf{L}$ is the negative cone of an Abelian ℓ-group. It follows for instance from [3, Corollary 1.18]. Thus we have the following characterization.

Theorem 2.3 *Let $\mathbf{L}$ be a cancellative ICRC. Then $\mathbf{L} \in \mathcal{CLG}^-$ iff $\mathbf{L}$ is divisible.*

The variety $\mathcal{CLG}^-$ is generated by $\mathbf{Z}^-$ which is a 1-generated algebra as a monoid. Since we are interested in covers of $\mathcal{CLG}^-$, it is natural to look at ICRCs arising on 2-generated submonoids of $\mathbf{Z}^-$. However, it turns out that the investigation of such ICRCs looks unfamiliar because it needs a lot of notions which are dual to well-known notions for natural numbers. Due to this fact we decide to work in dual setting.

A dual ICRL is an algebra $\mathbf{L} = \langle L, \wedge, \vee, +, \dot{-}, 0\rangle$ where $\langle L, \wedge, \vee, 0\rangle$ is a lattice with a bottom element 0, $\langle L, +, 0\rangle$ is a commutative monoid and for all $x, y, z \in L$ we have

$$x + y \ge z \text{ iff } x \ge z \dot{-} y\,.$$

Note that $x \le y$ iff $x \dot{-} y = 0$. The dual ICRLs are term equivalent to ICRLs. If $\mathbf{L} = \langle L, \vee, \wedge, \cdot, \to, 1\rangle$ is an ICRL then $\langle L, \sqcap, \sqcup, +, \dot{-}, 0\rangle$ is a dual ICRL, where $x \sqcap y := x \vee y$, $x \sqcup y := x \wedge y$, $x + y := x \cdot y$, $x \dot{-} y := y \to x$, and $0 := 1$. Conversely, if $\mathbf{L} = \langle L, \wedge, \vee, +, \dot{-}, 0\rangle$ is a dual ICRL then $\langle L, \sqcap, \sqcup, \cdot, \to, 1\rangle$ is an ICRL, where $x \sqcap y := x \vee y$, $x \sqcup y := x \wedge y$, $x \cdot y := x + y$, $x \to y := y \dot{-} x$, and $1 := 0$.

Similarly as for CRLs we adopt the convention that $+$ is performed first followed by $\dot{-}$ and then $\wedge, \vee$. The n-fold products of x in ICRLs are denoted as $x^n = x \cdots x$. In the dual setting we use the additive notation instead, i.e., $nx = x + \cdots + x$. We say that a dual ICRL is *cancellative* if its monoidal reduct is cancellative, i.e., if it satisfies the identity $x + y \dot{-} y \approx x$. A dual ICRL is called *divisible* if the join is definable as $x \vee y \approx (y \dot{-} x) + x$. By $\mathbf{N}$ we denote the dual of $\mathbf{Z}^-$, i.e., $\mathbf{N} = \langle \mathbb{N}, \wedge, \vee, +, \dot{-}, 0\rangle$, where $x \dot{-} y = (x - y) \vee 0$.

In dual setting Theorem 2.3 can be restated as follows.

Theorem 2.4 *Let $\mathbf{L}$ be a cancellative dual ICRC. Then $\mathbf{L} \in \mathcal{V}(\mathbf{N})$ iff $\mathbf{L}$ is divisible.*

3 2-generated submonoids of $\mathbb{N}$

The main aim of this section is to introduce dual ICRCs arising on 2-generated submonoids of $\langle \mathbb{N}, +, 0\rangle$ and prove some of their properties. Before we define them, we recall several useful facts on submonoids of $\mathbb{N}$. Let $a_1, \dots, a_k \in \mathbb{N}$. Then $M(a_1, \dots, a_k) = \{\sum_{i=1}^{k} n_i a_i \mid n_i \in \mathbb{N}\}$ denotes the subuniverse of $\langle \mathbb{N}, +, 0\rangle$ generated by $\{a_1, \dots, a_k\}$. If $a_1, \dots, a_n$ are coprime, then $\mathbb{N} \setminus M(a_1, \dots, a_k)$ is always finite and the greatest number which does not belong to $M(a_1, \dots, a_k)$ is called the *Frobenius number* (see e.g. [1]). This number is usually denoted $g(a_1, \dots, a_k)$. For $k = 2$ Sylvester found an explicit expression $g(a_1, a_2) = a_1a_2 - a_1 - a_2$ (see [10]). If $a_1, \dots, a_k$ are not coprime and $d = \gcd(a_1, \dots, a_k)$, then it follows from the above-mentioned facts that for any $b > g(a_1/d, \dots, a_k/d)$ we have $bd \in M(a_1, \dots, a_k)$.

Lemma 3.1 *Let a, b be coprime nonzero natural numbers. Then we have the following:*

1. *Every $c \in \mathbb{Z}$ can be expressed as $ma + nb$ for $m \in \mathbb{Z}$ and $n \in [0, a-1]$. Moreover, this representation is unique.*
2. *Let $c \in \mathbb{Z}$ and let $c = ma + nb$ be its unique representation from the previous statement. Then $c \in M(a, b)$ iff $m \geq 0$.*

PROOF: Since a, b are coprime, any $c \in \mathbb{Z}$ can be expressed as $c = ka + lb$ for some $k, l \in \mathbb{Z}$. Thus $c = ka + (\lfloor l/a \rfloor a + \rho_a(l))b = (k + \lfloor l/a \rfloor b)a + \rho_a(l)b$. The uniqueness follows easily since $ma + nb = m'a + n'b$ implies $a|(n' - n)$ because a, b are coprime. Moreover, $n, n' \in [0, a-1]$ implies $-a + 1 \leq n' - n \leq a - 1$. Thus $n = n'$ and $m = m'$. The second claim is obvious. □

Let $a, b \in \mathbb{N}$. Then $\mathbf{M}(a, b) = \langle M(a, b), \wedge, \vee, +, \dot{-}, 0\rangle$ is the dual ICRC living on $M(a, b)$, i.e., $\wedge, \vee$ induce the usual order on $\mathbb{N}$ and

$$x \dot{-} y = \min\{z \in M(a, b) \mid z \geq x - y\}\,.$$

The minimum in the last expression always exists since $\mathbb{N}$ is a well-ordered set. Note that for any $x, y \in M(a, b)$ we have $x \dot{-} y \geq x - y$, and $x \dot{-} y = x - y$ iff $x - y \in M(a, b)$.

It is clear that if a, b are not coprime then $\mathbf{M}(a, b) \cong \mathbf{M}(a/d, b/d)$ for $d = \gcd(a, b)$ since $\dot{-}$ is fully determined by the monoidal operation and the order. Thus we consider only coprime generators a, b. Let $a, b \in \mathbb{N}$ such that $a < b$. It is obvious that $a \in \{0, 1\}$ implies $\mathbf{M}(a, b) \cong \mathbf{N}$. Thus we exclude also these possibilities. The following is the set of interesting generators:

$$Gen = \{\langle a, b\rangle \in \mathbb{N}^2 \mid 1 < a < b,\ a, b \text{ coprime}\}\,.$$

Lemma 3.2 *Let $\langle a, b\rangle \in Gen$. Then for all $x, y \in M(a, b)$ such that $x \leq y$ we have*

$$y - x \leq y \dot{-} x < y - x + a\,.$$

Moreover, if $y - x < y \dot{-} x$ then $a = (y \dot{-} x) + x \dot{-} y$.

PROOF: Since $x \leq y$, we have $\lceil (y - x)/a \rceil \geq 0$. Thus $\lceil (y - x)/a \rceil a \in M(a, b)$. Moreover, $\lceil (y - x)/a \rceil < (y - x)/a + 1$ by Lemma 1.2. Consequently, we have

$$y \dot{-} x = \min\{z \in M(a, b) \mid z \geq y - x\} \leq \lceil (y - x)/a \rceil a < y - x + a\,.$$

To see the second claim, we have $0 < (y \dot{-} x) + x - y < a$ using the first claim and the assumption. Since a covers 0, we must have $a = (y \dot{-} x) + x \dot{-} y$. □

Proposition 3.3 *For each $\langle a,b\rangle \in Gen$ the algebra $\mathbf{M}(a,b)$ is a simple, non-divisible, cancellative, dual ICRC.*

PROOF: The dual ICRC $\mathbf{M}(a,b)$ is clearly cancellative since each submonoid of $\mathbb{N}$ is cancellative. The fact that it is simple follows from its Archimedean property, i.e., it contains only one nontrivial convex subalgebra (see Theorem 2.1). Finally, we have $b-a \notin M(a,b)$ by Lemma 3.1. Consequently, $(b \dot{-} a)+a > b = a \vee b$, i.e., $\mathbf{M}(a,b)$ is not divisible. $\square$

We want to show that different tuples from Gen generate different varieties. This can be done for tuples with different first components by using the following identities for $n \in \mathbb{N}$:

$$nx \vee ny \approx (ny \dot{-} nx) + nx\,,$$

which we call *n-divisibility.*

Proposition 3.4 *Let $\langle a,b\rangle \in Gen$. Then $\mathbf{M}(a,b)$ satisfies the a-divisibility. In addition, $\mathbf{M}(a,b)$ does not satisfy k-divisibility for any $k \in [1, a-1]$.*

PROOF: Let c be any element in $M(a,b)$. Then ac belongs to the subalgebra of $\mathbf{M}(a,b)$ generated by a. Since this algebra is isomorphic to $\mathbf{N}$ (i.e., it is divisible), the identity holds. The second claim follows immediately from Lemma 3.1 showing that $kb-ka \notin M(a,b)$, i.e., $(kb \dot{-} ka)+ka > kb = ka \vee kb$. $\square$

In order to separate the varieties generated by tuples with the same first component, we introduce the following identities for each $n \in \mathbb{N}$:

$$\Big(z \wedge \big((y \dot{-} x) + x \dot{-} (x \vee y)\big)\Big) \vee (x + nz \dot{-} y) \approx x + nz \dot{-} y\,. \tag{1n}$$

Note that for each $n \in \mathbb{N}$ the identity (1n) is equivalent to the following inequality:

$$z \wedge \big((y \dot{-} x) + x \dot{-} (x \vee y)\big) \leq x + nz \dot{-} y\,. \tag{2n}$$

Every algebra $\mathbf{M}(a,b)$ satisfying the identity (1n) has the property that whenever a difference $y-x$ is greater than or equal to na, then the divisibility law holds for x and y in $\mathbf{M}(a,b)$.

Proposition 3.5 *Let $\langle a,b\rangle \in Gen$. Then $\mathbf{M}(a,b)$ satisfies the identity (1n) for $n = \lceil g(a,b)/a\rceil$.*

PROOF: We check the validity of the equivalent inequality (2n). If $z=0$ then (2n) is trivially valid. Also if $x \geq y$ then the left-hand side of (2n) clearly equals 0 since $y \dot{-} x = 0$. Thus assume that $z \geq a$ and $x < y$. Since $(y \dot{-} x) + x \dot{-} y \leq a$ holds in $\mathbf{M}(a,b)$ by Lemma 3.2, the only nontrivial case is the case when $x + nz \dot{-} y = 0$, i.e., $x + nz \leq y$. Then $y - x \geq nz \geq na > g(a,b)$. Since $y-x > g(a,b)$, it follows that $y - x \in M(a,b)$. Consequently, we have $(y \dot{-} x) + x \dot{-} y = y \dot{-} y = 0$. $\square$

Now it remains to show that if we have two different pairs $\langle a,b\rangle, \langle a,c\rangle \in Gen$, then $\mathbf{M}(a,c)$ does not satisfy the identity (1n) for $n = \lceil g(a,b)/a\rceil$. We need the following technical lemma.

Lemma 3.6 *Let $\langle a,b\rangle, \langle a,c\rangle \in Gen$ and $b < c$. Then $\lceil g(a,b)/a\rceil a \leq g(a,c)$.*

PROOF: First, observe that $b+1 \leq c$. Thus we have $(a-1)(b+1) \leq (a-1)c$ because $a-1 \geq 0$. Subtracting a from both the sides of $(a-1)(b+1) \leq (a-1)c$, we obtain $ab-b-1 \leq ac-a-c$. Consequently,

$$g(a,b) + a - 1 = ab - b - 1 \leq ac - a - c = g(a,c)\,. \tag{3}$$

Second, we can express $g(a,b) = pa+r$ by integer division where $p, r \in \mathbb{N}$ and $r \in [0, a-1]$. Moreover, since a does not divide $g(a,b)$, we have $r \geq 1$. Thus $\lceil g(a,b)/a \rceil = p+1$. Consequently using (3), we get $g(a,c) \geq pa + r + a - 1 \geq (p+1)a = \lceil g(a,b)/a \rceil a$. $\square$

Proposition 3.7 *Let $\langle a,b \rangle, \langle a,c \rangle \in Gen$ and $b < c$. Then $\mathbf{M}(a,c)$ does not satisfy the identity (1n) for $n = \lceil g(a,b)/a \rceil$.*

PROOF: Let $x = z = a$, $y = g(a,c) + a \in M(a,c)$. Then clearly $y - x = g(a,c) \notin M(a,c)$, i.e., $y \dot{-} x > y - x$. Thus the left-hand-side of inequality $(2n)$ equals a by Lemma 3.2. Hence it is sufficient to show that the right-hand-side equals 0, i.e., $x + nz \leq y$. By Lemma 3.6 we have $x + nz = a + \lceil g(a,b)/a \rceil a \leq a + g(a,c) = y$. $\square$

Summing up Propositions 3.4, 3.5, 3.7, we get the following corollary.

Corollary 3.8 *Let $\langle a,b \rangle, \langle c,d \rangle$ be two different pairs from the set Gen. Then $\mathcal{V}(\mathbf{M}(a,b)) \neq \mathcal{V}(\mathbf{M}(c,d))$.*

4 Subalgebras of $\mathbf{M}(a,b)$

In order to describe the mutual position of varieties $\mathcal{V}(\mathbf{M}(a,b))$ in the subvariety lattice, we need to characterize subalgebras of $\mathbf{M}(a,b)$. We will prove in Theorem 4.9 that the proper nontrivial subalgebras of $\mathbf{M}(a,b)$ are isomorphic to $\mathbf{N}$ if a is prime or $\rho_a(b) \neq 1$. In the remaining cases (i.e., a is not prime and $\rho_a(b) = 1$) the subalgebras of $\mathbf{M}(a,b)$ are completely determined by the divisors of a (see Theorem 4.5). We start with a description of divisible subalgebras.

Lemma 4.1 *Let $a, b \in \mathbb{N}$. Then each divisible nontrivial subalgebra $\mathbf{B}$ of $\mathbf{M}(a,b)$ is isomorphic to $\mathbf{N}$.*

PROOF: Since each 2-generated submonoid of $\mathbb{N}$ is well ordered and infinite, the elements of B can be indexed by natural numbers, i.e., $B = \{c_0, c_1, c_2, \ldots\}$ and $0 = c_0 < c_1 < c_2 < \cdots$. We will prove by induction that each non-zero $c_k \in B$ equals kc_1. Assume that all $c_k = kc_1$ for $k \leq i$. Clearly, $c_i < c_{i+1} \leq c_1 + c_i = (i+1)c_1$, since $(i+1)c_1 \in B$ and c_{i+1} is the successor of c_i. Consequently, $0 < c_{i+1} \dot{-} c_i \leq (i+1)c_1 \dot{-} ic_1 = c_1$. Since c_1 is an atom, we get $c_{i+1} \dot{-} c_i = c_1$. Since $\mathbf{B}$ is divisible, we have $c_{i+1} = c_{i+1} \vee c_i = (c_{i+1} \dot{-} c_i) + c_i = (i+1)c_1$. Thus $\mathbf{B} \cong \mathbf{N}$. $\square$

The following lemma shows that if a subalgebra of $\mathbf{M}(a,b)$ is not divisible that we can focus only on generating sets containing a and nb for some $n \in [1, a-1]$.

Lemma 4.2 *Let $\langle a,b \rangle \in Gen$ and $\mathbf{B}$ a subalgebra of $\mathbf{M}(a,b)$. If $\mathbf{B}$ is not divisible then $a \in B$ and $nb \in B$ for some $n \in [1, a-1]$.*

PROOF: If $\mathbf{B}$ is not divisible then there are elements $c, d \in B$ such that $c > d$ and $c \dot{-} d > c - d$. Thus the atom of $\mathbf{M}(a, b)$ can be expressed as $a = (c \dot{-} d) + d \dot{-} c \in B$ by Lemma 3.2.

Furthermore, B has to contain an element e such that a does not divide e (otherwise $\mathbf{B}$ would be divisible). Let $e = ma + nb$ be the unique representation from Lemma 3.1 for some $m \in \mathbb{N}$ and $n \in [1, a-1]$. Then $nb = e \dot{-} ma \in B$. $\square$

Now we will deal with the description of subalgebras of $\mathbf{M}(a, b)$ when a is not prime and $\rho_a(b) = 1$. Let us first demonstrate the general results on an example.

Example 4.3 Consider the algebra $\mathbf{M}(4, 5)$. Then $\mathbb{N} \setminus M(4, 5) = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{1, 2, 3\}, \quad C_2 = \{6, 7\}, \quad C_3 = \{11\}.$$

The sets C_i's are convex subsets of missing elements in $M(4, 5)$. We will prove in Theorem 4.5 that each divisor d of 4 determines a subalgebra of $\mathbf{M}(4, 5)$ isomorphic to $\mathbf{M}(4/d, 5)$. The only interesting case here is $d = 2$. Consider the subset $M(4, 10) \subseteq M(4, 5)$. Note that 2 divides each element in $M(4, 10)$. We will show that $M(4, 10)$ forms a subuniverse of $\mathbf{M}(4, 5)$. To see this, we have to prove that $M(4, 10)$ is closed under $\dot{-}$. Let $x, y \in M(4, 10)$. If $x \geq y$, then $y \dot{-} x = 0 \in M(4, 10)$. Thus assume that $x < y$. Now we distinguish two cases. First, suppose that $y - x \in M(4, 5)$. Then $y - x = 4m + 5n$ for some $m, n \in \mathbb{N}$. Since 2 divides $x, y, 4$, we get $2|5n$. Thus $2|n$. Consequently, $y \dot{-} x = y - x = 4m + 10(n/2) \in M(4, 10)$. Second, suppose that $y - x \notin M(4, 5)$. Then $y - x \in C_i$ for some $i \in \{1, 2, 3\}$. Consequently, $y \dot{-} x = \min\{z \in M(4, 5) \mid z \geq y - x\} = 1 + \max C_i$. Observe that for each $i \in \{1, 2, 3\}$ the number $1 + \max C_i$ is a multiple of 4. Thus $1 + \max C_i \in M(4, 10)$. We have shown that the set $M(4, 10)$ forms a subuniverse of $\mathbf{M}(4, 5)$. Moreover, $\mathbf{M}(4, 10) \cong \mathbf{M}(2, 5)$.

Now we will characterize generally the subalgebras of $\mathbf{M}(a, b)$ when a is not prime and $\rho_a(b) = 1$.

Lemma 4.4 *Let $\langle a, b \rangle \in Gen$ such that $\rho_a(b) = 1$. Then for any $x, y \in M(a, b)$ such that $y - x \notin M(a, b)$ we have $y \dot{-} x = ka$ for $k = \lceil (y - x)/a \rceil$.*

PROOF: Clearly $ka \geq y - x$. We have to show that ka is the least such element in $M(a, b)$. Since $y - x \notin M(a, b)$, we have $y - x = ma + nb$ for $m < 0$ and $n \in [0, a-1]$ by Lemma 3.1, i.e., $y - x \leq nb - a$. As $k < (y - x)/a + 1$, we get $ka < y - x + a \leq nb$. Moreover, $y - x = (m + \lfloor b/a \rfloor n)a + n\rho_a(b) = (m + \lfloor b/a \rfloor n)a + n$ showing that $n = \rho_a(y - x)$. Let $z \in M(a, b)$ such that $z < ka < nb$. We have $z = sa + tb = (s + \lfloor b/a \rfloor t)a + t$. Then $s + \lfloor b/a \rfloor t < k = \lceil (y - x)/a \rceil$ and $t \in [0, n-1]$. Thus $s + \lfloor b/a \rfloor t \leq \lfloor (y - x)/a \rfloor$ by Lemma 1.2(2) and $t \leq n - 1$. Consequently, $z \leq \lfloor (y - x)/a \rfloor a + n - 1 < \lfloor (y - x)/a \rfloor a + n = y - x$. $\square$

Theorem 4.5 *Let $\langle a, b \rangle \in Gen$ such that $\rho_a(b) = 1$. For each divisor d of a there is a nontrivial subalgebra of $\mathbf{M}(a, b)$ isomorphic to $\mathbf{M}(a/d, b)$ and each nontrivial subalgebra of $\mathbf{M}(a, b)$ is isomorphic to $\mathbf{M}(a/d, b)$ for a divisor d of a.*

PROOF: Let d be a divisor of a. Consider the submonoid B generated by $\{a, db\}$. We will show that B is closed under $\dot{-}$, i.e., $\mathbf{B}$ is a subalgebra of $\mathbf{M}(a, b)$. Let $x, y \in B$. If $y - x \notin M(a, b)$, then $y \dot{-} x \in B$ by Lemma 4.4. Suppose that $y - x \in M(a, b)$. Then we have $y - x = ma + nb$. Since d divides both x and y, we get $d|nb$ and hence $d|n$ because a, b are coprime. Consequently, $y \dot{-} x = y - x = ma + (n/d)db \in B$. Moreover, the subalgebra $\mathbf{B}$ is clearly isomorphic to $\mathbf{M}(a/d, b)$.

Let $\mathbf{B}$ be a subalgebra of $\mathbf{M}(a,b)$. If $\mathbf{B}$ is divisible, then by Lemma 4.1 $\mathbf{B} \cong \mathbf{N} \cong \mathbf{M}(1,b)$. Let C be a set of generators of a non-divisible subalgebra $\mathbf{B}$ of $\mathbf{M}(a,b)$. Then $a \in C$ by Lemma 4.2. Further, we can assume that $C \subseteq \{a, b, 2b, \dots, (a-1)b\}$. Indeed, each element $c \in C$ can be expressed as $ma + nb$ for $m \in \mathbb{N}$ and $n \in [0, a-1]$ by Lemma 3.1. This implies $nb = (ma + nb) \dot{-} ma \in B$. Let $C = \{a, c_1 b, \dots, c_n b\}$ and $d = \gcd(a, c_1, \dots, c_n)$. Then there is $x \in B$ such that $\rho_a(x) = d$. Indeed, for any $k > g(a/d, c_1/d, \dots, c_n/d)$ and $\rho_a(k) = 1$ we can take $x = kd = (\lfloor k/a \rfloor a + 1)d = \lfloor k/a \rfloor da + d$. Consequently, we have $bx = b(\lfloor x/a \rfloor a + d) = \lfloor x/a \rfloor ba + db$. Thus $db = bx \dot{-} \lfloor x/a \rfloor ba \in B$. Since the submonoid generated by $\{a, db\}$ forms a subuniverse of $\mathbf{M}(a,b)$ isomorphic to $\mathbf{M}(a/d, b)$ and contains all $c_i b$'s, we are done. □

Now we will focus on the subalgebras of $\mathbf{M}(a,b)$ when a is prime or $\rho_a(b) \neq 1$. We will prove that the only proper nontrivial subalgebras in this case are divisible (thus isomorphic to $\mathbf{N}$ by Lemma 4.1). To show this, it is sufficient by Lemma 4.2 to prove that each set of the form $\{a, kb\}$ for some $k \in [1, a-1]$ generates the whole algebra $\mathbf{M}(a,b)$. If a is prime then a and kb are coprime. In this case $\{a, kb\}$ generates $\mathbf{M}(a,b)$ due to the following lemma.

Lemma 4.6 *Let $\langle a, b \rangle \in Gen$ and $c, d \in M(a,b)$. If c and d are coprime, then $\{c, d\}$ generates $\mathbf{M}(a,b)$.*

PROOF: Let $\mathbf{B}$ denote the subalgebra generated by $\{c, d\}$. Any number $x > g(c,d)$ can be expressed as a nonnegative integer combination of c, d and so $x, x + a \in B$. Consequently, $x + a \dot{-} x = a$. Similarly we can generate the element b. Thus $B = M(a,b)$. □

Now we will focus on the case when a is not prime and $\rho_a(b) \neq 1$. We start again with an example.

Example 4.7 Consider the algebra $\mathbf{M}(4,7)$. Similarly as in Example 4.3 one could expect that $M(4,14)$ forms a subuniverse. However, this is not the case since $M(4,14)$ is not closed under $\dot{-}$. In fact, we will show that $\{4, 14\}$ generates $\mathbf{M}(4,7)$. The following are the convex subsets of missing elements in $M(4,7)$:

$$C_1 = \{1,2,3\}\,, \quad C_2 = \{5,6\}\,, \quad C_3 = \{9,10\}\,, \quad C_4 = \{13\}\,, \quad C_5 = \{17\}\,.$$

Note that $1 + \max C_2 = 7$. If we are able to find elements $x, y \in M(4,14)$ such that $y - x \in C_2$ then $y \dot{-} x = 7 \notin M(4,14)$ showing that $\{4,14\}$ generates the whole algebra $\mathbf{M}(4,7)$. This is satisfied e.g. for $x = 8$ and $y = 14$.

In the example above it was sufficient to use $\dot{-}$ only once to obtain the second generator 7. Now we will present one more example illustrating that sometimes it is necessary to use $\dot{-}$ several times. Consider $\mathbf{M}(9,11)$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{M}(9,11)$ generated by $\{9,33\}$. Thus $M(9,33) \subseteq B$. We will prove in two steps that $B = M(9,11)$. First, we will show that $22 \in B$. Observe that $21 \notin M(9,11)$ by Lemma 3.1 since $21 = -5 \cdot 9 + 6 \cdot 11$. Thus we need to find elements $x, y \in B$ such that $y - x = 21$. Then $y \dot{-} x = 22$. Note that $\rho_9(21) = 3$. Thus it suffices to find an element $y \in B$ such that $\rho_9(y) = 3$. Then we can take for $x \in B$ a suitable multiple of 9. For instance, let $y = 33 \cdot 9 + 3 = 300$ and $x = 31 \cdot 9 = 279$. Then $\rho_9(y) = 3$ and $y \in M(9,33) \subseteq B$ since $y = 3 \cdot 100$ and $100 > 19 = g(3,11)$. Thus $22 = y \dot{-} x \in B$.

Consequently, $M(9,22) \subseteq B$. Now we repeat the process replacing 33 by 22. Again note that $10 = -5 \cdot 9 + 5 \cdot 11 \notin M(9,11)$ and $\rho_9(10) = 1$. Let $y = 22 \cdot 9 + 1 = 199$ and $x = 21 \cdot 9 = 189$. Then $y - x = 10$ and $y \in B$ since $y > 167 = g(9,22)$. Thus $11 = y \dot{-} x \in B$ and $B = M(9,11)$.

Lemma 4.8 *Let $\langle a,b\rangle \in Gen$, $\rho_a(b) \geq 2$, and $k \in [1, a-1]$. Then $\{a, kb\}$ generates $\mathbf{M}(a,b)$.*

PROOF: Let B be the universe of the subalgebra generated by $\{a, kb\}$. We will inductively construct a finite sequence $k_0, \ldots, k_n$ of natural numbers such that $k = k_0 > \cdots > k_n = 1$ and for all i's we have $k_i b \in B$.

Assume that $k_i b \in B$. If $k_i = 1$ then we are done. Thus suppose that $k_i \geq 2$. We will construct k_{i+1} such that $1 \leq k_{i+1} < k_i$. Let $k_{i+1} \in \mathbb{N}$ be minimal such that

$$(k_{i+1} - 1)\rho_a(b) < \gcd(a, k_i) \leq k_{i+1}\rho_a(b)\,.$$

Clearly $k_{i+1} \geq 1$ since $\gcd(a, k_i) \geq 1$. Now we will prove a series of claims.

Claim I: We have $k_{i+1} < k_i$.

First, if $k_{i+1} = 1$ then $k_{i+1} < k_i$ because we assume that $k_i \geq 2$. Suppose that $k_{i+1} \geq 2$. Then we get

$$k_{i+1} \leq 2(k_{i+1} - 1) \leq (k_{i+1} - 1)\rho_a(b) < \gcd(a, k_i) \leq k_i\,.$$

Claim II: We have $ak_i b + \gcd(a, k_i) \in B$.

Let $q = \gcd(a, k_i)$. We have

$$g(a/q, k_i b/q) = (ak_i b/q - a - k_i b)/q \leq ak_i b/q - a - k_i b < ak_i b/q + 1\,,$$

showing that $ak_i b/q + 1 \in M(a/q, k_i b/q)$. Consequently, $ak_i b + q \in M(a, k_i b) \subseteq B$.

Claim III: We have $k_{i+1}\lfloor b/a\rfloor = \lfloor k_{i+1} b/a\rfloor$.

If we prove that $k_{i+1}\rho_a(b) < a$ then the claim follows from Lemma 1.2(3). Since a does not divide k_i, there is $p \geq 2$ such that $a = p\gcd(a, k_i) \geq 2\gcd(a, k_i)$. Thus we have $\gcd(a, k_i) \leq a/2$. If $\rho_a(b) \geq a/2$ then $k_{i+1} = 1$ since $\gcd(a, k_i) \leq a/2$. Consequently, $k_{i+1}\rho_a(b) = \rho_a(b) < a$. Now assume that $\rho_a(b) < a/2$. Since $(k_{i+1} - 1)\rho_a(b) < \gcd(a, k_i) \leq a/2$, we have $k_{i+1}\rho_a(b) < a/2 + \rho_a(b) < a$.

Claim IV: Let $d = ak_i b + \gcd(a, k_i)$. The element $k_{i+1} b \in B$ and can be expressed as follows:

$$k_{i+1} b = d \dot{-} (k_i b - k_{i+1}\lfloor b/a\rfloor)a\,.$$

First, $d \in B$ by Claim II. The element $(k_i b - k_{i+1}\lfloor b/a\rfloor)a$ belongs to B as well because $k_i b > k_{i+1} b > k_{i+1} b/a \geq k_{i+1}\lfloor b/a\rfloor$. Further, we have

$$d - (k_i b - k_{i+1}\lfloor b/a\rfloor)a = k_{i+1}\lfloor b/a\rfloor a + \gcd(a, k_i) \leq k_{i+1}\lfloor b/a\rfloor a + k_{i+1}\rho_a(b) = k_{i+1} b\,.$$

Now we have to show that $k_{i+1} b$ is the least element in $M(a,b)$ greater than or equal to $d - (k_i b - k_{i+1}\lfloor b/a\rfloor)a$. Let $x \in M(a,b)$ such that $x < k_{i+1} b$. Then $x = ma + nb$ for $n \in [0, k_{i+1} - 1]$ and $m \in \mathbb{N}$. We have $x = (m + n\lfloor b/a\rfloor)a + n\rho_a(b)$. Further, $m + n\lfloor b/a\rfloor \leq k_{i+1}\lfloor b/a\rfloor$. Indeed, if $m + n\lfloor b/a\rfloor > k_{i+1}\lfloor b/a\rfloor = \lfloor k_{i+1} b/a\rfloor$ (see Claim III) then $m + n\lfloor b/a\rfloor \geq \lceil k_{i+1} b/a\rceil$ by

Lemma 1.2(2). Consequently, we get a contradiction since $x \geq (m+n\lfloor b/a \rfloor)a \geq \lceil k_{i+1}b/a \rceil a \geq k_{i+1}b$. Thus

$$x \leq k_{i+1}\lfloor b/a \rfloor a + (k_{i+1}-1)\rho_a(b) < k_{i+1}\lfloor b/a \rfloor a + \gcd(a, k_i) = d - (\lfloor d/a \rfloor - k_{i+1}\lfloor b/a \rfloor)a$$

showing that $k_{i+1}b$ is the least element in $M(a,b)$ greater than or equal to $d - (k_i b - k_{i+1}\lfloor b/a \rfloor)a$. Thus $k_{i+1}b = d \dot{-} (k_i b - k_{i+1}\lfloor b/a \rfloor)a \in B$.

Since we cannot have an infinite sequence $k_0 > k_1 > \cdots$ such that all $k_i \geq 2$, there must be $n \in \mathbb{N}$ satisfying $k_n = 1$. Consequently, $b = k_n b \in B$, i.e., $B = M(a,b)$. □

Theorem 4.9 *Let $\langle a, b\rangle \in Gen$. If a is prime or $\rho_a(b) \neq 1$ then each nontrivial proper subalgebra of $\mathbf{M}(a,b)$ is isomorphic to $\mathbf{N}$.*

PROOF: Each nontrivial divisible subalgebra of $\mathbf{M}(a,b)$ is isomorphic to $\mathbf{N}$ by Lemma 4.1. Assume that there is a nontrivial non-divisible subalgebra $\mathbf{B}$. Then by Lemma 4.2 we have $a \in B$ and $nb \in B$ for some $n \in [1, a-1]$. If a is prime then a and nb are coprime. Thus by Lemma 4.6 $\{a, nb\}$ generates $\mathbf{M}(a,b)$. If $\rho_a(b) \neq 1$ then the same follows from Lemma 4.8. In both the cases the subalgebra generated by $\{a, nb\}$ is not proper. □

5 Varieties generated by $\mathbf{M}(a,b)$

In this section we will prove that algebras $\mathbf{M}(a,b)$ for $\langle a,b\rangle \in Gen$ which have only divisible subalgebras generate covers of $\mathcal{V}(\mathbf{N})$. In order to do this, we will need well-known Jónsson's Lemma [9] stating that subdirectly irreducible members in a congruence distributive variety $\mathcal{V}(\mathcal{K})$ belong to $\mathrm{HSP_U}(\mathcal{K})$.

Let $\mathbf{M}(a,b)^I/U$ be an ultrapower of $\mathbf{M}(a,b)$. It is obvious that $\mathbf{M}(a,b)^I/U$ contains an isomorphic copy $\mathbf{A}$ of $\mathbf{M}(a,b)$ via the embedding sending $x \in M(a,b)$ to the congruence class $(x)/U$ where (x) denotes the constant mapping with value x. In the following we will identify $M(a,b)$ with its isomorphic copy A, i.e., $M(a,b) \subseteq M(a,b)^I/U$.

Lemma 5.1 *The algebra $\mathbf{M}(a,b)$ forms an initial segment of $\mathbf{M}(a,b)^I/U$, i.e., $M(a,b)$ is a downward closed subset of $M(a,b)^I/U$.*

PROOF: Let $x \in M(a,b)$ and $y/U \in M(a,b)^I/U$ such that $y/U \leq x$. Thus there is a subset $J \subseteq I$ belonging to U such that $y(i) \leq x$ for all $i \in J$. Since y is bounded on J from above by the constant mapping with the value x, it can attain on J only finitely many values $c_1, \ldots, c_k \in M(a,b)$. Since U is an ultrafilter, one of the subsets $\{i \in J \mid y(i) = c_j\}$ for $j \in \{1, \ldots, k\}$ has to belong to U. Thus y is equivalent to a constant mapping, i.e., $y/U \in M(a,b)$. □

Lemma 5.2 *Let $\mathbf{B}$ be a subdirectly irreducible algebra in $\mathcal{V}(\mathbf{M}(a,b))$ for $\langle a,b\rangle \in Gen$. Then $\mathbf{B} \in \mathcal{V}(\mathbf{N})$ or $\mathbf{B} \in \mathrm{ISP_U}(\mathbf{M}(a,b))$.*

PROOF: By Jonsson's lemma $\mathbf{B} \in \mathrm{HSP_U}(\mathbf{M}(a,b))$. Thus $\mathbf{B}$ is a homomorphic image of a subalgebra $\mathbf{C}$ of an ultrapower $\mathbf{M}(a,b)^I/U$, i.e., $\mathbf{B} \cong \mathbf{C}/\theta$ for a congruence θ. Assume that $\mathbf{B} \notin \mathrm{ISP_U}(\mathbf{M}(a,b))$, i.e., θ is nontrivial. Using Theorem 2.1 in the dual language, there is an element $e \in C$ such that $e \neq 0$ and $e/\theta = 0/\theta$. We will prove that $\mathbf{B}$ has to be divisible. Then the lemma follows by Theorem 2.4.

Let $c/\theta, d/\theta \in B$. We have to prove that $(d/\theta \dot{-} c/\theta) + c/\theta = c/\theta \vee d/\theta$. This equality is trivially satisfied when $c/\theta \geq d/\theta$. Hence suppose that $c/\theta < d/\theta$. Observe that by Lemma 3.2 the algebra $\mathbf{M}(a,b)$ satisfies the following first-order sentence:

$$\forall x \forall y \forall z \Big(\big(x \leq y \ \ \& \ \ z \neq 0\big) \ \ \Rightarrow \ \ \big(y \leq (y \dot{-} x) + x \ \ \& \ \ (y \dot{-} x) + x \leq y + z\big)\Big)\,, \tag{4}$$

where $\&, \Rightarrow$ denote the classical conjunction and implication respectively. Since universal sentences are preserved under taking ultrapowers and subalgebras, $\mathbf{C}$ satisfies (4) as well. The validity of (4) in $\mathbf{C}$ implies $d \leq (d \dot{-} c) + c \leq d + e$ because $c < d$ and $e \neq 0$. Consequently,

$$d/\theta \leq (d/\theta \dot{-} c/\theta) + c/\theta \leq d/\theta + e/\theta = d/\theta + 0/\theta = d/\theta = c/\theta \vee d/\theta\,,$$

showing that $\mathbf{B}$ is divisible. □

Thus if we want to prove that an $\mathbf{M}(a,b)$ generates a cover of $\mathcal{V}(\mathbf{N})$, it suffices to prove that each algebra in $\mathrm{SP_U}(\mathbf{M}(a,b))$ belongs either to $\mathcal{V}(\mathbf{N})$ or it contains a subalgebra isomorphic to $\mathbf{M}(a,b)$.

Lemma 5.3 *Let $\langle a,b\rangle \in Gen$ and $\mathbf{B}$ a subalgebra of an ultrapower $\mathbf{M}(a,b)^I/U$. If $\mathbf{B}$ is not divisible then $a, nb \in B$ for some $n \in [1, a-1]$.*

PROOF: Since $\mathbf{B}$ is not divisible, there are elements $x/U, y/U \in B$ such that $x/U < y/U$ and $y/U < (y/U \dot{-} x/U) + x/U$. Thus $x(i) < y(i)$ and $y(i) \dot{-} x(i) > y(i) - x(i)$ hold on a subset $J \subseteq I$ belonging to the ultrafilter U. Consequently, $a = (y(i) \dot{-} x(i)) + x(i) \dot{-} y(i)$ by Lemma 3.2 for all $i \in J$. Hence $a \in B$.

Further, we can bound $y(i) - x(i)$ from above. Namely, $y(i) - x(i) < g(a,b)$ for $i \in J$ otherwise $y(i) - x(i) \in M(a,b)$, i.e., $y(i) \dot{-} x(i) = y(i) - x(i)$. Consequently, $y(i) - x(i)$ can attain only finitely many values on J. Thus $y(i) - x(i)$ is constant on a subset $K \subseteq J$ belonging to U, i.e., $y(i) - x(i) = d$ for some $d \in \mathbb{N}$. By Lemma 3.1 there are $m \in \mathbb{Z}$ and $n \in [0, a-1]$ such that $d = ma + nb$. Moreover, $m < 0$ and $n > 0$ since $0 \leq d \notin M(a,b)$. Now, for each $i \in K$ we have

$$(y(i) + (-m)a) \dot{-} x(i) = (x(i) + d - ma) \dot{-} x(i) = (x(i) + nb) \dot{-} x(i) = nb\,.$$

Consequently, $nb \in B$. □

Theorem 5.4 *Let $\langle a,b\rangle \in Gen$. If a is prime or $\rho_a(b) \neq 1$, then $\mathcal{V}(\mathbf{M}(a,b))$ is a cover of $\mathcal{V}(\mathbf{N})$.*

PROOF: Clearly $\mathcal{V}(\mathbf{M}(a,b)) \supsetneq \mathcal{V}(\mathbf{N})$ since $\mathbf{M}(a,b)$ is not divisible by Proposition 3.3. Assume that $\mathbf{B}$ is a nontrivial subdirectly irreducible algebra in $\mathcal{V}(\mathbf{M}(a,b))$ such that $\mathbf{B} \notin \mathcal{V}(\mathbf{N})$ (i.e., $\mathbf{B}$ is not divisible). Then $\mathbf{B}$ is isomorphic to an algebra $\mathbf{C} \in \mathrm{SP_U}(\mathbf{M}(a,b))$ by Lemma 5.2. Using Lemma 5.3, we get $a, nb \in C$ for some $n \in [1, a-1]$. Finally, Lemmata 4.6, 4.8 show that $\mathbf{C}$ contains a subalgebra isomorphic to $\mathbf{M}(a,b)$, i.e., $\mathcal{V}(\mathbf{B}) \supseteq \mathcal{V}(\mathbf{M}(a,b))$. $\square$

Rephrasing the latter result in the dual language, we obtain the following corollary.

Corollary 5.5 *There are infinitely many representable integral covers of* $\mathcal{CLG}^-$ *in* $\mathbf{\Lambda}(\mathcal{CanCRL})$.

We have found infinitely many covers of $\mathcal{V}(\mathbf{N})$ among varieties $\mathcal{V}(\mathbf{M}(a,b))$ for $\langle a,b\rangle \in Gen$. In fact, the remaining varieties of this type do not generate covers of $\mathcal{V}(\mathbf{N})$. We will prove it by describing their mutual position in the subvariety lattice.

Theorem 5.6 *Let* $\langle a,b\rangle, \langle c,d\rangle \in Gen$ *such that* $\rho_a(b) = \rho_c(d) = 1$. *Then* $\mathcal{V}(\mathbf{M}(c,d)) \subseteq \mathcal{V}(\mathbf{M}(a,b))$ *iff* c *divides* a *and* $d = b$.

PROOF: The right-to-left implication follows easily from Theorem 4.5. To see the other implication suppose that $\mathcal{V}(\mathbf{M}(c,d)) \subseteq \mathcal{V}(\mathbf{M}(a,b))$. Then $\mathbf{M}(c,d) \in \mathcal{V}(\mathbf{M}(a,b))$. Since $\mathbf{M}(c,d) \notin \mathcal{V}(\mathbf{N})$, it follows from Lemma 5.2 that $\mathbf{M}(c,d)$ is isomorphic to a subalgebra $\mathbf{B}$ of an ultrapower $\mathbf{M}(a,b)^I/U$ via an isomorphism $f\colon M(c,d) \to B$. Furthermore, by Lemma 5.3 we have $a \in B$. Clearly, $f(c) = a$. Since there is $n \in \mathbb{N}$ such that $nc \geq d$, we have $na = nf(c) \geq f(d)$. Thus $\mathbf{B}$ is a subalgebra of $\mathbf{M}(a,b)$ by Lemma 5.1. Using Theorem 4.5, it follows that $\mathbf{B} \cong \mathbf{M}(a/e,b)$ for a divisor e of a. Obviously, $e < a$ since $\mathbf{M}(c,d) \ncong \mathbf{N}$. Consequently, $\langle a/e,b\rangle \in Gen$ and $\mathcal{V}(\mathbf{M}(c,d)) = \mathcal{V}(\mathbf{M}(a/e,b))$. By Corollary 3.8 we have $c = a/e$ and $d = b$. $\square$

Corollary 5.7 *Let* $\langle a,b\rangle \in Gen$. *Then* $\mathcal{V}(\mathbf{M}(a,b))$ *is a cover of* $\mathcal{V}(\mathbf{N})$ *iff* a *is prime or* $\rho_a(b) \neq 1$.

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