MINIMAL VARIETIES OF REPRESENTABLE COMMUTATIVE RESIDUATED LATTICES

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ABSTRACT. We solve several open problems on the cardinality of atoms in the subvariety lattice of residuated lattices and FL-algebras [4, Problems 17–19, pp. 437]. Namely, we prove that the subvariety lattice of residuated lattices contains continuum many 4-potent commutative representable atoms. Analogous results apply also to atoms in the subvariety lattice of FL_i-algebras and FL_o-algebras. On the other hand, we show that the subvariety lattice of residuated lattices contains only five 3-potent commutative representable atoms and two integral commutative representable atoms. Inspired by the construction of atoms, we are also able to prove that the variety of integral commutative representable residuated lattices is generated by its 1-generated finite members.

1. Introduction

It is well known (see e.g. [4]) that the subvariety lattice $\mathbf{\Lambda}(\mathsf{FL})$ of pointed residuated lattices (FL-algebras) is dually isomorphic to the lattice of substructural logics (i.e., the lattice of axiomatic extensions of Full Lambek Calculus). This paper focuses on the cardinality of minimal varieties of FL-algebras whose members satisfy some additional properties like commutativity, representability etc. Recall that a nontrivial variety K of FL-algebras is called *minimal* if it has only one, trivial, proper subvariety, i.e., K is an atom in $\mathbf{\Lambda}(\mathsf{FL})$. Note that minimal varieties of FL-algebras correspond to maximally consistent substructural logics.

Recall that the class of all residuated lattices can be viewed as a subvariety of FL-algebras (namely, every residuated lattice can be viewed as an FL-algebra satisfying the identity $1 = 0$). In [5, Problem 8.6] the authors posed a question whether there are uncountably many minimal varieties of residuated lattices that satisfy $x \cdot y = y \cdot x$ or $x^2 = x^3$. This question was answered in [3] by giving continuum many minimal varieties whose members are representable and satisfy $x = x^2$. Concerning the identity $x \cdot y = y \cdot x$, [3] gives only a partial answer by showing that there are at least countably many minimal varieties of residuated lattices, whose members are representable and commutative, leaving as an open question whether there are uncountable many of them or not. The same question appears also in [4] together with related problems on FL-algebras; see [4, Problems 17–19, pp. 437]. In Section 3 we solve the first problem by constructing continuum many minimal varieties of residuated lattices whose members are 4-potent, commutative and representable. The related problems on FL-algebras can be solved by easy modifications.

Section 4 shows that 4-potency is crucial in the above-mentioned result by proving that there are only five minimal varieties of residuated lattices whose members are 3-potent, commutative and representable. Another reasonable question is what happens if we replace the k-potency by integrality. In this case it is possible to show that there are only two minimal varieties of residuated lattices.

Finally, inspired by the construction of minimal varieties, we are also able to prove that the variety of integral commutative representable residuated lattices is generated by its 1-generated finite members (Section 5).

2. Preliminaries

The sets of natural numbers, integers and non-positive integers are denoted respectively $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}^-$.

This paper deals only with commutative residuated lattices and commutative FL-algebras. For definitions of general residuated lattices and FL-algebras see [4]. A *commutative residuated lattice*

(CRL) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, 1\rangle$, where $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a commutative monoid and for all $x, y, z \in A$ we have

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x \rightarrow z\,.$$

From the definition of CRL it readily follows that the multiplication $\cdot$ is monotone in both arguments while the residuum $\rightarrow$ is antitone in the first argument and monotone in the second one. The residuum is fully determined by the multiplication and the order since $x \rightarrow y = \max\{z \mid x \cdot z \leq y\}$. We adopt the usual convention of writing xy for $x \cdot y$. In the absence of parentheses, we assume that the multiplication is performed first followed by the residuum and the lattice operations. The n-fold products are defined inductively by $x^0 = 1$ and $x^{n+1} = xx^n$.

It is well known that the class of CRLs forms a variety (see e.g. [4]). Let $\mathbf{A}$ be a CRL. If 1 is also a top element, we call $\mathbf{A}$ an *integral commutative residuated lattice* (ICRL). Let $k \in \mathbb{N}$. Then a CRL $\mathbf{A}$ is called *k-potent* if it satisfies $x^{k+1} = x^k$. A CRL $\mathbf{A}$ is referred to as a *commutative residuated chain* (CRC), if the lattice reduct of $\mathbf{A}$ forms a chain. Analogously an integral CRC is called shortly ICRC. A CRL $\mathbf{A}$ is said to be *representable* if $\mathbf{A}$ is isomorphic to a subdirect product of CRCs. The varieties of representable and integral CRLs are denoted respectively RCRL and ICRL.

An *FL_e-algebra* is a pointed CRL, i.e., an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$, where the reduct $\langle A, \wedge, \vee, \cdot, \rightarrow, 1\rangle$ is a CRL. If this reduct is an ICRL then $\mathbf{A}$ is called *FL_{ei}-algebra.* An FL_e-algebra is called *FL_{eo}-algebra* if 0 is a bottom element. Finally, we call an FL_e-algebra *representable* (resp. *k-potent*) if the underlying CRL is representable (resp. k-potent). For details on CRLs and FL_e-algebras see [4]. The varieties of representable FL_{eo}-algebras and representable FL_{ei}-algebras are denoted respectively $\mathsf{RFL_{eo}}$ and $\mathsf{RFL_{ei}}$.

Let $\mathbf{A}$ be a CRL or an FL_e-algebra. Given $S \subseteq A$, the upset generated by S is denoted $\uparrow S$, i.e., $\uparrow S = \{b \in A \mid (\exists a \in S)(a \leq b)\}$. A subset F of A is called a *deductive filter* of $\mathbf{A}$ if

- $1 \in F$,
- $x \in F$ and $x \leq y$ implies $y \in F$ (i.e., F is an upset),
- $x, y \in F$ implies $xy \in F$,
- $x \in F$ implies $x \wedge 1 \in F$.

Clearly, $\uparrow\{1\}$ is the least deductive filter of $\mathbf{A}$. A deductive filter F is called *nontrivial* if $F \neq \uparrow\{1\}$. The collection of all deductive filters of $\mathbf{A}$ forms a lattice ordered by the inclusion. Moreover, it is known that this lattice is isomorphic to the congruence lattice of $\mathbf{A}$ (see [4]). Given a congruence θ, the corresponding deductive filter is $\uparrow(1/\theta)$. Conversely, let F be a deductive filter of $\mathbf{A}$. Then the corresponding congruence $\theta = \{\langle a, b\rangle \in A^2 \mid a \rightarrow b, b \rightarrow a \in F\}$. Let $a \in A$ such that $a \leq 1$. It is easy to see that the deductive filter generated by a (i.e., the least deductive filter of $\mathbf{A}$ containing a) is the set $F(a) = \uparrow\{a^n \mid n \in \mathbb{N}\}$.

In order to recognize minimal varieties of CRLs and FL_e-algebras, we will use [4, Lemma 9.1] which we restate here. A nontrivial algebra $\mathbf{A}$ is said to be *strictly simple*, if it lacks nontrivial proper subalgebras and congruences. By proper subalgebra of $\mathbf{A}$ we mean a subalgebra $\mathbf{B}$ which is not isomorphic to $\mathbf{A}$. An element $b \in A$ is called *nearly term definable* if there is an n-ary term-operation $t(x_1, \ldots, x_n)$ such that $t(a_1, \ldots, a_n) = b$ holds unless $a_1 = \cdots = a_n = 1$. Given an algebra $\mathbf{A}$, $\mathsf{V}(\mathbf{A})$ denotes the variety generated by $\mathbf{A}$.

Lemma 2.1 ([4])**.** *Let $\mathbf{A}$ be a strictly simple FL_e-algebra or CRL with a bottom element $\perp$ nearly term definable by an n-ary term t. Then, $\mathsf{V}(\mathbf{A})$ is a minimal variety. Moreover, if $\mathbf{A}'$ is a strictly simple FL_e-algebra or CRL with a bottom element nearly term definable by the same term t, then $\mathsf{V}(\mathbf{A}) \subseteq \mathsf{V}(\mathbf{A}')$ if and only if $\mathbf{A}$ and $\mathbf{A}'$ are isomorphic.*

In the rest of this section we recall several useful constructions which we will need at the sequel. Let $\mathbf{A}$ be a CRL. A *conucleus* on $\mathbf{A}$ is an interior operator σ on $\mathbf{A}$ (i.e., $\sigma\colon A \rightarrow A$ is a monotone map satisfying $\sigma(x) \leq x$ and $\sigma(\sigma(x)) = \sigma(x)$ for all $x \in A$) such that $\sigma(x)\sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$ for all $x, y \in A$. We denote the image of σ by A_σ, i.e., $A_\sigma = \sigma[A]$. It follows immediately from the definition that A_σ forms a submonoid of $\mathbf{A}$.

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 1\rangle$ be a CRL and σ a conucleus on $\mathbf{A}$. Then the structure $\mathbf{A}_\sigma = \langle A_\sigma, \wedge_\sigma, \vee, \cdot, \to_\sigma, 1\rangle$, where $x \wedge_\sigma y = \sigma(x \wedge y)$ and $x \to_\sigma y = \sigma(x \to y)$, is called a *$\sigma$-contraction* of $\mathbf{A}$. Observe that $\wedge_\sigma = \wedge$ if $\langle A, \wedge, \vee\rangle$ is a chain.

Theorem 2.2 ([4])**.** *Let $\mathbf{A}$ be a CRL and σ a conucleus on $\mathbf{A}$. Then $\mathbf{A}_\sigma$ is a CRL.*

Each interior operator σ on a poset $\mathbf{P}$ is fully determined by its image as follows: $\sigma(x) = \max\{a \in P_\sigma \mid a \leq x\}$. In fact, there exists a bijective correspondence between all interior operators σ on a poset $\mathbf{P}$ and all subposets $\mathbf{O}$ of $\mathbf{P}$ satisfying the condition:

$$\max\{a \in O \mid a \leq x\} \text{ exists for all } x \in P. \tag{1}$$

Due to this one can characterize σ-contractions of a CRL $\mathbf{A}$ as subposets which are at the same time submonoids and satisfy the condition (1), see [6].

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 1\rangle$ be an ICRL and $a \in A$. Note that the interval $[a, 1]$ is closed under all the operations except of the multiplication. If we define a new operation $\circ_a$ on $[a, 1]$ by $x \circ_a y = xy \vee a$, we obtain an ICRL $[\mathbf{A}]_a = \langle [a, 1], \wedge, \vee, \circ_a, \to, 1\rangle$ (see e.g. [4]). The algebra $[\mathbf{A}]_a$ is in fact a nuclear retraction of $\mathbf{A}$ for the nucleus $\gamma(x) = x \vee a$.

Let us illustrate the above notions on several examples of CRLs which we will need later on. Let $\mathbf{G}$ be an Abelian ℓ-group. Then $\mathbf{G}$ can be viewed as a CRL if we define $x \to y = yx^{-1}$. Consider the additive ℓ-group of integers $\mathbf{Z} = \langle \mathbb{Z}, \wedge, \vee, +, \to, 0\rangle$ viewed as a CRC, i.e., $x \to y = y - x$. Its negative cone $\mathbf{Z}^-$ is the image of an interior operator $\sigma(x) = x \wedge 0$. Since $\mathbb{Z}^-$ is also closed under $+$, σ is a conucleus. Thus $\mathbf{Z}^- = \mathbf{Z}_\sigma$ forms an ICRC where $x \to_\sigma y = (y - x) \wedge 0$. Finally, let $n \in \mathbb{N}$. Consider the truncation $\mathbf{L}_n = [\mathbf{Z}^-]_{-n}$ to the interval $[-n, 0]$. Then $\mathbf{L}_n$ is an ICRC where the multiplication is given by $x \circ_{-n} y = -n \vee (x + y)$ and its residuum by $x \to_\sigma y = (y - x) \wedge 0$. The algebra $\mathbf{L}_n$ is in fact the well-known $(n + 1)$-valued Wajsberg hoop (see e.g. [1]).

3. Continuum many 4-potent minimal varieties

Let K be a variety. Its subvariety lattice is denoted $\mathbf{\Lambda}(\mathsf{K})$. In this section we are going to solve the open problems [4, Problems 17–19, pp. 437]. Namely, we will prove that the subvariety lattices $\mathbf{\Lambda}(\mathsf{RCRL})$, $\mathbf{\Lambda}(\mathsf{RFL_{ei}})$ and $\mathbf{\Lambda}(\mathsf{RFL_{eo}})$ contain continuum many minimal varieties.

In order to construct algebras generating a minimal variety in a subvariety lattice, we will need a method extending an ICRC by a new neutral element for the multiplication. Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 1^{\mathbf{A}}\rangle$ be an ICRC with a coatom $a = \max(A \setminus \{1^{\mathbf{A}}\})$. We will extend the 1-free reduct of $\mathbf{A}$ by adding a new neutral element $1^{\mathbf{A}'}$ in order to obtain a CRC $\mathbf{A}' = \langle A', \wedge, \vee, \cdot, \to, 1^{\mathbf{A}'}\rangle$, where $A' = A \cup \{1^{\mathbf{A}'}\}$. The new lattice order $\wedge, \vee$ is the extension of the original order letting $a \leq 1^{\mathbf{A}'} \leq 1^{\mathbf{A}}$. Thus $1^{\mathbf{A}}$ becomes a top element of $\mathbf{A}'$. Let $x \in A \cup \{1^{\mathbf{A}'}\}$ and $y \in A \setminus \{1^{\mathbf{A}}\}$. The operations are extended as follows:

$$1^{\mathbf{A}'} \cdot x = x = x \cdot 1^{\mathbf{A}'}\,, \quad 1^{\mathbf{A}'} \to x = x\,, \quad y \to 1^{\mathbf{A}'} = 1^{\mathbf{A}}\,, \quad 1^{\mathbf{A}} \to 1^{\mathbf{A}'} = a\,.$$

Lemma 3.1. *Let $\mathbf{A}$ be an ICRC with a coatom a. Then $\mathbf{A}'$ is an CRC.*

Proof. Clearly, $\langle A', \wedge, \vee\rangle$ is a chain and $\langle A', \cdot, 1^{\mathbf{A}'}\rangle$ forms a commutative monoid. We will check that $\mathbf{A}'$ satisfies the residuation property, i.e., $xy \leq z$ iff $y \leq x \to z$. Since A is a subuniverse of the 1-free reduct of $\mathbf{A}'$, we have $xy \leq z$ iff $y \leq x \to z$ for all $x, y, z \in A$. Let $x, y, z \in A \cup \{1^{\mathbf{A}'}\}$ and suppose that at least one of them equals $1^{\mathbf{A}'}$. If $x = 1^{\mathbf{A}'}$ then we have $1^{\mathbf{A}'} \cdot y = y \leq z$ iff $y \leq z = 1^{\mathbf{A}'} \to z$. Thus assume $x \neq 1^{\mathbf{A}'}$. If $z \neq 1^{\mathbf{A}'}$ (i.e., $y = 1^{\mathbf{A}'}$) then $x \cdot 1^{\mathbf{A}'} = x \leq z$ iff $1^{\mathbf{A}'} \leq 1^{\mathbf{A}} \leq x \to z$. Now assume $z = 1^{\mathbf{A}'}$. If $x = 1^{\mathbf{A}}$ then $1^{\mathbf{A}} \cdot y \leq 1^{\mathbf{A}'}$ iff $y \leq a = 1^{\mathbf{A}} \to 1^{\mathbf{A}'}$. If $x \leq a$ then $x \cdot y \leq x \leq 1^{\mathbf{A}'}$. Thus $x \cdot y \leq 1^{\mathbf{A}'}$ iff $y \leq 1^{\mathbf{A}} = x \to 1^{\mathbf{A}'}$. Summing up, $\mathbf{A}'$ is a CRC. □

Let $\mathbf{A} = \langle A, \wedge, \vee, +, \to, \langle 0, 0\rangle\rangle$ be the totally ordered Abelian ℓ-group (viewed as a CRC) given by the lexicographic product of two copies of $\mathbf{Z}$, i.e., $A = \mathbb{Z}^2$ ordered lexicographically, $+$ is computed component-wise and $\langle x, y\rangle \to \langle u, v\rangle = \langle u - x, v - y\rangle$. Recall that there are $2^{\aleph_0}$ infinite subsets of $\mathbb{Z}^-$. We construct for each infinite subset $S \subseteq -2 + \mathbb{Z}^- = \{-2 + z \mid z \in \mathbb{Z}^-\}$ an algebra

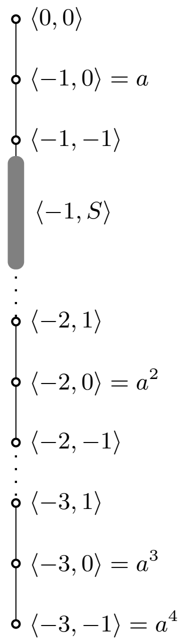

FIGURE 1. The structure of the algebra $[\mathbf{A}_{\sigma_S}]_{\langle -3,-1\rangle}$.

$\mathbf{A}_S$ from the algebra $\mathbf{A}$ by means of a conucleus σ_S. We define the conucleus σ_S by defining its image as follows:

$$A_{\sigma_S} = \{\langle 0,0\rangle, \langle -1,0\rangle, \langle -1,-1\rangle\} \cup \{\langle -1,z\rangle \in A \mid z \in S\} \cup \{\langle x,y\rangle \in A \mid x \le -2\}\,.$$

The set A_{σ_S} forms clearly a submonoid of $\mathbf{A}$ since $\langle -1,x\rangle + \langle -1,y\rangle = \langle -2,x+y\rangle \in A_{\sigma_S}$. Further we have to check the condition (1), i.e., if $\max\{x \in A_{\sigma_S} \mid x \le y\}$ exists for each $y \in A$. The existence is obvious for $y \in A_{\sigma_S}$. Suppose that $y = \langle u,v\rangle \notin A_{\sigma_S}$. Then u is greater than or equal to -1. If $u > 0$, then the maximum is $\langle 0,0\rangle$. If $u = 0$ then the maximum is $\langle -1,0\rangle$. If $u = -1$ then the maximum exists since the set $M = \{\langle w,z\rangle \in A_{\sigma_S} \mid w = -1\}$ is dually well-ordered and infinite (thus there is a lower bound of y in M). Hence A_{σ_S} is the image of a conucleus $\sigma_S(x) = \max\{y \in A_{\sigma_S} \mid y \le x\}$ and $\mathbf{A}_{\sigma_S}$ forms an ICRC. Note that $\langle x,y\rangle \to_{\sigma_S} \langle u,v\rangle = \sigma_S(u-x, v-y)$ in $\mathbf{A}_{\sigma_S}$. Finally, consider the truncated ICRC $[\mathbf{A}_{\sigma_S}]_{\langle -3,-1\rangle}$. Its structure is depicted in Figure 1. This ICRC is clearly 4-potent (i.e., it satisfies $x^5 = x^4$) because $x^4 = \langle -3,-1\rangle$ for every $x \neq \langle 0,0\rangle$.

Define $\mathbf{A}_S$ as the subalgebra of $[\mathbf{A}_{\sigma_S}]_{\langle -3,-1\rangle}$ generated by $a = \langle -1,0\rangle$. We will prove that each $\mathbf{A}_S$ contains some important elements. First, note that there is a term $r(x) = x^2 \to x^4$ such that $r^{\mathbf{A}_S}(a) = \langle -2,0\rangle \to_{\sigma_S} \langle -3,-1\rangle = \sigma_S(-1,-1) = \langle -1,-1\rangle$. Thus $\langle -1,-1\rangle \in A_S$. Since the set S is dually well-ordered, we can index its elements by natural numbers, i.e., $S = \{c_0 > c_1 > c_2 > \cdots\}$.

Lemma 3.2. *Let $n \in \mathbb{N}$. Then there is a term $s_n(x)$ such that $s_n^{\mathbf{A}_S}(a) = \langle -1,c_n\rangle$ for each $\mathbf{A}_S$. Thus $\langle -1,c_n\rangle \in A_S$ for all $n \in \mathbb{N}$.*

Proof. By induction on n. Let $s_0(x)$ be the term $x \to r(x)^2$. Then

$$s_0^{\mathbf{A}_S}(a) = \langle -1,0\rangle \to_{\sigma_S} \langle -2,-2\rangle = \sigma_S(-1,-2) = \langle -1,c_0\rangle\,,$$

since $\langle -1,c_0\rangle$ is the predecessor of $\langle -1,-1\rangle$. Now assume that there is a term $s_n(x)$ such that $s_n^{\mathbf{A}_S}(a) = \langle -1,c_n\rangle$. Let $s_{n+1}(x) = x \to r(x)\cdot s_n(x)$. Then

$$s_{n+1}^{\mathbf{A}_S}(a) = \langle -1,0\rangle \to_{\sigma_S} \langle -2,c_n-1\rangle = \sigma_S(-1,c_n-1) = \langle -1,c_{n+1}\rangle\,.$$

□

Lemma 3.3. *Let $n \in \mathbb{N}$. Then there is a term $t_n(x)$ such that $t_n^{\mathbf{A}_S}(a) = \langle -2, n\rangle$ for each $\mathbf{A}_S$. Thus $\langle -2, n\rangle \in A_S$ for all $n \in \mathbb{N}$.*

Proof. By induction on n. For $n = 0$ it is clear since $\langle -2, 0\rangle = a^2$. Assume that there is $t_n(x)$ such that $t_n^{\mathbf{A}_S}(a) = \langle -2, n\rangle$. Consider a term $t_{n+1}(x) = r(x) \to x \cdot t_n(x)$. Then

$$t_{n+1}^{\mathbf{A}_S}(a) = \langle -1, -1\rangle \to_{\sigma_S} \langle -3, n\rangle = \sigma_S(-2, n+1) = \langle -2, n+1\rangle .$$

□

The 4-potent ICRC $\mathbf{A}_S$ generated by a is simple since the only nontrivial deductive filter of $\mathbf{A}_S$ is $F(a) = \uparrow\{a^4\} = A_S$. However, $\mathbf{A}_S$ is not strictly simple because $\{a^4, \langle 0, 0\rangle\}$ forms a proper nontrivial subalgebra. In order to obtain a strictly simple CRC from $\mathbf{A}_S$, we use the construction from Lemma 3.1 and extend $\mathbf{A}_S$ by a new neutral element.

Lemma 3.4. *The algebra $\mathbf{A}'_S$ is strictly simple with a nearly term definable bottom element.*

Proof. We will show that any element $x \neq 1^{\mathbf{A}'_S}$ generates $\mathbf{A}'_S$. First, we can make the top element $\langle 0, 0\rangle = x \to x$. Second, we can produce $a = \langle -1, 0\rangle = \langle 0, 0\rangle \to 1^{\mathbf{A}'_S}$ since $\langle -1, 0\rangle$ is the coatom of $\mathbf{A}_S$. Since $\mathbf{A}_S$ is generated by a, we are done. Moreover the bottom element a^4 is nearly term definable by the term $x^4 \wedge (x \to 1)^4$. □

Finally, we have to prove that $\mathbf{A}'_R$ and $\mathbf{A}'_S$ are not isomorphic for different sets R, S so that we can invoke Lemma 2.1.

Lemma 3.5. *Let $R, S \subseteq -2 + \mathbb{Z}^-$ such that $R \neq S$. Then $\mathbf{A}_S$ is not isomorphic to $\mathbf{A}_R$. The same is true also for $\mathbf{A}'_S$ and $\mathbf{A}'_R$.*

Proof. Let us enumerate the elements of R, S as follows: $R = \{d_0 > d_1 > d_2 > \cdots\}$ and $S = \{c_0 > c_1 > c_2 > \cdots\}$. Suppose that f is an isomorphism from $\mathbf{A}_S$ to $\mathbf{A}_R$. Since f is an order-preserving bijection, f must be the identity when restricted to the set $\{a, a^3\}$, i.e., $f(a) = a$ and $f(a^3) = a^3$. It follows from Lemma 3.2 that $\langle -1, c_n\rangle \in A_S$, $\langle -1, d_n\rangle \in A_R$ and

$$f(-1, c_n) = f(s_n^{\mathbf{A}_S}(a)) = s_n^{\mathbf{A}_R}(f(a)) = s_n^{\mathbf{A}_R}(a) = \langle -1, d_n\rangle$$

for all $n \in \mathbb{N}$. Assume that k is the least natural number such that $c_k \neq d_k$. Without any loss of generality suppose that $c_k > d_k$. Lemma 3.3 has two consequences. First, $\langle -2, -c_k\rangle \in A_S, A_R$. Second, we have $f(-2, n) = f(t_n^{\mathbf{A}_S}(a)) = t_n^{\mathbf{A}_R}(f(a)) = t_n^{\mathbf{A}_R}(a) = \langle -2, n\rangle$ for all $n \in \mathbb{N}$. Thus we get

$$\begin{aligned} a^3 = f(a^3) = f(\langle -1, c_k\rangle \circ_{\langle -3,-1\rangle} \langle -2, -c_k\rangle) = \\ \langle -1, d_k\rangle \circ_{\langle -3,-1\rangle} \langle -2, -c_k\rangle = \langle -3, d_k - c_k\rangle \vee \langle -3, -1\rangle = \langle -3, -1\rangle = a^4 , \end{aligned}$$

which is a contradiction since $a^3 \neq a^4$.

The above argument also shows that 1-free reducts of $\mathbf{A}_S$ and $\mathbf{A}_R$ are not isomorphic either. Since the 1-free reduct of $\mathbf{A}_S$ (resp. $\mathbf{A}_R$) is a subalgebra of the 1-free reduct of $\mathbf{A}'_S$ (resp. $\mathbf{A}'_R$), it follows that $\mathbf{A}'_S$ and $\mathbf{A}'_R$ cannot be isomorphic. □

Using Lemma 2.1 and the lemmas above, we get the following theorem.

Theorem 3.6. *There are $2^{\aleph_0}$ minimal varieties in $\mathbf{\Lambda}(\mathsf{RCRL})$, $\mathbf{\Lambda}(\mathsf{RFL_{ei}})$ and $\mathbf{\Lambda}(\mathsf{RFL_{eo}})$. All the minimal varieties satisfy the identity $x^5 = x^4$, i.e., their members are 4-potent.*

Proof. The first claim follows immediately from Lemma 2.1. For the second observe that we can make $\mathbf{A}_S$ into an FL$_{ei}$-algebra $\mathbf{B}_S$ by interpreting the constant 0 as $a = \langle -1, 0\rangle$. Thus $\mathbf{B}_S$ is strictly simple since it is generated by 0. Moreover, the bottom element of $\mathbf{B}_S$ is term-definable by the term 0^4. The last claim can be proved as the first one since each $\mathbf{A}'_S$ can be made into an FL$_{eo}$-algebra by interpreting 0 as a^4. □

4. 3-potent and integral minimal varieties

In the previous section we have seen that there are continuum many minimal varieties in $\mathbf{\Lambda}(\mathsf{RCRL})$. However, all the constructed generators $\mathbf{A}'_S$ are 4-potent and non-integral. Thus there is a natural question how many minimal varieties we have if we restrict our attention on 3-potent (resp. integral) representable CRLs.

First, we will show that $\mathbf{\Lambda}(\mathsf{RCRL})$ contains only finitely many minimal varieties whose members are 3-potent. We start by defining a simple finite ICRC $\mathbf{U}_3 = \langle U_3, \wedge, \vee, \circ, \Rightarrow, 0\rangle$ which behaves almost like $\mathbf{L}_3 = \langle L_3, \wedge, \vee, \cdot, \to, 0\rangle$. Recall that $L_3 = \{-3, -2, -1, 0\}$, the multiplication on L_3 is given by $xy = -3 \vee (x + y)$ and the residuum by $x \to y = (y - x) \wedge 0$. Define $U_3 = L_3 \cup \{-2^*\}$. The order is given by $-3 < -2 < -2^* < -1 < 0$. The multiplication is defined by $x \circ y = xy$ for $x, y \in L_3$, $-2^* \circ x = -3 = x \circ -2^*$ for $x \neq 0$, and $-2^* \circ 0 = -2^* = 0 \circ -2^*$. The residuum $\Rightarrow$ is fully determined by $\circ$ and the order. The operations $\circ$ and $\Rightarrow$ are described in Figure 2.

$\circ$	-3	-2	-2^*	-1	0
-3	-3	-3	-3	-3	-3
-2	-3	-3	-3	-3	-2
-2^*	-3	-3	-3	-3	-2^*
-1	-3	-3	-3	-2	-1
0	-3	-2	-2^*	-1	0

$\Rightarrow$	-3	-2	-2^*	-1	0
-3	0	0	0	0	0
-2	-1	0	0	0	0
-2^*	-1	-1	0	0	0
-1	-2^*	-1	-1	0	0
0	-3	-2	-2^*	-1	0

Figure 2. The multiplication and residuum in $\mathbf{U}_3$.

Now we will characterize all subdirectly irreducible 3-potent ICRCs generated by its coatom. Such algebras have always coatom because of the following lemma which directly follows from [4, Lemma 3.59, 3.60].

Lemma 4.1. *Let $k \in \mathbb{N}$. A k-potent CRL is subdirectly irreducible iff it has a unique subcover of* 1.

Lemma 4.2. *Let $\mathbf{A}$ be a 3-potent ICRC. Suppose that $\mathbf{A}$ has a coatom a which generates $\mathbf{A}$. Then we have the following:*

1. *If $\mathbf{A}$ is 1-potent then $\mathbf{A} \cong \mathbf{L}_1$.*
2. *If $\mathbf{A}$ is 2-potent and not 1-potent then $\mathbf{A} \cong \mathbf{L}_2$.*
3. *If $\mathbf{A}$ is not 2-potent then $\mathbf{A} \cong \mathbf{L}_3$ or $\mathbf{A} \cong \mathbf{U}_3$.*

Proof. If $\mathbf{A}$ is 1-potent then $A = \{a < 1\}$ and $\mathbf{A}$ is clearly isomorphic to $\mathbf{L}_1$. If $\mathbf{A}$ is 2-potent and not 1-potent then $B = \{a^2 < a < 1\} \subseteq A$ is closed under multiplication. We will show that B is closed also under $\to$. Let $x, y \in B$. If $x \leq y$ then $x \to y = 1$. Further, $1 \to x = x$. The only remaining case is $a \to a^2$. Since $a \leq a \to a^2 < 1$ and a is a coatom, we have $a \to a^2 = a$. Thus $B = A$ and $\mathbf{A} \cong \mathbf{L}_2$. Finally, assume that $\mathbf{A}$ is not 2-potent. Let $B = \{a^3 < a^2 \leq a \to a^3 < a < 1\} \subseteq A$. The set B is closed under multiplication because $a(a \to a^3) = a^3$. We show that B is closed also under $\to$. Let $x, y \in B$. If $x \leq y$ then $x \to y = 1$. Further, $1 \to x = x$. Since $a \leq a \to a^2 < 1$ and a is a coatom, we have $a \to a^2 = a$. Similarly $a^2 \to a^3 = a$ and $(a \to a^3) \to a^3 = a$. Then $a \to (a \to a^3) = a^2 \to a^3 = a$. Finally, we have to check $(a \to a^3) \to a^2 \in B$. If $a \to a^3 = a^2$ then $(a \to a^3) \to a^2 = 1$ and $\mathbf{A} \cong \mathbf{L}_3$. If $a \to a^3 > a^2$ then $(a \to a^3) \to a^2 = a$ and $\mathbf{A} \cong \mathbf{U}_3$. □

The classification of 3-potent ICRCs generated by their coatom can be used in order to characterize 3-potent non-integral CRCs generated by their top element. The characterization uses the construction given in Lemma 3.1. The following proposition is a modification of [3, Proposition 4.6].

Proposition 4.3. *Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 1\rangle$ be a CRC with a top element $\top \neq 1$. If $\mathbf{A}$ is generated by $\top$, then $\mathbf{A} = \mathbf{B}'$ for an ICRC $\mathbf{B} = \langle B, \wedge, \vee, \cdot, \to, \top\rangle$, where $B = A \setminus \{1\}$. Furthermore, $\mathbf{B}$ is generated by its coatom $a = \top \to 1$. Finally, if $\mathbf{A}$ is k-potent then $\mathbf{B}$ is k-potent.*

Proof. Let $B = A \setminus \{1\}$. Since $\mathbf{A}$ is generated by $\top$, for each $b \in B$ there is a term t such that $t(\top) = b$. Moreover, we can assume that t does not contain $\wedge, \vee$ because $\langle A, \wedge, \vee \rangle$ is a chain. We will prove by induction on the complexity of t that $b\top = b$ for all $b \in B$. More formally we will prove the following statement: if $t(\top) \in B$ then $t(\top)\top = t(\top)$. To see the base case, assume that $t(x) = x$ (if $t(x) = 1$ then $t(\top) \notin B$). Since $\top\top = \top$, the statement holds. Now assume that $t(x) = r(x)s(x)$ and $t(\top) \in B$. Since $t(\top) \neq 1$, either $r(\top) \neq 1$ or $s(\top) \neq 1$. Without any loss of generality assume that $r(\top) \neq 1$. Then by induction hypothesis $r(\top)\top = r(\top)$. Consequently, $t(\top)\top = r(\top)s(\top)\top = r(\top)s(\top)$. Finally, assume that $t(x) = r(x) \to s(x)$ and $t(\top) \in B$. If $r(\top) = 1$ then $t(\top) = s(\top) \neq 1$. Thus by induction hypotheses we have $s(\top)\top = s(\top)$. Consequently, $t(\top)\top = s(\top)\top = s(\top) = t(\top)$. Now suppose that $r(\top) \neq 1$. Then by induction hypothesis $r(\top)\top = r(\top)$. Consequently,

$$\top \to t(\top) = \top \to (r(\top) \to s(\top)) = r(\top)\top \to s(\top) = r(\top) \to s(\top) = t(\top)\,.$$

Thus $t(\top)\top \leq t(\top)$. The other inequality holds always since $1 \leq \top$.

Next observe that $\top$ covers 1 in A since $\top \leq \top x = x$ for $x > 1$. Let $a = \top \to 1$. We claim that 1 covers a. Clearly $1 \not\leq \top \to 1$ otherwise $\top = 1\top \leq 1$. Moreover, if $x < 1$ then $x\top = x \leq 1$, so $x \leq \top \to 1$. Thus a is a coatom in B.

Now we will prove that B is a subalgebra of the 1-free reduct of $\mathbf{A}$. Let $x, y \in B$. Since $z\top = z$ for all $z \in B$, we have $xy \neq 1$. Thus B is closed under multiplication. If $x \leq y$ then $x = x\top \leq y$ which is equivalent to $\top \leq x \to y \in B$. If $x > y$ then $1 > x \to y \in B$. Thus B is also closed under $\to$. Moreover $\top$ acts like the monoidal identity on B. Consequently, $\mathbf{B} = \langle B, \wedge, \vee, \cdot, \to, \top \rangle$ forms an ICRC with a coatom a. Finally, it is easy to see that $\mathbf{A} = \mathbf{B}'$.

To see that $\mathbf{B}$ is generated by a, note that $a \to a = \top$ because $a\top = a$. Since $\mathbf{A}$ is generated by $\top$, we have a term $t(x)$ for each $b \in B$ such that $t(a) = b$. Again we may assume that $\wedge, \vee$ do not occur in t. The term t may contain the constant 1. Thus we have to show that it is possible to eliminate 1's from t. We will do it again by induction on the complexity of t, i.e., we are going to prove the following formal statement: if $t(a) \in B$ then there is a term t' such that $t'(a) = t(a)$ and 1 does not occur in t'. The base case is trivial because $t(a) \in B$ implies $t(x) = x$ which does not contain 1. Now assume that $t(x) = r(x)s(x)$ and $t(a) \in B$. Thus $r(a) \in B$ or $s(a) \in B$. Without any loss of generality suppose that $r(a) \in B$. Then by induction hypothesis we have a term r' not containing 1 such that $r'(a) = r(a)$. If $s(a) = 1$ then we can set $t' = r'$. If $s(a) \neq 1$ then again by induction hypothesis there is a term s' not containing 1 such that $s'(a) = s(a)$. Thus $t' = r's'$ is the desired term. Finally, assume that $t(x) = r(x) \to s(x)$ and $t(x) \in B$. If $r(a) = 1$ then $t(a) = s(a) \in B$. Consequently, we can set $t' = s'$ where s' is the 1-free term such that $s'(a) = s(a)$ whose existence follows from the induction hypothesis. If $r(a) \neq 1$ then by induction hypothesis we have a 1-free term r' such that $r'(a) = r(a)$. Clearly, if $s(a) \neq 1$ as well then we can set $t' = r' \to s'$ using the 1-free term s' from the induction hypothesis. If $s(a) = 1$ then there are two cases. First, $r(a) < 1$. Then $t(a) = r(a) \to 1 = \top$ because $r(a)\top = r(a) < 1$. Thus $t'(x) = x \to x$ is the desired term. Second, $r(a) = \top$. Then $t(a) = \top \to 1 = a$. Thus $t'(x) = x$ is the desired term.

The last claim of the proposition follows from the fact that $\langle B, \wedge, \vee, \cdot, \to \rangle$ is a subalgebra of $\langle A, \wedge, \vee, \cdot, \to \rangle$ and k-potency axiom does not involve the constant 1. □

Corollary 4.4. *Let $\mathbf{A}$ be a 3-potent CRC having a top element $\top \neq 1$. If $\mathbf{A}$ is generated by $\top$ then $\mathbf{A}$ is isomorphic to one of the algebras $\mathbf{L}_1', \mathbf{L}_2', \mathbf{L}_3', \mathbf{U}_3'$.*

Proof. Using Proposition 4.3, we get $\mathbf{A} = \mathbf{B}'$ for a 3-potent ICRC $\mathbf{B}$ generated by its coatom. Invoking Lemma 4.2, the algebra $\mathbf{B}$ has to be one of $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{U}_3$. □

Theorem 4.5. *There are exactly five 3-potent minimal varieties in the subvariety lattice $\mathbf{\Lambda}(\mathsf{RCRL})$, namely varieties generated by $\mathbf{L}_1, \mathbf{L}_1', \mathbf{L}_2', \mathbf{L}_3', \mathbf{U}_3'$.*

Proof. It is obvious that all of $\mathbf{L}_1, \mathbf{L}_1', \mathbf{L}_2', \mathbf{L}_3', \mathbf{U}_3'$ generate minimal variety in $\mathbf{\Lambda}(\mathsf{RCRL})$ since all of them are finite and strictly simple. Conversely, every minimal variety is generated by a subdirectly irreducible algebra. Thus assume that $\mathsf{V}(\mathbf{A})$ is a minimal variety generated by a subdirectly irreducible 3-potent representable CRL $\mathbf{A}$. To prove the claim, we will show that $\mathbf{A}$ contains one of $\mathbf{L}_1$, $\mathbf{L}_1'$, $\mathbf{L}_2'$, $\mathbf{L}_3'$, $\mathbf{U}_3'$ as a subalgebra.

Since $\mathbf{A}$ is subdirectly irreducible, it has to be a chain. We may also assume that $\mathbf{A}$ is generated by its unique subcover a of 1 in $\mathbf{A}$ (see Lemma 4.1). Then a^3 is an idempotent element. Using [4, Lemma 3.51], the interval $[a^3, a^3 \to 1]$ forms a subalgebra of $\mathbf{A}$. Since a belongs to this interval, we have $A = [a^3, a^3 \to 1]$, i.e., $\bot = a^3$ is a bottom element of $\mathbf{A}$ and $\top = a^3 \to 1$ is its top element. If $\mathbf{A}$ is integral then $\mathbf{A}$ is isomorphic to one of $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, $\mathbf{U}_3$ by Lemma 4.2. In any case, $\{\bot = a^3, \top = 1\}$ forms a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{L}_1$. Thus assume that $\mathbf{A}$ is not integral, i.e., $\top = a^3 \to 1 \neq 1$. Now consider a subalgebra $\mathbf{B}$ of $\mathbf{A}$ generated by $\top$. Using Corollary 4.4, $\mathbf{B}$ has to be isomorphic to one of $\mathbf{L}_1', \mathbf{L}_2', \mathbf{L}_3', \mathbf{U}_3'$. Thus the theorem follows. □

In the rest of this section we replace 3-potency with integrality and we show that there are only two minimal varieties in $\mathbf{\Lambda}(\mathsf{RCRL})$ whose members are integral. In fact, we obtain even a more general statement by showing that these two minimal varieties are the only minimal varieties in $\mathbf{\Lambda}(\mathsf{ICRL})$.

Theorem 4.6. *There are exactly two minimal varieties in $\mathbf{\Lambda}(\mathsf{ICRL})$, namely varieties generated by $\mathbf{L}_1$ and $\mathbf{Z}^-$.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible ICRL, i.e., $\mathbf{A}$ is nontrivial. Thus there is a strictly negative element $a < 1$ in $\mathbf{A}$. We will show that $\mathsf{V}(\mathbf{A})$ contains either $\mathbf{L}_1$ or $\mathbf{Z}^-$. Consider the non-increasing sequence $\langle a^k \rangle_{k\in\mathbb{N}}$. If $a^{k+1} = a^k$ for some $k \in \mathbb{N}$, then $k \geq 1$ and a^k is idempotent. Then it is easy to see that $\{a^k, 1\}$ forms a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{L}_1$. Thus in this case we have $\mathbf{L}_1 \in \mathsf{V}(\mathbf{A})$.

Now assume that $a^{k+1} < a^k$ holds for all $k \in \mathbb{N}$, i.e., $\langle a^k \rangle_{k\in\mathbb{N}}$ is a strictly decreasing sequence. We will show that $\mathbf{Z}^- \in \mathbf{ISHP}_{\mathrm{U}}(\mathbf{A}) \subseteq \mathsf{V}(\mathbf{A})$. Consider a non-principal ultrafilter U on $\mathbb{N}$ and the corresponding ultrapower $\mathbf{B} = \mathbf{A}^{\mathbb{N}}/U$. Set $\vec{1} = \langle 1, 1, \ldots \rangle/U$, $\vec{a} = \langle a, a, \ldots \rangle/U$, and $\vec{b} = \langle a^{k+1} \rangle_{k\in\mathbb{N}}/U$. Let θ be the congruence on $\mathbf{B}$ corresponding to the filter $F(\vec{a})$ generated by $\vec{a}$. In particular, we have $\vec{1}/\theta = \vec{a}/\theta$. Further, note that $\vec{b} \notin F(\vec{a})$. Indeed, if $\vec{b} \in F(\vec{a}) = \uparrow\{\vec{a}^m \mid m \in \mathbb{N}\}$, then for some $m \in \mathbb{N}$ and $J \in U$ we have $a^{k+1} \geq a^m$ for all $k \in J$. However, $\langle a^{k+1} \rangle_{k\in\mathbb{N}}$ is strictly decreasing. Thus J has to be finite. Consequently, U has to be principal (a contradiction).

We claim that $\mathbf{Z}^-$ is isomorphic to the subalgebra of $\mathbf{B}/\theta$ generated by $\vec{b}/\theta$ via the map $f(-n) = \vec{b}^n/\theta$. First, note that

$$\vec{b}^m \cdot \vec{b}^n = \langle a^{m(k+1)} \rangle_{k\in\mathbb{N}}/U \cdot \langle a^{n(k+1)} \rangle_{k\in\mathbb{N}}/U = \langle a^{(m+n)(k+1)} \rangle_{k\in\mathbb{N}}/U = \vec{b}^{m+n}\,.$$

Thus $\vec{b}^m/\theta \cdot \vec{b}^n/\theta = \vec{b}^{m+n}/\theta$. Second, we will check that $\vec{b}^m/\theta \to \vec{b}^n/\theta = \vec{b}^{n-m}/\theta$ for $m < n$. Note that for all $k \in \mathbb{N}$ we have

$$a^{(n-m)(k+1)} \leq a^{m(k+1)} \to a^{n(k+1)} < a^{(n-m)(k+1)-1} \leq a \to a^{(n-m)(k+1)}\,.$$

Thus $\vec{b}^{n-m} \leq \vec{b}^m \to \vec{b}^n \leq \vec{a} \to \vec{b}^{n-m}$. Consequently,

$$\vec{b}^{n-m}/\theta \leq \vec{b}^m/\theta \to \vec{b}^n/\theta \leq \vec{a}/\theta \to \vec{b}^{n-m}/\theta = \vec{1}/\theta \to \vec{b}^{n-m}/\theta = \vec{b}^{n-m}/\theta\,.$$

Finally, since $\vec{b} \notin F(\vec{a})$, we have $\vec{b}^n \notin F(\vec{a})$ for all $n \geq 1$. Consequently, $\vec{b}^m/\theta > \vec{b}^n/\theta$ for $m < n$ because $\vec{b}^m/\theta \to \vec{b}^n/\theta = \vec{b}^{n-m}/\theta \neq 1/\theta$. Thus f is an order-preserving bijection. □

5. Generating class for representable ICRLs

The construction of the ICRC $\mathbf{A}_S$ given in Section 3 using a conucleus can be modified in order to produce a large class of 1-generated algebras. In fact, we can prove that each finite ICRC embeds into one of them. Thus we obtain a new generating class for the variety of integral representable CRLs, namely the class of 1-generated finite members.

We start with a definition of the lexicographic product for ICRCs. Let $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, *^{\mathbf{A}}, \to^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle B, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, *^{\mathbf{B}}, \to^{\mathbf{B}}, 1^{\mathbf{B}} \rangle$ be ICRCs. Define an algebra $\mathbf{A} \overrightarrow{\times} \mathbf{B} = \langle A \times B, \wedge, \vee, \cdot, \to, \langle 1^{\mathbf{A}}, 1^{\mathbf{B}} \rangle \rangle$

such that the lattice operations $\wedge, \vee$ are determined by the lexicographic order $\leq_{\mathrm{lex}}$ on $A \times B$, the reduct $\langle A \times B, \cdot, \langle 1^{\mathbf{A}}, 1^{\mathbf{B}} \rangle \rangle$ is the direct product of the monoidal reducts of $\mathbf{A}$ and $\mathbf{B}$, and

$$\langle x, y \rangle \to \langle u, v \rangle = \begin{cases} \langle x \to^{\mathbf{A}} u, 1^{\mathbf{B}} \rangle & \text{if } x *^{\mathbf{A}} (x \to^{\mathbf{A}} u) <^{\mathbf{A}} u\,, \\ \langle x \to^{\mathbf{A}} u, y \to^{\mathbf{B}} v \rangle & \text{otherwise.} \end{cases}$$

We call $\mathbf{A}$ *cancellative* if its monoidal reduct is cancellative.

Proposition 5.1. *Let $\mathbf{A}, \mathbf{B}$ be nontrivial ICRCs. Then the lexicographic product $\mathbf{A} \overset{\rightarrow}{\times} \mathbf{B}$ is an ICRC iff $\mathbf{A}$ is cancellative.*

Proof. First, assume that $\mathbf{A}$ is cancellative. Clearly $\langle A \times B, \leq_{\mathrm{lex}} \rangle$ is a chain with the top element $\langle 1^{\mathbf{A}}, 1^{\mathbf{B}} \rangle$. We will show that

$$\langle x, y \rangle \cdot \langle w, z \rangle \leq_{\mathrm{lex}} \langle u, v \rangle \quad \text{iff} \quad \langle w, z \rangle \leq_{\mathrm{lex}} \langle x, y \rangle \to \langle u, v \rangle\,. \tag{2}$$

To see the left-to-right implication of (2), assume that $\langle x *^{\mathbf{A}} w, y *^{\mathbf{B}} z \rangle \leq_{\mathrm{lex}} \langle u, v \rangle$. Thus $x *^{\mathbf{A}} w \leq^{\mathbf{A}} u$, i.e., $w \leq^{\mathbf{A}} x \to^{\mathbf{A}} u$. If $w <^{\mathbf{A}} x \to^{\mathbf{A}} u$ or $\langle x, y \rangle \to \langle u, v \rangle = \langle x \to^{\mathbf{A}} u, 1^{\mathbf{B}} \rangle$, the right-hand side of (2) is obviously true. Thus suppose that $w = x \to^{\mathbf{A}} u$ and $x *^{\mathbf{A}} (x \to^{\mathbf{A}} u) = u$. Then $x *^{\mathbf{A}} w = u$. By our assumption we must have $y *^{\mathbf{B}} z \leq^{\mathbf{B}} v$. Consequently, $z \leq^{\mathbf{B}} y \to^{\mathbf{B}} v$ which means that the right-hand side of (2) is true also in this case. Conversely, assume that $\langle w, z \rangle \leq_{\mathrm{lex}} \langle x, y \rangle \to \langle u, v \rangle$. Then $w \leq^{\mathbf{A}} x \to^{\mathbf{A}} u$, i.e., $x *^{\mathbf{A}} w \leq^{\mathbf{A}} u$. If $x *^{\mathbf{A}} w <^{\mathbf{A}} u$ then the left-hand side of (2) is true. Thus suppose that $x *^{\mathbf{A}} w = u$. Then $u = x *^{\mathbf{A}} w \leq^{\mathbf{A}} x *^{\mathbf{A}} (x \to^{\mathbf{A}} u) \leq^{\mathbf{A}} u$. Thus $w = x \to^{\mathbf{A}} u$ because $\mathbf{A}$ is cancellative. Consequently, $z \leq^{\mathbf{B}} y \to^{\mathbf{B}} v$. Thus $y *^{\mathbf{B}} z \leq^{\mathbf{B}} v$, i.e., $\langle x *^{\mathbf{A}} w, y *^{\mathbf{B}} z \rangle \leq_{\mathrm{lex}} \langle u, v \rangle$.

Now, assume that $\mathbf{A}$ is not cancellative. Then there are $a, b, c \in A$ such that $a <^{\mathbf{A}} b$ and $a *^{\mathbf{A}} c = b *^{\mathbf{A}} c$. Let $u \in B$, $u < 1^{\mathbf{B}}$. Then $\langle a, 1^{\mathbf{B}} \rangle <_{\mathrm{lex}} \langle b, u \rangle$ but

$$\langle b, u \rangle \cdot \langle c, 1^{\mathbf{B}} \rangle = \langle b *^{\mathbf{A}} c, u \rangle = \langle a *^{\mathbf{A}} c, u \rangle <_{\mathrm{lex}} \langle a *^{\mathbf{A}} c, 1^{\mathbf{B}} \rangle = \langle a, 1^{\mathbf{B}} \rangle \cdot \langle c, 1^{\mathbf{B}} \rangle\,.$$

Thus the multiplication on $A \times B$ is not order-preserving, i.e., $\mathbf{A} \overset{\rightarrow}{\times} \mathbf{B}$ cannot be a CRL. □

Note that $\{\langle 1^{\mathbf{A}}, b \rangle \mid b \in B\}$ forms always a subalgebra of $\mathbf{A} \overset{\rightarrow}{\times} \mathbf{B}$ isomorphic to $\mathbf{B}$.

Now we will describe the above-mentioned construction of 1-generated ICRCs. Let $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, *^{\mathbf{A}}, \to^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ be a finitely generated ICRC. Let $G = \{g_1 < g_2 < \cdots < g_n\}$ be the generating set for $\mathbf{A}$. Extend $g_1 < g_2 < \cdots < g_n$ to a sequence $\langle g_i \rangle_{i \in \mathbb{N}}$ letting $g_0 = 1^{\mathbf{A}}$ and $g_{n+1} = g_{n+2} = \cdots = 1^{\mathbf{A}}$. We will define a conucleus σ on the lexicographic product $\mathbf{B} = \mathbf{Z}^{-} \overset{\rightarrow}{\times} \mathbf{A}$ by giving its image B_σ. Let $B_i = \{\langle -i, x \rangle \mid x \in A, x \leq g_i\}$ for $i \in \mathbb{N}$. Then $B_\sigma = \bigcup_{i \in \mathbb{N}} B_i$ (see Figure 3).

Lemma 5.2. *The set B_σ is the image of a conucleus.*

Proof. We will check that B_σ is a submonoid of $\mathbf{B}$. Let $\langle -u, x \rangle, \langle -v, y \rangle \in B_\sigma$. Then $\langle -u, x \rangle \in B_u$ and $\langle -v, y \rangle \in B_v$. Thus $x \leq g_u$ and $y \leq g_v$. Consequently, $x *^{\mathbf{A}} y \leq g_u *^{\mathbf{A}} g_v \leq g_u \wedge^{\mathbf{A}} g_v$. If $u = v = 0$ then $g_u = g_v = 1^{\mathbf{A}}$ and $\langle -u, x \rangle \cdot \langle -v, y \rangle = \langle 0, x *^{\mathbf{A}} y \rangle \in B_0$. Suppose that $u \geq 1$ or $v \geq 1$. Without any loss of generality we can assume $u \geq 1$. Then $g_u *^{\mathbf{A}} g_v \leq g_u \leq g_{u+v}$ because the sequence $\langle g_1, g_2, \ldots \rangle$ is non-decreasing. Thus $\langle -u, x \rangle \cdot \langle -v, y \rangle = \langle -u - v, x *^{\mathbf{A}} y \rangle \in B_{u+v}$.

Next, let us check that B_σ is the image of an interior operator. Let $\langle -u, x \rangle \notin B_\sigma$. Then $\langle -u, x \rangle \notin B_u$, i.e., $x > g_u$. Clearly $\max\{a \in B_\sigma \mid a \leq_{\mathrm{lex}} \langle -u, x \rangle\} = \langle -u, g_u \rangle$. □

Thus $\mathbf{B}_\sigma$ is an ICRC. Observe that B_0 forms a subalgebra of $\mathbf{B}_\sigma$ isomorphic to $\mathbf{A}$. Recall that n is the number of generators of $\mathbf{A}$ and consider the truncated algebra $[\mathbf{B}_\sigma]_{\langle -2(n+1), 1^{\mathbf{A}} \rangle}$. Let $\mathbf{C}$ be the subalgebra of $[\mathbf{B}_\sigma]_{\langle -2(n+1), 1^{\mathbf{A}} \rangle}$ generated by the element $g = \langle -1, g_1 \rangle$. In order to show that $\mathbf{A}$ can be embedded into $\mathbf{C}$, we have to prove that $B_0 \subseteq C$.

Lemma 5.3. *The ICRC $\mathbf{C}$ contains all $\langle 0, x \rangle$ for $x \in A$, i.e., $B_0 \subseteq C$.*

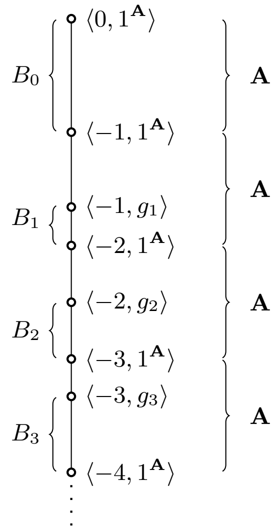

FIGURE 3. The structure of $\mathbf{B}$ and B_σ.

Proof. First, we have $g^{2(n+1)} = \langle -2(n+1), 1^{\mathbf{A}} \rangle \in C$. Further, for all $0 \leq u \leq 2(n+1)$ we have

$$g^{2(n+1)-u} \to_\sigma g^{2(n+1)} = \sigma\big(\langle -2(n+1)+u, g_1^{2(n+1)-u} \rangle \to \langle -2(n+1), 1^{\mathbf{A}} \rangle\big) = \sigma(-u, 1^{\mathbf{A}}) = \langle -u, g_u \rangle \,,$$

since $g_1^{2(n+1)-u} \to^{\mathbf{A}} 1^{\mathbf{A}} = 1^{\mathbf{A}}$ and $\langle -u, g_u \rangle = \max B_u$. Thus $\langle -u, g_u \rangle \in C$.

Let $0 \leq k \leq n$. Then $n < n+1+k < 2(n+1)$. Thus $g_{n+1+k} = g_{n+1} = 1^{\mathbf{A}}$ and $\langle -n-1-k, 1^{\mathbf{A}} \rangle, \langle -n-1, 1^{\mathbf{A}} \rangle \in C$. Consequently, we have

$$\langle -n-1-k, 1^{\mathbf{A}} \rangle \to_\sigma \langle -n-1, 1^{\mathbf{A}} \rangle \circ_{\langle -2(n+1), 1^{\mathbf{A}} \rangle} \langle -k, g_k \rangle = \sigma\big(\langle -n-1-k, 1^{\mathbf{A}} \rangle \to \langle -n-1-k, g_k \rangle\big) = \langle 0, g_k \rangle \,.$$

Thus $\langle 0, g_k \rangle \in C$ for each $g_k \in G$. Since A is generated by G, B_0 is generated by $\{\langle 0, g_k \rangle \mid g_k \in G\}$. Consequently, $B_0 \subseteq C$. □

Clearly $\mathbf{A}$ can be embedded into $\mathbf{C}$ via mapping $x \mapsto \langle 0, x \rangle$. Moreover, if $\mathbf{A}$ is finite then $\mathbf{C}$ is finite as well. Since the variety of representable ICRLs has the finite embeddability property [2] (i.e., it is generated by finite members as a quasi-variety), we get the following theorem.

Theorem 5.4. *The variety of representable ICRLs is generated as a quasi-variety by* 1*-generated finite members.*

Acknowledgement. The work of the author was partly supported by the grant P202/10/1826 of the Czech Science Foundation and partly by the Institutional Research Plan AV0Z10300504.

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