Distributive substructural logics as coalgebraic logics over posets

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Abstract

We show how to understand frame semantics of distributive substructural logics coalgebraically, thus opening a possibility to study them as coalgebraic logics. As an application of this approach we prove a general version of Goldblatt-Thomason theorem that characterizes definability of classes of frames for logics extending the distributive Full Lambek logic, as e.g. relevance logics, many-valued logics or intuitionistic logic. The paper is rather conceptual and does not claim to contain significant new results. We consider a category of frames as posets equipped with monotone relations, and show that they can be understood as coalgebras for an endofunctor of the category of posets. In fact, we adopt a more general definition of frames that allows to cover a wider class of distributive modal logics. Goldblatt-Thomason theorem for classes of resulting coalgebras for instance shows that frames for axiomatic extensions of distributive Full Lambek logic are modally definable classes of certain coalgebras, the respective modal algebras being precisely the corresponding subvarieties of distributive residuated lattices.

Keywords: Substructural logics, frame semantics, coalgebras, coalgebraic logic, Goldblatt-Thomason theorem.

1 Introduction

Modal logics are coalgebraic, the relational frames of classical modal logics can be seen as Set coalgebras for the powerset functor. Given an endofunctor T on Set, a conceptually clear setting of classical coalgebraic logic of T-coalgebras can be based on an adjunction called *logical connection*, linking categories Set and BA of sets and Boolean algebras [5,6] and capturing syntax and semantics of the propositional part of the language. Such connection can be "lifted" to a connection between categories of T-coalgebras and Boolean algebras with

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operators, which is in general "almost" an adjunction, capturing syntax and semantics of the modal part of the language. From certain properties of the lifted connection one automatically obtains soundness, completeness and expressivity of the modal language. One can also explore the connection to obtain the Goldblatt-Thomason definability theorem for classes of T-coalgebras for a reasonable class of Set functors [24].

In this paper, lead by a motivation to approach distributive substructural logics in a coalgebraic way, we use an (enriched) logical connection [25,29] between categories Pos of posets and DL of distributive lattices. We consider a general language of distributive lattices with operators, including the usual language of substructural logics as an instance. We start with requiring no additional axioms the operators should satisfy (not even the residuation laws), obtaining coalgebras for a certain endofunctor T on posets as semantics of this language. As an application of this setting we prove Goldblatt-Thomason definability theorem for classes of T-coalgebras. Classes of T-coalgebras definable by additional axioms of distributive substructural logics then precisely correspond to frames for these logics as surveyed and studied in [30]. Distributive modal logics have been treated coalgebraically before [7,27]. We see the main novelty of this paper in the fact that we use a weaker assumption than a duality of the category of algebras and certain topological spaces, thus resulting in non-topological coalgebras as semantics of distributive modal or substructural logics.

A leading example of a logic, semantics of which we want to cover, is the distributive full Lambek calculus **dFL** [15] in the following language

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \otimes \varphi \mid \varphi \to \varphi \mid \varphi \leftarrow \varphi \mid e \qquad (1)$$

where p ranges through a given poset of atomic propositions, $\wedge$ and $\vee$ are tied together by a distributive law, and the remaining four connectives $\otimes$, $\leftarrow$, $\to$, e satisfy additional equational axioms as, for example, the residuation laws. The algebraic semantics of **dFL** are residuated lattices.

We want to take the stance that $\wedge$ and $\vee$ are the only *propositional connectives* of the language, while the remaining four constructions $\otimes$, $\leftarrow$, $\to$, e are *modalities*. To prove that the study of relational models of the above language falls into the realm of *coalgebraic modal logic* it will be essential to start with a weaker setting, with *no* additional requirements on the modalities, apart from being monotone and preserving $\wedge$ or $\vee$, i.e. being operators over distributive lattices.

As it turns out, the natural environment for giving models of the above language is the one of *posets* and *monotone* relations. Namely, a relational model will consist of a poset $\mathscr{W}$ and four monotone relations $P_\otimes$, $P_\leftarrow$, $P_\to$ and P_e on $\mathscr{W}$. For example, $P_\otimes$ will be a monotone relation (i.e., a monotone map $P_\otimes : \mathscr{W}^{op} \times \mathscr{W}^{op} \times \mathscr{W} \longrightarrow \mathbb{2}$, where $\mathbb{2}$ is the two-element chain) that we will denote by $P_\otimes : \mathscr{W} \times \mathscr{W} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{W}$. Hence the "arity" of $P_\otimes$ mirrors the arity of the "modality" $\otimes$. Analogously, P_e will be a monotone relation of the form $P_e : \mathbb{1} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{W}$ where $\mathbb{1}$ denotes the one-element preorder. Hence P_e will appear as a "nullary" monotone relation, mirroring the fact that the "modality" e is nullary. We prove that the above quintuple $\mathbb{W} = (\mathscr{W}, P_\otimes, P_\leftarrow, P_\rightarrow, P_e)$ can be seen as a *coalgebra* for an endofunctor T of the category Pos of posets and monotone maps.

The reasoning does not change much if we incorporate slightly more general languages of the form

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \heartsuit(\varphi_0, \dots, \varphi_{n-1}) \mid (\varphi_0, \dots, \varphi_{l-1}) \multimap \psi \mid {\sim}\varphi \tag{2}$$

where p ranges through a poset At of atomic propositions, the connectives $\wedge$, $\vee$ are tied together by the distributive law, $\heartsuit$ is an n-ary *fusion-like* connective, $\multimap$ is an l-ary *implication-like* connective, and $\sim$ is a *negation-like* connective. These connectives are required to interact with $\wedge$ and $\vee$ in the sense that the following equalities are valid for each $0 \leq i \leq n$:

$$\begin{aligned}
\heartsuit(\dots, \varphi_i \vee \varphi_i' \dots) &= \heartsuit(\dots, \varphi_i, \dots) \vee \heartsuit(\dots, \varphi_i', \dots) \\
(\dots, \varphi_i \vee \varphi_i', \dots) \multimap \psi &= ((\dots, \varphi_i, \dots) \multimap \psi) \wedge ((\dots, \varphi_i', \dots) \multimap \psi) \\
(\varphi_0, \dots, \varphi_{l-1}) \multimap (\psi \wedge \psi') &= ((\varphi_0, \dots, \varphi_{l-1}) \multimap \psi) \wedge ((\varphi_0, \dots, \varphi_{l-1}) \multimap \psi') \\
{\sim}(\varphi_1 \vee \varphi_2) &= {\sim}\varphi_1 \wedge {\sim}\varphi_2
\end{aligned}$$

In slogans: $\heartsuit$ should preserve $\vee$ pointwise, $\multimap$ should pointwise transform $\vee$ in its premises to $\wedge$, and it should preserve $\wedge$ in its conclusion, $\sim$ should transform $\vee$ into $\wedge$. [2]

We will prove that:

(i) Relational models of the language (2) are precisely the coalgebras for an endofunctor $T : \mathsf{Pos} \longrightarrow \mathsf{Pos}$. Moreover, the construction of T copies the syntax of the "modalities" $\heartsuit$, $\multimap$, $\sim$ in (2).

(ii) The algebraic semantics of (2) will be given by a variety $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ of distributive lattices with operators $\heartsuit$, $\multimap$ and $\sim$.

(iii) It is essential to start with *no* requirements on the modalities in order to obtain a coalgebraic description. Any additional equational requirements on the modalities $\heartsuit$, $\multimap$ and $\sim$ will result in a *modally definable* class of T-coalgebras. We characterize modally definable classes in the spirit of *Goldblatt-Thomason Theorem* known from the classical modal logic.

As an illustration, we explain how various classes of frames for languages of the type (2) can be perceived as modally definable. In particular, we cover all the frames for the distributive substructural logics as studied in [30], namely:

- The class of frames modelling the *distributive full Lambek calculus* is modally definable by the equations for residuated distributive lattices. The modalities are $\otimes$, $\rightarrow$, $\leftarrow$ and e.
- The class of frames modelling the *intuitionistic logic* is modally definable by the equations for Heyting algebras. The modalities are $\otimes$ (coinciding with $\wedge$) and $\rightarrow$.

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[2] The language above, in its greatest generality, allows for finitely many connectives of each kind, all of various arities. In order not to make the notation too heavy, we will assume that there is just one connective of each kind in our signature. The results for the general case are straightforward generalisations of results for our simplification.

- The class of frames modelling *relevance logic* is modally definable. The modalities are $\otimes$, $\rightarrow$, $\leftarrow$, e and $\sim$.

Related work: Using relational models on posets for modelling semantics of various nonclassical logics goes back at least to the work of Routley and Meyer [31], and Dunn, see [10,11,12] or [30] for an overview. We see the novelty of our approach in the fact that we can systematically work with such frames as coalgebras, hence one has a canonical notion of a frame morphism as morphism of corresponding coalgebras. [3]

The original Goldblatt-Thomason theorem for modal logics [21] characterizes modally definable classes of Kripke *frames*. For positive modal logic it was proved in [9]. Our version of the theorem is an analog of coalgebraic Goldblatt-Thomason theorem for Set coalgebras [24, Theorem 3.15(2.)]. Possibilities to generalize the theorem to coalgebras over measurable spaces have been explored in [28]. Coalgebraic Goldblatt-Thomason theorem for classes of *models* can be found in [20] and [24, Theorem 3.15(1.)].

Our approach relates to, but significantly differs from extensive work relating algebraic and frame (or topological) semantics of modal and substructural logics, using dualities and discrete dualities for distributive lattices [17,18,19], distributive lattices with operators [32,33,23,27], or posets [13], most of it using canonical extensions: in contrast to this approach we do not use a dual equivalence of distributive lattices and certain topological spaces, a weaker kind of adjunction between DL and posets, called logical connection, is enough. The frames, and thus the coalgebras we consider are not topological as those obtained in [27], [7] or [1], they can however be seen as non-topological analogues of those.

Organisation of the paper: Section 2 is devoted to fixing the terminology and notation for monotone relations. In Section 3 we briefly recall how the *semantics of the propositional part of coalgebraic logic* is captured by an adjunction of a special kind, called *logical connection*. Relational *frames as coalgebras* are introduced in Section 4. *Complex algebras* and *canonical frames* are studied in Sections 5 and 6. Our main result: the *modal definability theorem* is proved in Section 7. We illustrate this result by examples of distributive *full Lambek calculus*, *relevance logic*, etc. We hint at possible generalizations of our approach in Section 8.

Remark on the notation we use: We work with posets and monotone relations as with categories enriched over the two-element chain $\mathbb{2}$, see Section 2. Therefore our formulas for manipulation monotone relations use the structure of the complete Boolean algebra $\mathbb{2}$ and are to be computed there. We think that the notation will become convenient in future generalizations to enriched categories, see Section 8.

Due to space limitations we have omitted most of the proofs. [§]

2 Preliminaries

Recall that a *poset* $\mathscr{W}$ is a set W equipped with a reflexive, transitive and antisymmetric relation $\leq$. Instead of writing $x \leq x'$ we will often write $\mathscr{W}(x, x') = 1$ (and writing $\mathscr{W}(x, x') = 0$, if $x \leq x'$ does not hold). This is in compliance with the fact that a poset $\mathscr{W}$ can be seen as a small category *enriched* in the two-element chain

[3] The usual notion of morphism for substructural frames is different — it requires equalities $a = f(x), b = f(y)$ in the back condition in 4.6. The same notion of a frame morphism as ours in the special case of frames for fuzzy logics is given in [8].

[§] For the purposes of the refereeing process, all the proofs are in the Appendix.

$\mathbb{2}$. Although we will not use any machinery of enriched category theory explicitly, we find the above notation convenient in the view of further generalizations, see Section 8 below.

An *opposite* $\mathscr{W}^{op}$ of the poset $\mathscr{W}$ has the same set of elements as $\mathscr{W}$, but we put $\mathscr{W}^{op}(x, x') = \mathscr{W}(x', x)$.

Recall further that a *monotone map* $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ consists of an assignment $x \mapsto fx$ such that, for any x and x', the inequality $\mathscr{W}_1(x, x') \leq \mathscr{W}_2(fx, fx')$ holds in $\mathbb{2}$. The poset of all monotone maps from $\mathscr{W}_1$ to $\mathscr{W}_2$, with the order defined pointwise, is denoted by $[\mathscr{W}_1, \mathscr{W}_2]$. A *product* $\mathscr{W}_1 \times \mathscr{W}_2$ of posets $\mathscr{W}_1$, $\mathscr{W}_2$ is an order on the pairs of elements, defined pointwise. We denote by $\mathscr{W}^n$ the product of n-many copies of $\mathscr{W}$ with itself, writing $\mathscr{W}^0 = \mathbb{1}$ — the one-element poset.

Given posets $\mathscr{W}_1$ and $\mathscr{W}_2$, a *monotone relation from* $\mathscr{W}_1$ *to* $\mathscr{W}_2$, denoted by

$$R : \mathscr{W}_1 \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\;\;\;\; \mathscr{W}_2$$

is a monotone map of the form $R : \mathscr{W}_1^{op} \times \mathscr{W}_2 \longrightarrow \mathbb{2}$. We write $R(x, x') = 1$ to denote that x is related to x'. In what follows we will omit the adjective 'monotone' and speak just of relations. A relation of the form

$$R : \mathscr{W}^n \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\;\;\;\; \mathscr{W}$$

is called an *n-ary relation* on $\mathscr{W}$, where $n \geq 0$. For $n = 0$ we obtain

$$R : \mathbb{1} \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\;\;\;\; \mathscr{W}$$

and it is easy to see that such a relation corresponds to an *upperset* of $\mathscr{W}$, i.e., the set $U = \{x \mid Rx = 1\}$ has the property: if $x \in U$ and $x \leq x'$, then $x' \in U$.

Relations compose in the usual way: the composite of the relations $R : \mathscr{W}_1 \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\;\;\;\; \mathscr{W}_2$ $\quad S : \mathscr{W}_2 \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\;\;\;\; \mathscr{W}_3$ is a relation $S \cdot R : \mathscr{W}_1 \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\;\;\;\; \mathscr{W}_3$ given by the formula

$$S \cdot R(x, z) = \bigvee_y S(y, z) \wedge R(x, y)$$

For every poset $\mathscr{W}$, the *identity relation* $id_{\mathscr{W}} : \mathscr{W} \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\;\;\;\; \mathscr{W}$ is defined by putting $id_{\mathscr{W}}(x, x') = 1$ iff $x \leq x'$. Hence it is consistent to write $\mathscr{W}$ instead of $id_{\mathscr{W}}$.

It is easy to see that the above composition is associative and that it has identity relations as units, hence we obtain a category (enriched in posets) of posets and relations. The above definitions are specializations of the theory of *profunctors* (also *distributors*, or, *modules*), known from enriched category theory. See, for example, [34] for more details.

3 The logical connection

The semantics of the propositional part of the language, i.e., of the language

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \tag{3}$$

where p ranges through a poset At of atomic propositions and $\wedge$ and $\vee$ are tied by the distributive law, will be given by a *logical connection* of the category Pos of *posets* and *monotone maps* and the category DL of *distributive lattices* and *lattice morphisms*. The logical connection

$$\mathit{Stone} \dashv \mathit{Pred} : \mathsf{Pos}^{op} \longrightarrow \mathsf{DL} \tag{4}$$

is given by the two-element chain $\mathbb{2}$ as a *schizophrenic object*. Recall how the above connection works (we refer to [29] or [25] for more details on logical connections):

(i) *Pred* sends a poset $\mathscr{W}$ to the distributive lattice $([\mathscr{W}, 2], \cap, \cup)$ of uppersets on $\mathscr{W}$. A monotone map f is sent to $[f, 2] : U \mapsto U \cdot f$.

For a poset $\mathscr{W}$, the distributive lattice $Pred(\mathscr{W})$ is to be considered as the "distributive lattice of truth-distributions on $\mathscr{W}$".

(ii) For a distributive lattice $\mathscr{A}$, $Stone(\mathscr{A})$ is the poset $\mathsf{DL}(\mathscr{A}, 2)$ of *prime filters* on $\mathscr{A}$. The mapping $Stone(h)$ is given by composition: a prime filter F is sent to the prime filter $F \cdot h$.

The poset $Stone(\mathscr{A})$ is the "Stone space" of the distributive lattice $\mathscr{A}$.

(iii) The unit $\eta_{\mathscr{A}} : \mathscr{A} \longrightarrow [\mathsf{DL}(\mathscr{A}, 2), 2]$ is the lattice homomorphism sending x in $\mathscr{A}$ to the upperset of all prime filters on $\mathscr{A}$ that contain x.

(iv) The counit $\varepsilon_{\mathscr{W}} : \mathscr{W} \longrightarrow \mathsf{DL}([\mathscr{W}, 2], 2)$ is the monotone map sending x in $\mathscr{W}$ to the prime filter of those uppersets on $\mathscr{W}$ that contain x.

The semantics of the propositional language (3) is given by the logical connection (4), together with another adjunction

$$F \dashv U : \mathsf{DL} \longrightarrow \mathsf{Pos} \tag{5}$$

where U denotes the obvious forgetful functor and F sends a poset $\mathscr{X}$ to the *free distributive lattice* on $\mathscr{X}$. More in detail, the semantics is given as follows:

(i) Fix a poset At of atomic propositions. The distributive lattice $F(\mathsf{At})$ is then the *Lindenbaum-Tarski* algebra of formulas.

(ii) Observe that $U(Pred(\mathscr{W})) = [\mathscr{W}, 2]$, for every poset $\mathscr{W}$. Hence, due to the adjunction $F \dashv U$, monotone maps of the form $\mathsf{val} : \mathsf{At} \longrightarrow [\mathscr{W}, 2]$ are in bijective correspondence with lattice morphisms $\|-\|_{\mathsf{val}} : F(\mathsf{At}) \longrightarrow Pred(\mathscr{W})$.

Of course, as the notation suggests, the monotone map val is the *valuation* of atomic propositions, assigning to every p the upperset $\mathsf{val}(p)$ of those x's in $\mathscr{W}$, where p is valid. The lattice homomorphism $\|-\|_{\mathsf{val}}$ is then the free extension of the valuation val. It can be described inductively as follows:

$$\|p\|_{\mathsf{val}} = \mathsf{val}(p), \ \|\varphi_1 \wedge \varphi_2\|_{\mathsf{val}} = \|\varphi_1\|_{\mathsf{val}} \cap \|\varphi_2\|_{\mathsf{val}}, \ \|\varphi_1 \vee \varphi_2\|_{\mathsf{val}} = \|\varphi_1\|_{\mathsf{val}} \cup \|\varphi_2\|_{\mathsf{val}}$$

We will later add more connectives (fusion-like, implication-like and negation-like) but we are going to consider them as *modal operators* on distributive lattices. In fact, as we will see, such extension of the language will yield extensions of the above two functors *Pred* and *Stone*.

4 Relational frames as coalgebras

We define structures that we call (relational) frames for the language of the type (2). Frames will consist of a poset of states and various relations reflecting the syntax of "modalities" of the language, compare to frames in [30]. We prove that frames are exactly the coalgebras for a certain endofunctor of the category of posets.

Notation 4.1 We will introduce the following "vector" conventions: for a relation $P : \mathscr{W}^n \longrightarrow\!\!\!\!\!\!\!\!\mapsto \mathscr{W}$ we will write $P(\vec{x}; x)$ instead of $P(x_0, \ldots, x_{n-1}; x)$. For $P : \mathscr{W} \longrightarrow\!\!\!\!\!\!\!\!\mapsto (\mathscr{W}^{op})^l \times \mathscr{W}$ we will write $P(x; \vec{y}, z)$ instead of $P(x; y_0, \ldots, y_{l-1}, z)$. Analogously we will write $\mathscr{W}_2(\vec{a}, f\vec{x})$ instead of $\mathscr{W}_2(a_0, fx_0) \wedge \cdots \wedge \mathscr{W}_2(a_{n-1}, fx_{n-1})$, etc.

Definition 4.2 A *relational frame* for the language (2) is a quadruple $\mathbb{W} = (\mathscr{W}, P_{\heartsuit}, P_{\multimap}, P_{\sim})$, consisting of a poset $\mathscr{W}$, and relations

$$P_{\heartsuit} : \mathscr{W}^n \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}, \qquad P_{\multimap} : \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto (\mathscr{W}^l)^{op} \times \mathscr{W}, \qquad P_{\sim} : \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}^{op}$$

A *morphism* from $\mathbb{W}_1 = (\mathscr{W}_1, P^1_{\heartsuit}, P^1_{\multimap}, P^1_{\sim})$ to $\mathbb{W}_2 = (\mathscr{W}_2, P^2_{\heartsuit}, P^2_{\multimap}, P^2_{\sim})$ is a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ such that the following three equations hold:

$$P^2_{\heartsuit}(\vec{a}; fy) = \bigvee_{\vec{x}} \mathscr{W}_2(\vec{a}, f\vec{x}) \wedge P^1_{\heartsuit}(\vec{x}; y) \tag{6}$$

$$P^2_{\multimap}(fx; \vec{b}, c) = \bigvee_{\vec{y}, z} \mathscr{W}_2(\vec{b}, f\vec{y}) \wedge \mathscr{W}_2(fz, c) \wedge P^1_{\multimap}(x; \vec{y}, z) \tag{7}$$

$$P^2_{\sim}(fx; b) = \bigvee_{y} \mathscr{W}_2(b, fy) \wedge P^1_{\sim}(x; y) \tag{8}$$

We write $f : \mathbb{W}_1 \longrightarrow \mathbb{W}_2$ to indicate that f is a morphism of relational frames.

Remark 4.3 We have not defined semantics in a relational frame yet, but the following intuitions about the "meaning" of the individual relations $P_{\heartsuit}$, $P_{\multimap}$ and $P_{\sim}$ on $\mathscr{W}$ might be useful (see Notation 4.1).

(i) $P_{\heartsuit}(\vec{x}; y) = 1$ holds, if $\vec{x} \Vdash \vec{\varphi}$ implies $y \Vdash \heartsuit\vec{\varphi}$.

(ii) $P_{\multimap}(x; \vec{y}, z) = 1$ holds, if $x \Vdash \vec{\varphi} \multimap \psi$ and $\vec{y} \Vdash \vec{\varphi}$ imply $z \Vdash \psi$.

(iii) $P_{\sim}(x; y) = 1$ holds, if $y \Vdash \varphi$ implies $x \nVdash {\sim}\varphi$.

See Remark 5.5 below for precising the above intuitions.

Example 4.4 A relational frame $\mathbb{W}$ for the language (1) consists of a poset $\mathscr{W}$, together with fusion-like relations $P_{\otimes} : \mathscr{W} \times \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}$, $P_e : \mathbb{1} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}$, and implication-like relations $P_{\to} : \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}^{op} \times \mathscr{W}$ and $P_{\leftarrow} : \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}^{op} \times \mathscr{W}$.

Let us stress that the relations $P_{\otimes}$, P_e, $P_{\to}$ and $P_{\leftarrow}$ are (as of yet) arbitrary. When one needs special properties as, for example, the frame to be the model of a distributive full Lambek calculus (for such frames see [30]), one needs to invoke modal definability theorem. This is shown in Example 7.7 below.

Example 4.5 Relational frames for the language $\wedge$, $\vee$, $\otimes$, $\to$, e and $\sim$ of *relevance logic*, see [12], are posets $\mathscr{W}$ equipped with relations $P_{\otimes} : \mathscr{W} \times \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}$, $P_e : \mathbb{1} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}$, $P_{\to} : \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}^{op} \times \mathscr{W}$ and $P_{\sim} : \mathscr{W} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{W}^{op}$. The above relations are as of yet arbitrary. Frames for various classes of relevance logic are modally definable, see Remark 7.8 below.

Remark 4.6 It is very easy to prove that the above equations (6)–(8) can be "split" into six inequalities, giving us the *back & forth* description of morphisms for fusion-like, implication-like and negation-like connectives. More precisely:

(i) The equation (6) is equivalent to the conjunction of the following two inequalities

$$P^1_{\heartsuit}(\vec{x}; y) \leq P^2_{\heartsuit}(f\vec{x}; fy) \tag{9}$$

$$P^2_{\heartsuit}(\vec{a}; fy) \leq \bigvee_{\vec{x}} \mathscr{W}_2(\vec{a}, f\vec{x}) \wedge P^1_{\heartsuit}(\vec{x}; y) \tag{10}$$

(ii) The equation (7) is equivalent to the conjunction of the following two inequalities

$$P^1_{\multimap}(x; \vec{y}, z) \leq P^2_{\multimap}(fx; f\vec{y}; fz) \tag{11}$$

$$P^2_{\multimap}(fx; \vec{b}, c) \leq \bigvee_{\vec{y}, z} \mathscr{W}_2(\vec{b}, f\vec{y}) \wedge \mathscr{W}_2(fz, c) \wedge P^1_{\multimap}(x; \vec{y}, z) \tag{12}$$

(iii) The equation (8) is equivalent to the conjunction of the following two inequalities

$$P^1_\sim(x;y) \leq P^2_\sim(fx;fy) \tag{13}$$

$$P^2_\sim(fx;b) \leq \bigvee_y \mathscr{W}_2(b,fy) \wedge P^1_\sim(x;y) \tag{14}$$

We define now three functors

$$T_\heartsuit : \mathsf{Pos} \longrightarrow \mathsf{Pos}, \quad T_\multimap : \mathsf{Pos} \longrightarrow \mathsf{Pos}, \quad T_\sim : \mathsf{Pos} \longrightarrow \mathsf{Pos}$$

and prove that their product $T = T_\heartsuit \times T_\multimap \times T_\sim$ gives rise to relational frames and their morphisms. Namely: frames are T-coalgebras and frame morphisms are T-coalgebra morphisms.

Definition 4.7

(i) The functor $T_\heartsuit$ sends $\mathscr{W}$ to the poset $[(\mathscr{W}^n)^{op}, 2]$ of lowersets on $\mathscr{W}^n$. For a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$, the map $T_\heartsuit(f)$ sends $\vec{L} : (\mathscr{W}_1^n)^{op} \longrightarrow 2$ to

$$\vec{b} \mapsto \bigvee_{\vec{x}} \mathscr{W}_2(\vec{b}, f\vec{x}) \wedge \vec{L}\vec{x}$$

(ii) The functor $T_\multimap$ sends $\mathscr{W}$ to the poset $[(\mathscr{W}^l)^{op} \times \mathscr{W}, 2]^{op}$. For a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$, the map $T_\multimap(f)$ sends $X : (\mathscr{W}_1^l)^{op} \times \mathscr{W}_1 \longrightarrow 2$ to

$$(\vec{b}, c) \mapsto \bigvee_{\vec{y},z} \mathscr{W}_2(\vec{b}, f\vec{y}) \wedge \mathscr{W}_2(fz, c) \wedge X(\vec{y}, z)$$

(iii) The functor $T_\sim$ sends $\mathscr{W}$ to the poset $[\mathscr{W}^{op}, 2]^{op}$. For a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$, the map $T_\sim(f)$ sends $X : \mathscr{W}_1^{op} \longrightarrow 2$ to

$$b \mapsto \bigvee_y \mathscr{W}_2(b, fy) \wedge Xy$$

Proposition 4.8 *Put $T = T_\heartsuit \times T_\multimap \times T_\sim$. The category of relational frames and their morphisms is isomorphic to the category Pos_T of T-coalgebras and their morphisms.*

Proof.

(i) To give a monotone map $\gamma : \mathscr{W} \longrightarrow T(\mathscr{W})$ is to give three monotone maps $\gamma_\heartsuit : \mathscr{W} \longrightarrow T_\heartsuit(\mathscr{W})$, $\gamma_\multimap : \mathscr{W} \longrightarrow T_\multimap(\mathscr{W})$ and $\gamma_\sim : \mathscr{W} \longrightarrow T_\sim(\mathscr{W})$. Each of the three maps, however, can be uncurried to produce monotone maps $P_\heartsuit : (\mathscr{W}^n)^{op} \times \mathscr{W} \longrightarrow 2$, $P_\multimap : \mathscr{W}^{op} \times (\mathscr{W}^l)^{op} \times \mathscr{W} \longrightarrow 2$ and $P_\sim : \mathscr{W}^{op} \times \mathscr{W}^{op} \longrightarrow 2$. To conclude: T-coalgebras are exactly the relational frames.

(ii) To give a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ such that the square

$$\begin{tikzcd} \mathscr{W}_1 \arrow[r, "\gamma_1"] \arrow[d, "f"'] & T(\mathscr{W}_1) \arrow[d, "T(f)"] \\ \mathscr{W}_2 \arrow[r, "\gamma_2"'] & T(\mathscr{W}_2) \end{tikzcd}$$

commutes, is, by Definition 4.7, to give a monotone map f such that equations (6)–(8) hold. To conclude: coalgebra homomorphisms are exactly the morphisms of relational frames.

□

5 Complex algebras

The complex algebra $\mathit{Pred}^\sharp(\mathbb{W})$ of the frame $\mathbb{W}$ will be a distributive lattice $\mathit{Pred}(\mathscr{W})$, equipped with extra operators $\heartsuit$, $\multimap$ and $\sim$.

We prove that taking a complex algebra defines a functor $\mathit{Pred}^\sharp$ from the (opposite of the) category of relational frames and their morphisms to the category $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ of distributive lattices equipped with extra operations. Moreover, this construction extends the predicate functor $\mathit{Pred} : \mathsf{Pos}^{op} \longrightarrow \mathsf{DL}$ in the sense that the square

$$\begin{tikzcd}
(\mathsf{Pos}_T)^{op} \arrow[r, "\mathit{Pred}^\sharp"] \arrow[d, "(V_T)^{op}"'] & \mathsf{DL}_{\heartsuit,\multimap,\sim} \arrow[d, "U_{\heartsuit,\multimap,\sim}"] \\
\mathsf{Pos}^{op} \arrow[r, "\mathit{Pred}"'] & \mathsf{DL}
\end{tikzcd} \tag{15}$$

commutes. Above, $V_T\colon \mathsf{Pos}_T \longrightarrow \mathsf{Pos}$ is the forgetful functor sending a coalgebra $(\mathscr{W},\gamma)$ to the poset $\mathscr{W}$.

Definition 5.1 The category $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ is defined as follows:

(i) Objects are distributive lattices $\mathscr{A} = (\mathscr{A}_o, \wedge, \vee)$ (where $\mathscr{A}_o$ denotes the underlying *poset*), together with *monotone* maps

$$[\![\heartsuit]\!]_{\mathscr{A}} : \mathscr{A}_o^n \longrightarrow \mathscr{A}_o, \quad [\![\multimap]\!]_{\mathscr{A}} : (\mathscr{A}_o^l)^{op} \times \mathscr{A}_o \longrightarrow \mathscr{A}_o, \quad [\![\sim]\!]_{\mathscr{A}} : \mathscr{A}_o^{op} \longrightarrow \mathscr{A}_o$$

called the *interpretations* of $\heartsuit$, $\multimap$ and $\sim$. We will usually omit the brackets $[\![-]\!]_{\mathscr{A}}$ and denote $(\mathscr{A}, \heartsuit, \multimap, \sim)$ by $\mathbb{A}$.

The operations are required to satisfy the following axioms, for each $0 \leq i \leq n$:

$$\begin{aligned}
\heartsuit(\dots, x_i \vee x_i', \dots) &= \heartsuit(\dots, x_i, \dots) \vee \heartsuit(\dots, x_i', \dots) \\
(\dots, x_i \vee x_i', \dots) \multimap y &= ((\dots, x_i, \dots) \multimap y) \wedge ((\dots, x_i', \dots) \multimap y) \\
\vec{x} \multimap (y_1 \wedge y_2) &= (\vec{x} \multimap y_1) \wedge (\vec{x} \multimap y_2) \\
\sim(x_1 \vee x_2) &= \sim x_1 \wedge \sim x_2
\end{aligned}$$

(ii) A *morphism* from $\mathbb{A}_1$ to $\mathbb{A}_2$ is a lattice morphism $h : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$ preserving the additional operations $\heartsuit$, $\multimap$ and $\sim$ on the nose.

The obvious underlying functor will be denoted by $U_{\heartsuit,\multimap,\sim} : \mathsf{DL}_{\heartsuit,\multimap,\sim} \longrightarrow \mathsf{DL}$.

Remark 5.2 It is clear that $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ is a finitary variety over Pos in the sense of categorical universal algebra. More precisely: the composite $U \cdot U_{\heartsuit,\multimap,\sim} : \mathsf{DL}_{\heartsuit,\multimap,\sim} \longrightarrow \mathsf{Pos}$ of the obvious forgetful functors is a monadic functor. In particular, the forgetful functor $U \cdot U_{\heartsuit,\multimap,\sim} : \mathsf{DL}_{\heartsuit,\multimap,\sim} \longrightarrow \mathsf{Pos}$ has a left adjoint, hence there also exists a left adjoint $F_{\heartsuit,\multimap,\sim} : \mathsf{DL} \longrightarrow \mathsf{DL}_{\heartsuit,\multimap,\sim}$ to $U_{\heartsuit,\multimap,\sim}$. Thus, given a poset At, we can form $F_{\heartsuit,\multimap,\sim}(F(\mathsf{At}))$. This is the *Lindenbaum-Tarski algebra* of formulas for the language (2) and we denote it by $\mathscr{L}(\mathsf{At})$.

Definition 5.3 The *complex algebra* $\mathit{Pred}^\sharp(\mathbb{W}) = (([\mathscr{W}, 2], \cap, \cup), \heartsuit, \multimap, \sim)$ is defined as follows:

(i) Given a vector $\vec{U}$ of uppersets $U_0, \dots, U_{n-1}$, the upperset $\heartsuit\vec{U}$ is defined by the formula

$$y \mapsto \bigvee_{\vec{x}} \vec{U}\vec{x} \wedge P_\heartsuit(\vec{x}; y)$$

(ii) Given a vector $\vec{U}$ of uppersets $U_0, \ldots, U_{l-1}$, and an upperset W, the upperset $\vec{U} \multimap W$ is defined by the formula

$$x \mapsto \bigwedge_{\vec{y},z} \vec{U}\vec{y} \wedge P_{\multimap}(x;\vec{y},z) \Rightarrow Wz$$

(iii) Given an upperset U, the upperset $\sim U$ is defined by the formula

$$x \mapsto \bigwedge_{y} P_{\sim}(x;y) \Rightarrow \neg Uy$$

where the $\neg$ sign is negation in $\mathbb{2}$.

The following result is easy to prove, when one uses the back & forth description of morphism of frames, see Remark 4.6.

Proposition 5.4 *The assignment* $\mathbb{W} \mapsto \mathit{Pred}^\sharp(\mathbb{W})$ *can be extended to a functor from* $(\mathsf{Pos}_T)^{op}$ *to* $\mathsf{DL}_{\heartsuit,\multimap,\sim}$. *Moreover, the square (15) commutes.*

Remark 5.5 The square (15) allows us to give *semantics* of the language. More precisely, we saw in Section 3 that the adjunction $F \dashv U : \mathsf{DL} \longrightarrow \mathsf{Pos}$, together with $\mathit{Stone} \dashv \mathit{Pred} : \mathsf{Pos}^{op} \longrightarrow \mathsf{DL}$, takes care of the semantics $\|-\|_{\mathsf{val}}$ of the propositional part of the logic.

The adjunction $F_{\heartsuit,\multimap,\sim} \dashv U_{\heartsuit,\multimap,\sim} : \mathsf{DL}_{\heartsuit,\multimap,\sim} \longrightarrow \mathsf{DL}$, together with square (15), allow us to define, for every frame $\mathbb{W}$, a *semantics morphism*

$$\|-\|_{\mathsf{val}} : \mathscr{L}(\mathsf{At}) \longrightarrow \mathit{Pred}^\sharp(\mathbb{W})$$

in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ as the transpose under the composite adjunction

$$\mathsf{DL}_{\heartsuit,\multimap,\sim} \underset{U_{\heartsuit,\multimap,\sim}}{\overset{F_{\heartsuit,\multimap,\sim}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{DL} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{Pos}$$

of a *valuation* $\mathsf{val} : \mathsf{At} \longrightarrow [\mathscr{W}, \mathbb{2}]$.

It is possible to give an inductive description of $\|-\|_{\mathsf{val}}$. Namely: the equations

$$\begin{array}{ll}
\|p\|_{\mathsf{val}} = \mathsf{val}(p), & \|\varphi_1 \wedge \varphi_2\|_{\mathsf{val}} = \|\varphi_1\|_{\mathsf{val}} \cap \|\varphi_2\|_{\mathsf{val}}, \\
\|\varphi_1 \vee \varphi_2\|_{\mathsf{val}} = \|\varphi_1\|_{\mathsf{val}} \cup \|\varphi_2\|_{\mathsf{val}} & \|\heartsuit\vec{\varphi}\|_{\mathsf{val}} = \heartsuit\|\vec{\varphi}\|_{\mathsf{val}}, \\
\|\vec{\varphi} \multimap \psi\|_{\mathsf{val}} = \|\vec{\varphi}\|_{\mathsf{val}} \multimap \|\psi\|_{\mathsf{val}}, & \|\sim\varphi\|_{\mathsf{val}} = \sim\|\varphi\|_{\mathsf{val}}
\end{array}$$

hold. Above, the symbols $\heartsuit$, $\multimap$ and $\sim$ on the right-hand sides are to be interpreted as the operations in the complex algebra $\mathit{Pred}^\sharp(\mathbb{W})$.

Let us call the pair $(\mathbb{W}, \mathsf{val})$, consisting of a frame and a valuation, a *model*. Then the morphism $\|-\|_{\mathsf{val}}$ defines the notion of *local truth* in the model $(\mathbb{W}, \mathsf{val})$ — we write $x \Vdash_{\mathbb{W},\mathsf{val}} \varphi$, if x belongs to the upperset $\|\varphi\|_{\mathsf{val}}$, or, equivalently, if $\|\varphi\|_{\mathsf{val}} x = 1$. If rewritten in terms of $\Vdash$, the above equations give the familiar inductive definition of validity. Namely (omitting the obvious cases of atomic propositions and $\wedge$ and $\vee$):

(i) $x \Vdash_{\mathbb{W},\mathsf{val}} \heartsuit\vec{\varphi}$ holds iff there exists $\vec{y}$ such that both $\vec{y} \Vdash \vec{\varphi}$ and $P_{\heartsuit}(\vec{y};x)$ hold.

(ii) $x \Vdash_{\mathbb{W},\mathsf{val}} \vec{\varphi} \multimap \psi$ holds iff for all $\vec{y}$ and z such that $\vec{y} \Vdash \vec{\varphi}$ and $P_{\multimap}(x;\vec{y},z)$ hold, $z \Vdash \psi$ holds.

(iii) $x \Vdash_{\mathbb{W},\mathsf{val}} \sim\varphi$ iff for all y such that $P_{\sim}(x;y)$ holds, $y \nVdash \varphi$ holds.

6 Canonical relational frames

The assignment of the *canonical frame* $Stone^\sharp(\mathbb{A})$ to an object $\mathbb{A}$ of $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ is, in a way, dual to the formation of complex algebras. We prove below that $\mathbb{A} \mapsto Stone^\sharp(\mathbb{A})$ is functorial and that the square

$$\begin{tikzcd} \mathsf{DL}_{\heartsuit,\multimap,\sim} \arrow[r, "Stone^\sharp"] \arrow[d, "U_{\heartsuit,\multimap,\sim}"'] & (\mathsf{Pos}_T)^{op} \arrow[d, "(V_T)^{op}"] \\ \mathsf{DL} \arrow[r, "Stone"'] & \mathsf{Pos}^{op} \end{tikzcd} \tag{16}$$

commutes.

Definition 6.1 Suppose $\mathbb{A} = (\mathscr{A}, \heartsuit, \multimap, \sim)$ is in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$. Define $Stone^\sharp(\mathbb{A})$ as follows:

(i) The underlying poset of $Stone^\sharp(\mathbb{A})$ is the poset $\mathsf{DL}(\mathscr{A}, 2)$ of prime filters on the distributive lattice $\mathscr{A}$.

(ii) The relation $P_\heartsuit$ is defined as follows:

$$P_\heartsuit(\vec{F}; G) = \bigwedge_{\vec{x}} \vec{F}\vec{x} \Rightarrow G(\heartsuit\vec{x})$$

(iii) The relation $P_\multimap$ is defined as follows:

$$P_\multimap(F; \vec{G}, H) = \bigwedge_{\vec{x},y} F(\vec{x} \multimap y) \wedge \vec{G}\vec{x} \Rightarrow Hy$$

(iv) The relation $P_\sim$ is defined as follows:

$$P_\sim(F; G) = \bigwedge_{x} Gx \Rightarrow \neg F(\sim x)$$

where the $\neg$ sign is the negation in 2.

The above definitions clearly make sense if we work with mere *uppersets* in lieu of prime filters. We will need the following three technical results that slightly generalize the results originating in the work on relevance logic, see Section 6 of [11].

Lemma 6.2 (Squeeze Lemma for $\heartsuit$) *Suppose $P_\heartsuit(\vec{F'}; G) = 1$ holds, where $\vec{F'}$ is a vector of filters and G a prime filter. Then there is a vector $\vec{F}$ of prime filters that extends $\vec{F'}$ and $P_\heartsuit(\vec{F}; G) = 1$.*

Lemma 6.3 (Squeeze Lemma for $\multimap$) *Suppose $P_\multimap(F; \vec{G'}, \overline{I'}) = 1$, where F is a prime filter, $\vec{G'}$ is a vector of filters and $\overline{I'}$ is a complement of an ideal I'. Then there exists a vector $\vec{G}$ of prime filters such that $\vec{G}$ extends $\vec{G'}$ and a prime ideal I that extends I' and $P_\multimap(F; \vec{G}, \overline{I}) = 1$, where $\overline{I}$ denotes the complement of I.*

Lemma 6.4 (Squeeze Lemma for $\sim$) *Suppose $P_\sim(F; G') = 1$, where F is a prime filter and G' is a filter. Then there exists a prime filter G extending G' such that $P_\sim(F; G) = 1$.*

The above three lemmata allow us to prove that the computation of a canonical frame is a functorial process.

Proposition 6.5 *The assignment $\mathbb{A} \mapsto Stone^\sharp(\mathbb{A})$ can be extended to a functor from $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ to $(\mathsf{Pos}_T)^{op}$. Moreover, the square (16) commutes.*

7 Modal definability

Our modal definability theorem (Theorem 7.6 below) will identify classes $\mathbf{C}$ of frames such that the image of $\mathbf{C}$ under $Pred^\sharp$ is an "HSP" class in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$, i.e., it is a variety (compare with the version of Goldblatt-Thomason theorem for modal logics [3, Theorem 5.54] and [24, Theorem 3.15/2.]). Since we work over posets, the notion of HSP-closedness has to take this fact under consideration. Namely, we will use the factorization system $(\mathcal{E}, \mathcal{M})$ on Pos where $\mathcal{E}$ consists of surjective monotone maps and $\mathcal{M}$ of monotone maps reflecting order, i.e., $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ is in $\mathcal{M}$ if $\mathscr{W}_1(x, x') = \mathscr{W}_2(fx, fx')$ holds for every x and x'. That $(\mathcal{E}, \mathcal{M})$ is indeed a factorization system on Pos is proved in [4]. We will use the HSP Theorem w.r.t. a factorization system, see [26]:

A class $\mathbf{A}$ of algebras in a variety $\mathscr{V}$ over Pos is definable by equations in $\mathscr{V}$ iff $\mathbf{A}$ satisfies the following conditions ($U : \mathscr{V} \longrightarrow \mathsf{Pos}$ denotes the underlying functor):

(H) If $e : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ is such that $U(e)$ is a split epi in Pos and $\mathbb{A}_1$ is in $\mathscr{A}$, then $\mathbb{A}_2$ is in $\mathbf{A}$.

(S) If $m : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ is such that $U(m)$ is in $\mathcal{M}$ and $\mathbb{A}_2$ is in $\mathscr{A}$, then $\mathbb{A}_1$ is in $\mathbf{A}$.

(P) If $\mathbb{A}_i$, $i \in I$, are in $\mathscr{A}$, then $\prod_{i\in I} \mathbb{A}_i$ is in $\mathbf{A}$.

In fact, since the algebraic semantics of our logic takes place in (distributive) lattices, we may as well replace equationally defined classes by inequationally defined. We prefer to introduce the inequational description, since it is often more useful in applications.

Definition 7.1 Suppose $\mathbb{W}$ is a relational frame. We say that α *entails* β, and denote this fact by $\alpha \models_{\mathbb{W}} \beta$, provided that $\|\alpha\|_{\mathsf{val}} \leq \|\beta\|_{\mathsf{val}}$ holds, for every valuation $\mathsf{val} : \mathsf{At} \longrightarrow [\mathscr{W}, 2]$.

Given a class Σ of pairs of formulas, we denote by $\mathsf{Mod}(\Sigma)$ the class of frames $\mathbb{W}$ such that $\alpha \models_{\mathbb{W}} \beta$, for all $(\alpha, \beta) \in \Sigma$.

The following result is trivial.

Lemma 7.2 *$\alpha \models_{\mathbb{W}} \beta$ holds iff $Pred^\sharp(\mathbb{W}) \models \alpha \wedge \beta = \alpha$, where the $\models$ sign on the right denotes validity in the sense of universal algebra.*

Although the notation might suggest it, *it is not the case* that the logical connection $Stone \dashv Pred$ lifts to an adjunction $Stone^\sharp \dashv Pred^\sharp$. The unit of $Stone \dashv Pred$ does lift, however, and we will need this technicality in the proof of Theorem 7.6.

Lemma 7.3 *The unit η of $Stone \dashv Pred$ is a morphism in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$, i.e., η lifts along the functor $U_{\heartsuit,\multimap,\sim} : \mathsf{DL}_{\heartsuit,\multimap,\sim} \longrightarrow \mathsf{DL}$ to a natural transformation $\eta^\sharp : Id_{\mathsf{DL}_{\heartsuit,\multimap,\sim}} \longrightarrow Pred^\sharp Stone^\sharp$.*

Another technical result that we need for Theorem 7.6 is the following one.

Lemma 7.4 *The functor Stone sends maps reflecting order to surjective monotone maps.*

Finally, before stating Theorem 7.6, we need to introduce the concept of a *prime extension* of a frame.

Definition 7.5 The frame $\mathbb{W}^* = Stone^\sharp Pred^\sharp(\mathbb{W})$ is called the *prime extension* of $\mathbb{W}$.

Theorem 7.6 *Suppose $\mathbf{C}$ is a class of relational frames that is closed under prime extensions (if $\mathbb{W}$ is in $\mathbf{C}$, then $\mathbb{W}^*$ is in $\mathbf{C}$). Then the following are equivalent:*

(i) *There is Σ such that $\mathbf{C} = \mathsf{Mod}(\Sigma)$.*

(ii) **C** *satisfies the following four conditions:*
  (a) **C** *is closed under "surjective coalgebraic quotients", i.e., if* $e : \mathbb{W}_1 \longrightarrow \mathbb{W}_2$ *is surjective and* $\mathbb{W}_1$ *is in* **C***, so is* $\mathbb{W}_2$.
  (b) **C** *is closed under "subcoalgebras", i.e., if* $m : \mathbb{W}_1 \longrightarrow \mathbb{W}_2$ *reflects order and* $\mathbb{W}_2$ *is in* **C***, so is* $\mathbb{W}_1$.
  (c) **C** *is closed under coproducts.*
  (d) **C** *reflects prime extensions: if* $\mathbb{W}^*$ *is in* **C***, so is* $\mathbb{W}$.

Proof. For proof see Appendix. □

Example 7.7 The *distributive and associative full Lambek calculus* (denoted by **dFL**) is given by the grammar (1), where $\otimes$ is required to be associative, to have e as a unit and to satisfy the residuation laws $\varphi \otimes \psi \leq \chi$ iff $\psi \leq \varphi \to \chi$ iff $\varphi \leq \chi \leftarrow \psi$. Thus, the subvariety of $\mathsf{DL}_{\otimes,e,\to,\leftarrow}$ that we want to deal with is exactly that of distributive residuated lattices.

The frames that are definable by the above (in)equations are precisely the quintuples $(\mathscr{W}, P_\otimes, P_\to, P_\leftarrow, P_e)$ that satisfy the following conditions (for details see [30, Chapter 11]):

(i) $P_\otimes$ is associative: $\bigvee_z (P_\otimes(x, y; z) \wedge P_\otimes(z, u; v)) = \bigvee_w (P_\otimes(y, u; w) \wedge P_\otimes(x, w; v))$
(ii) and has P_e as a (left and right) unit:
$W(x, y) = \bigvee_z (P_e(z) \to P_\otimes(z, x; y)) = \bigvee_z (P_e(z) \to P_\otimes(x, z; y))$
(iii) The equalities $P_\otimes(x_0, x_1; y) = P_\to(x_1; x_0, y) = P_\leftarrow(x_0; x_1, y)$ hold.

Class **C** of frames satisfying the above conditions is easily seen to verify the conditions in Theorem 7.6.

Example 7.8 Many interesting examples can be found among the extensions of (associative) **dFL** with, e.g., the structural rules, or when expanding the language by negation. Instances of the first possibility are: **dFL** extended with any combination of: exchange, weakening, contraction. See [30] for details on what follows.

(i) The exchange rule corresponds to the commutativity of $P_\otimes$, i.e. to the equality $P_\otimes(x, y; z) = P_\otimes(y, x; z)$.
(ii) Weakening corresponds to: $P_\otimes(x_0, x_1; y)$ implies $x_0 \leq y$ and $x_1 \leq y$.
(iii) Contraction corresponds to the equality $P_\otimes(x, x; x) = 1$.

This includes, for example, intuitionistic logic, obtained as an extension of **dFL** with all the three structural rules.[4] Instances of the second possibility include, e.g., the relevance logic **R**, see [12] or [30]. Here the language $\otimes$, $\to$, $\leftarrow$, e is extended by a negation connective $\sim$.

The frames $(\mathscr{W}, P_\otimes, P_\to, P_\leftarrow, P_e, P_\sim)$ for the relevance logic **R** are the frames for **dFL** satisfying, in addition, the contraction equality together with the following three axioms ([30]):

(a) $P_\sim(x; y) = P_\sim(y; x)$,
(b) $\bigvee_y P_\otimes(x_0, x_1; y) \wedge P_\sim(y; u) \leq \bigvee_s P_\otimes(u, x_0; s) \wedge P_\sim(x_1; s)$,
(c) $\bigvee_y (P_\sim(x; y) \wedge \bigwedge_z (P_\sim(y; z) \Rightarrow \mathscr{W}(z, x))) = 1$.

The class **C** of frames satisfying these axioms is easily seen to verify the conditions of Theorem 7.6. It is modally definable by corresponding axioms of **R**.

---

[4] A usual frame $(X, \leq)$ for intuitionistic logic can be perceived as a relational frame defining $P(x, y; z) = x \leq z \wedge y \leq z$. Then coalgebraic morphisms correspond precisely to bounded morphisms.

8 Conclusions and further work

We have shown that frames for various kinds of distributive substructural logic can be perceived naturally as modally definable classes of poset coalgebras. It seems natural to construct first frames for logics that have minimal necessary restrictions on the modalities — these frames are exactly the coalgebras for a certain endofunctor of the category of posets. Such an approach yields the notion of frame morphisms for free: the morphisms of frames are exactly the coalgebra morphisms. Any (in)equational requirement on the modalities results in singling out a subclass of frames that is modally definable in the sense of Goldblatt-Thomason Theorem. Hence any subvariety of modal algebras (= distributive lattices with operators) defines a Goldblatt-Thomason subclass of frames, and vice versa, which has been illustrated by well-known examples of frames for distributive full Lambek calculus, relevance logic, etc.

The limitation of our result lies certainly in the presence of the distributive law for the propositional part of the logic since it leaves out nondistributive substructural logics. We believe that this can be easily overcome by passing to general lattices and using a two-sorted representation of lattices in the sense of [22]. The underlying logical connection will be two-sorted, hence the "state space" will consist of two posets connected with a monotone relation. This is in compliance with various notions of generalized frames, as studied, e.g., in [16] and [14]. Furthermore, this approach will also allow to pass naturally from posets to categories enriched in a general commutative quantale. In the latter framework, we believe to be able to study, e.g., many-valued modal and substructural logics in a rather conceptual way.

A natural further direction would be to prove a more general Goldblatt-Thomason theorem for coalgebras over posets or categories enriched in a general commutative quantale, obtaining an analogue of [24, Theorem 3.15]. Another line of research explores the fact that the coalgebraic functor we obtained is easily seen to satisfy the Beck-Chevalley Condition in the sense of [2]. Hence it will be possible to develop the theory of cover modalities over coalgebras for distributive substructural logics.

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Appendix

The verification of the "back & forth" description of frame morphisms — Remark 4.6

(i) Suppose (6) holds. Then the computations

$$\begin{aligned} P^2_{\heartsuit}(f\vec{x}; fy) &= \bigvee_{\vec{x'}} \mathscr{W}_2(f\vec{x}, f\vec{x'}) \wedge P^1_{\heartsuit}(\vec{x'}; y) \\ &\geq \bigvee_{\vec{x'}} \mathscr{W}_1(\vec{x}, \vec{x'}) \wedge P^1_{\heartsuit}(\vec{x'}; y) \\ &= P^1_{\heartsuit}(\vec{x}; y) \end{aligned}$$

verify the "forth" condition (9) for $\heartsuit$ and the "back" condition (10) for $\heartsuit$ is trivial.

Conversely, suppose inequalities (9) and (10) hold. Then the inequalities

$$\begin{aligned} \bigvee_{\vec{x}} \mathscr{W}_2(\vec{a}, f\vec{x}) \wedge P^1_{\heartsuit}(\vec{x}; y) &\leq \bigvee_{\vec{x}} \mathscr{W}_2(\vec{a}, f\vec{x}) \wedge P^2_{\heartsuit}(f\vec{x}; fy) \\ &\leq P^2_{\heartsuit}(\vec{a}; fy) \end{aligned}$$

hold by (9) and monotonicity of $P^2_{\heartsuit}$. This proves (6).

(ii) Suppose (7) holds. We only need to verify the inequality (11):

$$\begin{aligned} P^2_{\multimap}(fx; f\vec{y}, fz) &= \bigvee_{\vec{y'}, z'} \mathscr{W}_2(f\vec{y}, f\vec{y'}) \wedge \mathscr{W}_2(fz', fz) \wedge P^1_{\multimap}(x; \vec{y'}, z') \\ &\geq \bigvee_{\vec{y'}, z'} \mathscr{W}_1(\vec{y}, \vec{y'}) \wedge \mathscr{W}_1(z', z) \wedge P^1_{\multimap}(x; \vec{y'}, z') \\ &= P^1_{\multimap}(x; \vec{y}, z) \end{aligned}$$

Conversely, suppose inequalities (11) and (12) hold. Then the inequalities

$$\begin{aligned} \bigvee_{\vec{y}, z} \mathscr{W}_2(\vec{b}, f\vec{y}) \wedge \mathscr{W}_2(fz, c) \wedge P^1_{\multimap}(x; \vec{y}, z) &\leq \bigvee_{\vec{y}, z} \mathscr{W}_2(\vec{b}, f\vec{y}) \wedge \mathscr{W}_2(fz, c) \wedge P^2_{\multimap}(fx; f\vec{y}, fz) \\ &\leq P^2_{\multimap}(fx; b, c) \end{aligned}$$

prove (7).

(iii) Suppose (8) holds. Then we have inequalities

$$\begin{aligned} P^2_{\sim}(fx; fy) &= \bigvee_{y'} \mathscr{W}_2(fy, fy') \wedge P^1_{\sim}(x; y') \\ &\geq \bigvee_{y'} \mathscr{W}_1(y, y') \wedge P^1_{\sim}(x; y') \\ &= P^1_{\sim}(x; y) \end{aligned}$$

and inequality (13) hold.

Conversely, suppose inequalities (13) and (14) hold. Then we have inequalities

$$\begin{aligned} \bigvee_{y} \mathscr{W}_2(b, fy) \wedge P^1_{\sim}(x; y) &\leq \bigvee_{y} \mathscr{W}_2(b, fy) \wedge P^2_{\sim}(fx; fy) \\ &\leq P^2_{\sim}(fx; b) \end{aligned}$$

proving (8).

Proof of Proposition 5.4

It is easy to verify that, given a frame $\mathbb{W}$, the algebra $\mathit{Pred}^\sharp(\mathbb{W})$ is an object of $\mathsf{DL}_{\heartsuit,\multimap,\sim}$.

For a frame morphism $f : \mathbb{W}_1 \longrightarrow \mathbb{W}_2$, put $\mathit{Pred}^\sharp(f)$ to be the mapping $[f, 2] : [\mathscr{W}_2, 2] \longrightarrow [\mathscr{W}_1, 2]$. We verify that the three operations are preserved on the nose:

(i) The commutativity of the square

$$\begin{tikzcd} {[\mathscr{W}_2, 2]^n} \arrow[r, "{[f,2]^n}"] \arrow[d, "\heartsuit"'] & {[\mathscr{W}_1, 2]^n} \arrow[d, "\heartsuit"] \\ {[\mathscr{W}_2, 2]} \arrow[r, "{[f,2]}"'] & {[\mathscr{W}_1, 2]} \end{tikzcd}$$

is the requirement that the equality

$$\bigvee_{\vec{a}} \vec{U}\vec{a} \wedge P^2_\heartsuit(\vec{a}; fy) = \bigvee_{\vec{x}} \vec{U} f\vec{x} \wedge P^1_\heartsuit(\vec{x}; y)$$

holds for every y. The inequality $\geq$ is obvious: put $\vec{a} = f\vec{x}$ and use that $P^1_\heartsuit(\vec{x}; y) \leq P^2_\heartsuit(f\vec{x}; fy)$ holds, see (9). The converse inequality follows from the inequality (10).

(ii) The commutativity of the square

$$\begin{tikzcd} {([\mathscr{W}_2, 2]^l)^{op} \times [\mathscr{W}_2, 2]} \arrow[r, "{([f,2]^l)^{op} \times [f,2]}"] \arrow[d, "\multimap"'] & {([\mathscr{W}_1, 2]^l)^{op} \times [\mathscr{W}_1, 2]} \arrow[d, "\multimap"] \\ {[\mathscr{W}_2, 2]} \arrow[r, "{[f,2]}"'] & {[\mathscr{W}_1, 2]} \end{tikzcd}$$

is the requirement that the equality

$$\bigwedge_{\vec{b},c} \vec{U}\vec{b} \wedge P^2_\multimap(fx; \vec{b}, c) \Rightarrow Wc = \bigwedge_{\vec{y},z} \vec{U} f\vec{y} \wedge P^1_\multimap(x; \vec{y}, z) \Rightarrow Wfz$$

holds for every x. The inequality $\leq$ follows from $P^1_\multimap(x; \vec{y}, z) \leq P^2_\multimap(fx; f\vec{y}, fz)$, see (11). For the converse inequality, use inequality (12).

(iii) The commutativity of the square

$$\begin{tikzcd} {[\mathscr{W}_2, 2]^{op}} \arrow[r, "{[f,2]^{op}}"] \arrow[d, "\sim"'] & {[\mathscr{W}_1, 2]^{op}} \arrow[d, "\sim"] \\ {[\mathscr{W}_2, 2]} \arrow[r, "{[f,2]}"'] & {[\mathscr{W}_1, 2]} \end{tikzcd}$$

is the requirement that the equality

$$\bigwedge_{b} P^2_\sim(fx; b) \Rightarrow \neg Ub = \bigwedge_{y} P^1_\sim(x; y) \Rightarrow \neg Ufy$$

holds for every x. The inequality $\leq$ follows from inequality (13). For the converse inequality, use inequality (14).

Proof of Lemma 6.2

Consider the following system

$$\mathbf{E} = \{\vec{P} \mid \vec{P} \text{ extends } \vec{F'} \text{ and } P_{\heartsuit}(\vec{P}; G) = 1 \}$$

of vectors of filters, ordered by inclusion. The set $\mathbf{E}$ is nonempty by assumption and every nonempty chain in $\mathbf{E}$ has clearly a supremum. By Zorn's Lemma, there exists a maximal element $\vec{F} = (F_0, \dots, F_{n-1})$ of $\mathbf{E}$. We prove that it is a vector of prime filters. We only prove that F_0 is a prime filter, the reasoning about the remaining cases is the same.

Suppose $a \notin F_0$ and $b \notin F_0$ and denote by F_a the filter generated by $F_0 \cup \{a\}$ and by F_b the filter generated by $F_0 \cup \{b\}$. We can write

$$F_a = \{y \mid \text{ there exists } x \in F_0 \text{ such that } a \wedge x \leq y \}$$

$$F_b = \{y \mid \text{ there exists } x \in F_0 \text{ such that } b \wedge x \leq y \}$$

By maximality of $\vec{F}$, neither $\vec{F}^a = (F_a, F_1 \dots, F_{n-1})$ nor $\vec{F}^b = (F_b, F_1, \dots, F_{n-1})$ is in $\mathbf{E}$. It only holds if both vectors violate the conditions $P_{\heartsuit}(\vec{F}^a; G) = 1$ and $P_{\heartsuit}(\vec{F}^b; G) = 1$. Thus we have witnesses $\vec{y}^a = (y^a, y_1^a, \dots, y_{n-1}^a)$ in $\vec{F}^a$, and $\vec{y}^b = (y^b, y_1^b, \dots, y_{n-1}^b)$ in $\vec{F}^b$ such that $\heartsuit\vec{y}^a \notin G$ and $\heartsuit\vec{y}^b \notin G$. In particular, there are $x^a \in F_0$, $x^b \in F_0$ such that $a \wedge x^a \leq y^a$ and $b \wedge x^b \leq y^b$. Put $x = x^a \wedge x^b$ and $y_1 = y_1^a \wedge y_1^b$, $\dots$, $y_{n-1} = y_{n-1}^a \wedge y_{n-1}^b$. Use that F_a, F_b and G are filters to obtain $a \wedge x \leq y^a$ and $\heartsuit(y^a, y_1, \dots, y_{n-1}) \notin G$, and $b \wedge x \leq y^b$ and $\heartsuit(y^b, y_1, \dots, y_{n-1}) \notin G$. Thus $\heartsuit(a \wedge x, y_1, \dots, y_{n-1}) \notin G$ and $\heartsuit(b \wedge x, y_1, \dots, y_{n-1}) \notin G$. Since G is assumed to be prime, we have proved

$$\heartsuit(a \wedge x, y_1, \dots, y_{n-1}) \vee \heartsuit(b \wedge x, y_1, \dots, y_{n-1}) \notin G$$

Since $\heartsuit$ preserves joins by Definition 5.1, and the lattice is distributive, we obtain

$$\begin{aligned}\heartsuit(a \wedge x, y_1, \dots, y_{n-1}) \vee \heartsuit(b \wedge x, y_1, \dots, y_{n-1}) &= \heartsuit((a \wedge x) \vee (b \wedge x), y_1, \dots, y_{n-1}) \\ &= \heartsuit((a \vee b) \wedge x, y_1, \dots, y_{n-1}) \notin G\end{aligned}$$

If we assume that $a \vee b \in F_0$, then $(a \vee b) \wedge x \in F_0$, since $x \in F_0$ and F_0 is a filter. But then $((a \vee b) \wedge x, y_1, \dots, y_{n-1}) \in \vec{F}$, yielding $\heartsuit((a \vee b) \wedge x, y_1, \dots, y_{n-1}) \in G$, a contradiction.

Proof of Lemma 6.3

Consider the set

$$\mathbf{E} = \{(\vec{Q}, J) \mid \vec{Q} \text{ extends } \vec{G'},\, J \text{ extends } I', \text{ and } P_{\multimap}(F; \vec{Q}, \overline{I}) = 1\}$$

of pairs (vector of filters, ideal), ordered by inclusion. The set is nonempty by assumption and it clearly has suprema of nonempty chains. By Zorn's Lemma, there exists a maximal element $(\vec{G}, I)$ in $\mathbf{E}$.

We prove that $\vec{G}$ is a vector of prime filters and that I is a prime ideal.

(i) To prove that G_i in $\vec{G} = (G_0, \dots, G_{l-1})$ is a prime filter is analogous to previous lemma.

We prove that G_0 in $\vec{G} = (G_0, \dots, G_{l-1})$ is a prime filter, the reasoning for the remaining cases is the same.

Suppose $a \notin G_0$ and $b \notin G_0$. Denote by G_a the filter generated by $G_0 \cup \{a\}$ and by G_b the filter generated by $G_0 \cup \{b\}$. Moreover, we can write

$G_a = \{x \mid \text{ there exists } z \in G_0 \text{ such that } a \wedge z \leq x\ \}$

$G_b = \{x \mid \text{ there exists } z \in G_0 \text{ such that } b \wedge z \leq x\ \}$

By maximality of $(\vec{G}, I)$, neither the pair $(\vec{G}^a, I) = ((G_a, G_1 \dots, G_{n-1}), I)$ nor the pair $(\vec{G}^b, I) = ((G_b, G_1, \dots, G_{n-1}), I)$ is in $\mathbf{E}$. It only holds if both pairs violate the conditions $P_{\multimap}(F; \vec{G}^a, \overline{I}) = 1$ and $P_{\multimap}(F; \vec{G}^b, \overline{I}) = 1$.

Denote by $((x^a, x_1^a, \dots, x_{l-1}^a), y^a)$ and $((x^b, x_1^b, \dots, x_{l-1}^b), y^b)$ the witnesses of the above failures, i.e., $(x^a, x_1^a, \dots, x_{l-1}^a) \multimap y^a \in F$, $(x^a, x_1^a, \dots, x_{l-1}^a) \in \vec{G}^a$, $y^a \in I$, and $(x^b, x_1^b, \dots, x_{l-1}^b) \multimap y^b \in F$, $(x^b, x_1^b, \dots, x_{l-1}^b) \in \vec{G}^b$, $y^b \in I$ hold, where $a \wedge z^a \leq x^a$ and $b \wedge z^b \leq x^b$ for some z^a, z^b in G_0.

Since I is an ideal, $y = y^a \vee y^b \in I$. By Definition 5.1, $\multimap$ is monotone in its last argument. Hence both $(x^a, x_1^a, \dots, x_{l-1}^a) \multimap y \in F$ and $(x^b, x_1^b, \dots, x_{l-1}^b) \multimap y \in F$ hold.

Put $z = z^a \wedge z^b$, $x_1 = x_1^a \wedge x_1^b$, $\dots$, $x_{l-1} = x_{l-1}^a \wedge x_{l-1}^b$. Then we have $(z, x_1, \dots, x_{l-1}) \in \vec{G}$ and since $\multimap$ is antitone in its first l arguments (see Definition 5.1), both $(a \wedge z, x_1, \dots, x_{l-1}) \multimap y \in F$ and $(b \wedge z, x_1, \dots, x_{l-1}) \multimap y \in F$ hold.

Thus, using Definition 5.1 and distributivity of the lattice, we obtain

$$((a \wedge z, x_1, \dots, x_{l-1}) \multimap y) \wedge ((b \wedge z, x_1, \dots, x_{l-1}) \multimap y) =$$
$$((a \wedge z) \vee (b \wedge z), x_1, \dots, x_{l-1}) \multimap y) =$$
$$((a \vee b) \wedge z, x_1, \dots, x_{l-1}) \multimap y) \in F$$

Suppose $a \vee b \in G_0$. Then $(a \vee b) \wedge z \in G_0$ and hence $((a \vee b) \wedge z, x_1, \dots, x_{l-1}) \in \vec{G}$. Thus $y \in \overline{I}$. But $y \in I$, a contradiction.

(ii) We prove that I is a prime ideal. Consider $a \notin I$, $b \notin I$. Denote by I_a the ideal generated by $I \cup \{a\}$ and by I_b the ideal generated by $I \cup \{b\}$. We have the formulas

$I_a = \{y \mid \text{ there exists } x \in I \text{ such that } y \leq a \vee x\ \}$

$I_b = \{y \mid \text{ there exists } x \in I \text{ such that } y \leq b \vee x\ \}$

By maximality of I, neither $P_{\multimap}(F; \vec{G}, \overline{I_a}) = 1$ nor $P_{\multimap}(F; \vec{G}, \overline{I_b}) = 1$ holds.

Denote by $(\vec{x}^a, y^a)$ and $(\vec{x}^b, y^b)$ the witnesses of the above failures, i.e., $\vec{x}^a \multimap y^a \in F$, $\vec{x}^a \in \vec{G}$, $y^a \in I_a$, and $\vec{x}^b \multimap y^b \in F$, $\vec{x}^b \in \vec{G}$, $y^b \in I_b$ hold, where $y^a \leq a \vee z^a$ and $y^b \leq b \vee z^b$ for some z^a, z^b in I. Put $z = z^a \vee z^b \in I$, $\vec{x} = \vec{x}^a \wedge \vec{x}^b$.

Then $\vec{x} \in \vec{G}$, $\vec{x} \multimap (a \vee z) \in F$ and $\vec{x} \multimap (b \vee z) \in F$. Hence, by Definition 5.1, $\vec{x} \multimap ((a \vee z) \wedge (b \vee z)) \in F$. Using distributivity of the lattice, we have proved $\vec{x} \multimap ((a \wedge b) \vee z) \in F$.

Suppose $a \wedge b \in I$. Since, by the above, $(a \wedge b) \vee z \in \overline{I}$, we obtain a contradiction.

Proof of Lemma 6.4

Consider the set

$$\mathbf{E} = \{P \mid P \text{ extends } G' \text{ and } P_{\sim}(F; P) = 1\}$$

of filters, ordered by inclusion. The set $\mathbf{E}$ is nonempty, has suprema of nonempty chains, and by Zorn's Lemma it therefore possesses a maximal element G. To prove that G is a prime filter is analogous to previous cases.

Suppose $a \notin G$ and $b \notin G$. Denote by G_a the filter generated by $G \cup \{a\}$ and by G_b the filter generated by $G \cup \{b\}$. In formulas:

$$G_a = \{y \mid \text{ there exists } x \in G \text{ such that } a \wedge x \leq y \}$$

$$G_b = \{y \mid \text{ there exists } x \in G \text{ such that } b \wedge x \leq y \}$$

By maximality, neither $P_{\sim}(F; G_a) = 1$ nor $P_{\sim}(F; G_b) = 1$ holds. Thus we have some $y_1 \in G_a$ and $y_2 \in G_b$ with ${\sim}y_1 \in F$ and ${\sim}y_2 \in F$. Since F is a filter and by Definition 5.1, ${\sim}y_1 \wedge {\sim}y_2 = {\sim}(y_1 \vee y_2) \in F$. As before, there is $x \in G$ such that $a \wedge x \leq y_1$ and $b \wedge x \leq y_2$. By distributivity and Definition 5.1 again, $(a \vee b) \wedge x \leq y_1 \vee y_2$ and ${\sim}(y_1 \vee y_2) \leq {\sim}((a \vee b) \wedge x)$. Thus, ${\sim}((a \vee b) \wedge x) \in F$.

Now suppose for contradiction that $a \vee b \in G$. Then $(a \vee b) \wedge x \in G$, contradicting $G \in \mathbf{E}$.

Proof of Proposition 6.5

Given $h : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$, we define $\mathit{Stone}^\sharp(h)$ as $\mathsf{DL}(h, 2) : \mathsf{DL}(\mathscr{A}_2, 2) \longrightarrow \mathsf{DL}(\mathscr{A}_1, 2)$. We only need to prove that equations (6)–(8) are satisfied. For the purposes of better readability we denote $[h, 2]$ by $h^\dagger$ in what follows.

(i) The required equality

$$P^1_{\heartsuit}(\vec{K}; h^\dagger G) = \bigvee_{\vec{F}} \mathsf{DL}(\mathscr{A}_1, 2)(\vec{K}, h^\dagger \vec{F}) \wedge P^2_{\heartsuit}(\vec{F}; G)$$

can be rewritten, using the definition of $h^\dagger$, to the equation

$$P^1_{\heartsuit}(\vec{K}; Gh) = \bigvee_{\vec{F}} \mathsf{DL}(\mathscr{A}_1, 2)(\vec{K}, \vec{F}h) \wedge P^2_{\heartsuit}(\vec{F}; G)$$

We prove inequalities (9) and (10):

(a) To prove $P^2_{\heartsuit}(\vec{F}; G) \leq P^1_{\heartsuit}(\vec{F}h; Gh)$, suppose $\vec{F}hx = 1$. Then $G(\heartsuit(hx)) = Gh(\heartsuit x) = 1$ and we are done.

(b) We prove $P^1_{\heartsuit}(\vec{K}; Gh) \leq \bigvee_{\vec{F}} \mathsf{DL}(\mathscr{A}_1, 2)(\vec{K}, \vec{F}h) \wedge P^2_{\heartsuit}(\vec{F}; G)$.

Define a vector $\vec{K}'$ of *filters* on $\mathscr{A}_2$ by putting

$$\vec{K}'\vec{a} = \bigvee_{\vec{x}} \mathscr{A}_2(h\vec{x}, \vec{a}) \wedge \vec{K}\vec{x}$$

We prove $P^2_{\heartsuit}(\vec{K}'; G) = 1$, supposing $P^1_{\heartsuit}(\vec{K}; Gh) = 1$. To that end, suppose $\vec{K}'\vec{a} = 1$ and choose $\vec{x}$ such that $\mathscr{A}_2(h\vec{x}, \vec{a}) \wedge \vec{K}\vec{x} = 1$. Then $Gh(\heartsuit\vec{x}) = G(\heartsuit(h\vec{x})) = 1$, hence $G(\heartsuit\vec{a}) = 1$, since $\heartsuit$ is monotone.

Now use Lemma 6.2 to find a vector $\vec{F}$ of *prime filters* such that $\vec{F}$ extends $\vec{K}'$ and $P^2_{\heartsuit}(\vec{F}; G) = 1$ holds. It remains to prove the equality $\mathsf{DL}(\mathscr{A}_1, 2)(\vec{K}, \vec{F}h) = 1$. This follows immediately from the fact that $\vec{F}$ extends $\vec{K}'$: if $\vec{K}\vec{x} = 1$, then $\vec{K}'(h\vec{x}) = 1$, hence $\vec{F}h\vec{x} = 1$.

(ii) The required equality

$$P^1_{\multimap}(h^\dagger F; \vec{L}, M) = \bigvee_{\vec{G}, H} \mathsf{DL}(\mathscr{A}_1, 2)(\vec{L}, h^\dagger \vec{G}) \wedge \mathsf{DL}(\mathscr{A}_1, 2)(h^\dagger H, M) \wedge P^2_{\multimap}(F; \vec{G}, H)$$

can be rewritten to the equality

$$P^1_{\multimap}(Fh; \vec{L}, M) = \bigvee_{\vec{G}, H} \mathsf{DL}(\mathscr{A}_1, 2)(\vec{L}, \vec{G}h) \wedge \mathsf{DL}(\mathscr{A}_1, 2)(Hh, M) \wedge P^2_{\multimap}(F; \vec{G}, H)$$

We prove inequalities (11) and (12):

(a) For proving the inequality $P^2_{\multimap}(F;\vec{G},H) \leq P^1_{\multimap}(Fh;\vec{G}h,Hh)$, assume $P^2_{\multimap}(F;\vec{G},H) = 1$. If $Fh(\vec{x} \multimap y) \wedge \vec{G}h\vec{x} = F(h\vec{x} \multimap hy) \wedge \vec{G}h\vec{x} = 1$, then $Hhy = 1$, which was to be proved.

(b) We prove the inequality

$$P^1_{\multimap}(Fh;\vec{L},M) \leq \bigvee_{\vec{G},H} \mathsf{DL}(\mathscr{A}_1,2)(\vec{L},\vec{G}h) \wedge \mathsf{DL}(\mathscr{A}_1,2)(Hh,M) \wedge P^2_{\multimap}(F;\vec{G},H)$$

Define

$$\vec{G}'\vec{b} = \bigvee_{\vec{y}} \mathscr{A}_2(h\vec{y},\vec{b}) \wedge \vec{L}\vec{y}, \quad I'c = \bigvee_{z} \mathscr{A}_2(c,hz) \wedge \neg Mz$$

and observe that $\vec{G}'$ is a vector of filters and I' is an ideal. Moreover, the complement $\overline{I'}$ of I' is given by the formula

$$\overline{I'}c = \bigwedge_{z} \mathscr{A}_2(c,hz) \Rightarrow Mz$$

We will prove that $P^2_{\multimap}(F;\vec{G}',\overline{I'}) = 1$, if we suppose $P^1_{\multimap}(Fh;\vec{L},M) = 1$.

To that end, suppose $F(\vec{b} \multimap c) \wedge \vec{G}'\vec{b} = 1$ and suppose z is such that $\mathscr{A}_2(c,hz) = 1$ holds. We need to prove $Mz = 1$.

Pick $\vec{y}$ witnessing $\vec{G}'\vec{b} = 1$. Then $F(h\vec{y} \multimap hz) = Fh(\vec{y} \multimap z) = 1$ and $\vec{L}\vec{y} = 1$. Therefore $Mz = 1$, since we assumed $P^1_{\multimap}(Fh;\vec{L},M) = 1$.

By Lemma 6.3 there exist $\vec{G}$ and I such that $\vec{G}$ is a vector of prime filters extending $\vec{G}'$, I is a prime ideal extending I', and $P^2_{\multimap}(F,\vec{G},\overline{I}) = 1$ holds. Since a complement of a prime ideal is a prime filter, we can put $H = \overline{I}$.

It remains to show that $\vec{G}h$ extends L and Hh is extended by M.

Since $\vec{G}'h$ clearly extends L, so does $\vec{G}h$ (use that $\vec{G}$ extends $\vec{G}'$).

Since $\overline{I'}h$ is extended by M, so is $\overline{I}h$. This follows from the fact that I extends I'.

(iii) The required equality

$$P^1_{\sim}(h^\dagger F;L) = \bigvee_{G} \mathsf{DL}(\mathscr{A}_1,2)(L,h^\dagger G) \wedge P^2_{\sim}(F;G)$$

can be rewritten to the equality

$$P^1_{\sim}(Fh;L) = \bigvee_{G} \mathsf{DL}(\mathscr{A}_1,2)(L,Gh) \wedge P^2_{\sim}(F;G)$$

We prove inequalities in (13) and (14):

(a) To prove the inequality $P^2_{\sim}(F;G) \leq P^1_{\sim}(Fh;Gh)$, suppose that $P^2_{\sim}(F;G) = 1$ and $Fhx = 1$. Then $\neg G({\sim}hx) = \neg Gh({\sim}x) = 1$, which had to be proved.

(b) We prove the inequality $P^1_{\sim}(Fh;L) \leq \bigvee_G \mathsf{DL}(\mathscr{A}_1,2)(L,Gh) \wedge P^2_{\sim}(F;G)$.

Define the filter G' by the formula

$$G'b = \bigvee_{y} \mathscr{A}_2(hy,b) \wedge Ly$$

and observe that $P^2_{\sim}(F;G') = 1$ holds, if we assume $P^1_{\sim}(Fh;L) = 1$.

Indeed: suppose $G'b = 1$ and let y witness this equality. We need to prove $\neg F(\sim b) = 1$. But we know $\neg Fh(\sim y) = \neg F(\sim(hy)) = 1$. Therefore $\neg F(\sim b) = 1$, since $\mathscr{A}_2(hy, b) = 1$ and F is an upperset.

By Lemma 6.4 we can find a prime filter G extending G' such that $P^2_{\sim}(F; G) = 1$ holds. Moreover, Gh extends L, since $G'h$ does.

Proof of Lemma 7.3

Recall from Section 3 that $\eta_{\mathscr{A}}$ is the lattice homomorphism that maps an element x to the set of all prime filters on $\mathscr{A}$ that contain x.

We want to prove that, for any $\mathbb{A} = ((\mathscr{A}, \wedge, \vee), \heartsuit, \multimap, \sim)$, the *underlying monotone mapping* $\iota_{\mathscr{A}}$ of $\eta_{\mathscr{A}}$ preserves the operations $\heartsuit$, $\multimap$ and $\sim$. That is, $\iota_{\mathscr{A}}$ works exactly like $\eta_{\mathscr{A}}$, we only "forget" that it is a lattice homomorphism.

For better readability, we denote the poset $[\mathsf{DL}(\mathscr{A}, 2), 2]$ by $\mathscr{A}^*$ and the operations thereon by $\heartsuit^*$, $\multimap^*$ and $\sim^*$.

(i) The commutativity of the square

$$\begin{tikzcd} \mathscr{A}^n \arrow[r, "\iota^n_{\mathscr{A}}"] \arrow[d, "\heartsuit"'] & (\mathscr{A}^*)^n \arrow[d, "\heartsuit^*"] \\ \mathscr{A} \arrow[r, "\iota_{\mathscr{A}}"'] & \mathscr{A}^* \end{tikzcd}$$

means, when chasing the elements, the requirement

$$\begin{tikzcd} \vec{x} \arrow[r, mapsto] \arrow[d, mapsto] & \{\vec{F} \mid \vec{F}\vec{x} = 1\} \arrow[d, mapsto] \\ \heartsuit\vec{x} \arrow[r, mapsto] & \{G \mid G(\heartsuit\vec{x}) = 1\} = \{G \mid \bigvee_{\vec{F}} \vec{F}\vec{x} \wedge P_{\heartsuit}(\vec{F}; G) = 1\} \end{tikzcd}$$

i.e., the requirement on the equality

$$G(\heartsuit\vec{x}) = \bigvee_{\vec{F}} \vec{F}\vec{x} \wedge P_{\heartsuit}(\vec{F}; G)$$

to hold, where $P_{\heartsuit}(\vec{F}; G) = \bigwedge_{\vec{x}} \vec{F}\vec{x} \Rightarrow G(\heartsuit\vec{x})$.

The inequality $\geq$ holds: suppose $\vec{F}\vec{x} \wedge P_{\heartsuit}(\vec{F}; G) = 1$. Then $G(\heartsuit\vec{x}) = 1$ by the definition of $P_{\heartsuit}$.

The inequality $\leq$ holds: define a "vector"

$$\vec{F'} = \mathscr{A}^n(\vec{x}, -)$$

of uppersets and observe that $\vec{F'}$ is a vector of filters on $\mathscr{A}$. Observe further that $P_{\heartsuit}(\vec{F'}; G) = 1$ holds. Use Lemma 6.2 to produce a vector of prime filters such that $\vec{F'} \leq \vec{F}$ and $P_{\heartsuit}(\vec{F}; G) = 1$. This finishes the proof of $\leq$.

(ii) The commutativity of the square

$$\begin{tikzcd} (\mathscr{A}^l)^{op} \times \mathscr{A} \arrow[r, "(\iota^l_{\mathscr{A}})^{op} \times \iota_{\mathscr{A}}"] \arrow[d, "\multimap"'] & ((\mathscr{A}^*)^l)^{op} \times \mathscr{A}^* \arrow[d, "\multimap^*"] \\ \mathscr{A} \arrow[r, "\iota_{\mathscr{A}}"'] & \mathscr{A}^* \end{tikzcd}$$

means, when chasing the elements, the requirement

$$\begin{tikzcd}
(\vec{y}, z) \arrow[r, mapsto] \arrow[d, mapsto] & \{(\vec{G}, H) \mid \vec{G}\vec{y} \wedge Hz = 1\} \arrow[d, mapsto] \\
\vec{y} \multimap z \arrow[r, mapsto] & \{F \mid F(\vec{y} \multimap z) = 1\} = \{F \mid \bigwedge_{\vec{G},H} \vec{G}\vec{y} \wedge P_{\multimap}(F; \vec{G}, H) \Rightarrow Hz = 1\}
\end{tikzcd}$$

i.e., the requirement on the equality

$$F(\vec{y} \multimap z) = \bigwedge_{\vec{G},H} \vec{G}\vec{y} \wedge P_{\multimap}(F; \vec{G}, H) \Rightarrow Hz$$

to hold, where $P_{\multimap}(F; \vec{G}, H) = \bigwedge_{\vec{y},z} F(\vec{y} \multimap z) \wedge \vec{G}\vec{y} \Rightarrow Hz$.

The inequality $\leq$ holds: suppose $F(\vec{y} \multimap z) = 1$ and $\vec{G}\vec{y} \wedge P_{\multimap}(F; \vec{G}, H) = 1$. Then $Hz = 1$ holds by the definition of $P_{\multimap}$.

The inequality $\geq$ holds: define a "vector"

$$\vec{G'} = \mathscr{A}^{l}(\vec{y}, -)$$

of uppersets, and an upperset

$$H' = F(\vec{y} \multimap -)$$

Then $\vec{G'}$ is a vector of filters and H' is a filter. Observe further that $P_{\multimap}(F; \vec{G'}, H') = 1$. Now use Lemma 6.3 to produce prime filters $\vec{G}$, H such that $\vec{G'} \leq \vec{G}$ and $H \leq H'$ and $P_{\multimap}(F; \vec{G}, H) = 1$. Hence $\vec{G}\vec{y} \wedge P_{\multimap}(F; \vec{G}, H) = 1$, therefore $Hz = 1$. Finally $H'z = 1$, which we were supposed to prove.

(iii) The commutativity of the square

$$\begin{tikzcd}
\mathscr{A}^{op} \arrow[r, "\iota_{\mathscr{A}}^{op}"] \arrow[d, "\sim"'] & (\mathscr{A}^{*})^{op} \arrow[d, "\sim^{*}"] \\
\mathscr{A} \arrow[r, "\iota_{\mathscr{A}}"'] & \mathscr{A}^{*}
\end{tikzcd}$$

means, when chasing the elements, the requirement

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & \{G \mid Gx = 1\} \arrow[d, mapsto] \\
\sim x \arrow[r, mapsto] & \{F \mid F(\sim x) = 1\} = \{F \mid \bigwedge_{G} P_{\sim}(F; G) \Rightarrow \neg Gx\}
\end{tikzcd}$$

i.e., the requirement on the equality

$$F(\sim x) = \bigwedge_{G} P_{\sim}(F; G) \Rightarrow \neg Gx$$

to hold, where $P_{\sim}(F; G) = \bigwedge_{x} Gx \Rightarrow \neg F(\sim x)$.

The inequality $\leq$ holds: suppose $F(\sim x) = 1$ and $P_{\sim}(F; G) = 1$. Suppose further that $\neg Gx = 0$, or, equivalently $Gx = 1$ Then $\neg F(\sim x) = 1$, which contradicts $F(\sim x) = 1$.

The inequality $\geq$ holds: suppose $\bigwedge_G P_{\sim}(F;G) \Rightarrow \neg Gx = 1$ and $F(\sim x) = 0$. Then $G' = \mathscr{A}(x;-)$ is a filter and $P_{\sim}(F;G') = 1$ holds: if $x \leq x'$ in $\mathscr{A}$, then $\neg F(\sim x') = 1$ holds since F is an upperset. Use Lemma 6.4 to produce a prime filter G extending G' such that $P_{\sim}(F;G) = 1$. Then $\neg Gx = 1$, i.e., $Gx = 0$. This is a contradiction.

Proof of Lemma 7.4

Suppose $m : \mathscr{A} \longrightarrow \mathscr{B}$ is a lattice homomorphism that reflects order. We need to prove that the monotone map $Stone(m) : Stone(\mathscr{B}) \longrightarrow Stone(\mathscr{A})$ is surjective. To that end, fix a prime filter F on $\mathscr{A}$. Define the set

$$\mathbf{E} = \{G \mid G \cdot m = F\}$$

of filters on $\mathscr{B}$, ordered by inclusion. The set $\mathbf{E}$ is nonempty, since m reflects order: put $Gb = \bigvee_a \mathscr{B}(ma, b) \wedge Fa$ and observe that G is in $\mathbf{E}$. Furthermore, the union of a nonempty chain of elements of $\mathbf{E}$ is an element of $\mathbf{E}$. By Zorn's Lemma, $\mathbf{E}$ has a maximal element G_0. It is easy to prove that it is a prime filter.

Choose b_1 and b_2 such that $b_1 \vee b_2 \in G_0$ and $b_1 \notin G_0$, $b_2 \notin G_0$. Define

$$G_{b_1} = \{x \mid \text{ there exists } z \in G_0 \text{ such that } b_1 \wedge z \leq x \}$$

$$G_{b_2} = \{x \mid \text{ there exists } z \in G_0 \text{ such that } b_2 \wedge z \leq x \}$$

By maximality of G_0, neither $G_{b_1} \cdot m = F$, nor $G_{b_2} \cdot m = F$ holds. Hence there are z^{b_1}, z^{b_2} in G_0 and a_1, a_2 in $\mathscr{A}$, both not in F, such that $b_1 \wedge z^{b_1} \leq ma_1$ and $b_2 \wedge z^{b_2} \leq ma_2$ hold. Since G_0 is a filter, $z = z^{b_1} \wedge z^{b_2}$ is in G_0. Moreover, $b_1 \wedge z \leq ma_1$ and $b_2 \wedge z \leq ma_2$. Since m is monotone, the inequalities $b_1 \wedge z \leq m(a_1 \vee a_2)$ and $b_2 \wedge z \leq m(a_1 \vee a_2)$ hold. Hence, using distributivity, the inequality $(b_1 \wedge z) \vee (b_2 \wedge z) = (b_1 \vee b_2) \wedge z \leq m(a_1 \vee a_2)$ holds. This proves that $a_1 \vee a_2$ is in F, hence a_1 or a_2 is in F, since F is supposed to be prime. This is a contradiction.

Proof of Theorem 7.6

1 implies 2. Suppose $\mathbf{C} = \mathsf{Mod}(\Sigma)$. We will verify the four conditions for $\mathbf{C}$.

(a) Suppose $e : \mathbb{W}_1 \longrightarrow \mathbb{W}_2$ is a surjective coalgebra morphism. We prove that if $\alpha \models_{\mathbb{W}_1} \beta$, then $\alpha \models_{\mathbb{W}_2} \beta$.

Consider $y \in \mathscr{W}_2$ and a valuation $\mathsf{val} : \mathsf{At} \longrightarrow [\mathscr{W}_2, 2]$. We can define a new valuation $\mathsf{val}' : \mathsf{At} \longrightarrow [\mathscr{W}_1, 2]$ by the composition

$$\mathsf{At} \xrightarrow{\mathsf{val}} [\mathscr{W}_2, 2] \xrightarrow{[e,2]} [\mathscr{W}_1, 2]$$

Then the diagram

$$\begin{tikzcd} \mathscr{L}(\mathsf{At}) \arrow[r, "\|-\|_{\mathsf{val}}"] \arrow[rr, bend right, "\|-\|^{\mathsf{val}'}"'] & Pred^{\sharp}(\mathbb{W}_2) \arrow[r, "Pred^{\sharp}(e)"] & Pred^{\sharp}(\mathbb{W}_1) \end{tikzcd}$$

commutes in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$.

Let x be such that $ex = y$. Then, by assumption, $x \Vdash_{\mathsf{val}'} \alpha \leq \beta$, hence

$$\begin{aligned} \|\alpha \wedge \beta\|_{\mathsf{val}} ex &= [e, 2](\|\alpha \wedge \beta\|_{\mathsf{val}})x = \|\alpha \wedge \beta\|_{\mathsf{val}'} x = \|\alpha\|_{\mathsf{val}'} x = [e, 2](\|\alpha\|_{\mathsf{val}})x \\ &= \|\alpha\|_{\mathsf{val}} ex \end{aligned}$$

Therefore $ex \Vdash_{\mathsf{val}} \alpha \leq \beta$, i.e., $y \Vdash_{\mathsf{val}} \alpha \leq \beta$.

(b) Suppose $m : \mathbb{W}_1 \longrightarrow \mathbb{W}_2$ is a coalgebra morphism with m reflecting order. We prove that if $\alpha \models_{\mathbb{W}_2} \beta$, then $\alpha \models_{\mathbb{W}_1} \beta$.

Observe that $[m, 2] : [\mathscr{W}_2, 2] \longrightarrow [\mathscr{W}_1, 2]$ is a split epimorphism in Pos. Indeed: there exists a monotone map $z : [\mathscr{W}_1, 2] \longrightarrow [\mathscr{W}_2, 2]$ such that $[m, 2] \cdot z = id$. Given $u : \mathscr{W}_1 \longrightarrow 2$, define $v : \mathscr{W}_2 \longrightarrow 2$ by the formula

$$vy = \bigvee_x \mathscr{W}_2(mx, y) \wedge ux$$

Then $z : u \mapsto v$ is monotone and the equalities

$$vmx' = \bigvee_x \mathscr{W}_2(mx, mx') \wedge ux = \bigvee_x \mathscr{W}_1(x, x') \wedge ux = ux'$$

prove $[m, 2] \cdot z = id$ (above, we have used that m reflects order).

Suppose $\mathsf{val} : \mathsf{At} \longrightarrow [\mathscr{W}_1, 2]$ is given. To prove $x \in \|\alpha\|_{\mathsf{val}}$, consider

$$\mathsf{val}' \equiv \mathsf{At} \xrightarrow{\mathsf{val}} [\mathscr{W}_1, 2] \xrightarrow{z} [\mathscr{W}_2, 2]$$

By assumption, $\|\alpha \wedge \beta\|_{\mathsf{val}'} mx = \|\alpha\|_{\mathsf{val}'} mx$. But the diagram

$$\begin{tikzcd} \mathscr{L}(\mathsf{At}) \arrow[r, "\|-\|_{\mathsf{val}'}"] \arrow[rr, bend right, "\|-\|_{\mathsf{val}}"'] & \mathit{Pred}^\sharp(\mathbb{W}_2) \arrow[r, "\mathit{Pred}^\sharp(m)"] & \mathit{Pred}^\sharp(\mathbb{W}_1) \end{tikzcd}$$

commutes in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$ due to $[m, 2] \cdot z = id$. Hence $\|\alpha \wedge \beta\|_{\mathsf{val}} x = \|\alpha\|_{\mathsf{val}} x$.

(c) Suppose $\alpha \models_{\mathbb{W}_i} \beta$, for all $i \in I$. We prove that $\alpha \models_{\coprod_{i \in I} \mathbb{W}_i} \beta$.

The functor $\mathit{Pred}^\sharp$ preserves products (in fact, it preserves all limits). Products in $(\mathsf{Pos}_T)^{op}$ are, of course, coproducts in Pos_T.

Consider x in $\coprod_{i \in I} \mathscr{W}_i$. Since coproducts of frames are formed on the level of posets, there is $i \in I$ such that x is in $\mathscr{W}_i$. Let $\mathsf{val} : \mathsf{At} \longrightarrow \prod_{i \in I} [\mathscr{W}_i, 2]$ be any valuation. Then, by assumption, $x \Vdash_{\mathsf{val}_i} \alpha \wedge \beta = \alpha$, where

$$\mathsf{val}_i \equiv \mathsf{At} \xrightarrow{\mathsf{val}} \textstyle\prod_{i \in I} [\mathscr{W}_i, 2] \xrightarrow{p_i} [\mathscr{W}_i, 2]$$

and where p_i denotes the i-th projection.

This proves $\|\alpha \wedge \beta\|_{\mathsf{val}} x = \|\alpha\|_{\mathsf{val}} x$.

(d) Suppose $\alpha \models_{\mathbb{W}^*} \beta$. We prove that $\alpha \models_{\mathbb{W}} \beta$.

Take x in $\mathscr{W}$ and $\mathsf{val} : \mathsf{At} \longrightarrow [\mathscr{W}, 2]$. Recall that, by Lemma 7.3, η lifts to $\eta^\sharp$, hence we can consider the valuation

$$\mathsf{val}' \equiv \mathsf{At} \xrightarrow{\mathsf{val}} [\mathscr{W}, 2] \xrightarrow{UU_{\heartsuit,\multimap,\sim}(\eta^\sharp_{\mathit{Pred}^\sharp(\mathbb{W})})} [\mathit{Stone}\,\mathit{Pred}(\mathscr{W}), 2]$$

and therefore the diagram

$$\begin{tikzcd} \mathscr{L}(\mathsf{At}) \arrow[r, "\|-\|_{\mathsf{val}}"] \arrow[rr, bend right, "\|-\|_{\mathsf{val}'}"'] & \mathit{Pred}^\sharp(\mathbb{W}) \arrow[r, "\eta^\sharp_{\mathit{Pred}^\sharp(\mathbb{W})}"] & \mathit{Pred}^\sharp\,\mathit{Stone}^\sharp\,\mathit{Pred}^\sharp(\mathbb{W}) \end{tikzcd} \tag{i}$$

commutes in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$. Thus, we obtain a commutative diagram

$$\begin{tikzcd}
U_{\heartsuit,\multimap,\sim}\mathscr{L}(\mathsf{At}) \arrow[r, "U_{\heartsuit,\multimap,\sim}(\|-\|_{\mathsf{val}})"] \arrow[rr, bend right, "U_{\heartsuit,\multimap,\sim}(\|-\|_{\mathsf{val}'})"'] & \mathit{Pred}(\mathscr{W}) \arrow[r, "\eta_{\mathit{Pred}(\mathscr{W})}"] & \mathit{Pred}\,\mathit{Stone}\,\mathit{Pred}(\mathscr{W})
\end{tikzcd}$$

in DL (apply $U_{\heartsuit,\multimap,\sim}$ to diagram (i) and use that $U_{\heartsuit,\multimap,\sim}(\eta^{\sharp}_{\mathit{Pred}^{\sharp}(\mathbb{W})}) = \eta_{\mathit{Pred}(\mathscr{W})}$). Hence also the diagram

$$\begin{tikzcd}
U_{\heartsuit,\multimap,\sim}\mathscr{L}(\mathsf{At}) \arrow[r, "U_{\heartsuit,\multimap,\sim}(\|-\|_{\mathsf{val}})"] \arrow[dr, "U_{\heartsuit,\multimap,\sim}(\|-\|_{\mathsf{val}'})"] \arrow[ddr, "U_{\heartsuit,\multimap,\sim}(\|-\|_{\mathsf{val}})"'] & \mathit{Pred}(\mathscr{W}) \arrow[d, "\eta_{\mathit{Pred}(\mathscr{W})}"] \arrow[dd, bend left=60, "id"] \\
& \mathit{Pred}\,\mathit{Stone}\,\mathit{Pred}(\mathscr{W}) \arrow[d, "\mathit{Pred}(\varepsilon_{\mathscr{W}})"] \arrow[r, phantom, "(*)"] & {} \\
& \mathit{Pred}(\mathscr{W})
\end{tikzcd} \qquad \text{(ii)}$$

commutes in DL. In fact, the area $(*)$ in the above diagram is just one of the triangle equalities for $\mathit{Stone} \dashv \mathit{Pred}$.

By assumption, $\varepsilon_{\mathscr{W}}(x) \Vdash_{\mathsf{val}'} \alpha \wedge \beta = \alpha$. From the lower triangle in (ii) it follows that $x \Vdash_{\mathsf{val}} \alpha \wedge \beta = \alpha$:

$$\begin{aligned}
\|\alpha \wedge \beta\|_{\mathsf{val}} x &= [\varepsilon_{\mathscr{W}}, 2](\|\alpha \wedge \beta\|_{\mathsf{val}'} x) = \|\alpha \wedge \beta\|_{\mathsf{val}'} \varepsilon_{\mathscr{W}}(x) = \|\alpha\|_{\mathsf{val}'} \varepsilon_{\mathscr{W}}(x) \\
&= [\varepsilon_{\mathscr{W}}, 2](\|\alpha\|_{\mathsf{val}'} x) = \|\alpha\|_{\mathsf{val}} x
\end{aligned}$$

2 implies 1. Denote by Σ the set of pairs (α, β) such that $\alpha \models_{\mathbb{W}} \beta$, for all $\mathbb{W}$ in $\mathbf{C}$. Hence $\mathbf{C} \subseteq \mathsf{Mod}(\Sigma)$ by definition.

Suppose $\mathbb{W}_0$ is in $\mathsf{Mod}(\Sigma)$, we want to prove that $\mathbb{W}_0$ is in $\mathbf{C}$.

Define $\mathbf{A}$ to be the closure of $\{\mathit{Pred}^{\sharp}(\mathbb{W}) \mid \mathbb{W} \in \mathbf{C}\}$ under products, subalgebras along monotone maps reflecting order and images along split epis in Pos. Therefore $\mathit{Pred}^{\sharp}(\mathbb{W}_0)$ is in $\mathbf{A}$ and there is a diagram

$$\mathit{Pred}^{\sharp}(\mathbb{W}_0) \xleftarrow{\ e\ } \mathbb{A} \xrightarrow{\ m\ } \textstyle\prod_{i \in I} \mathit{Pred}^{\sharp}(\mathbb{W}_i)$$

in $\mathsf{DL}_{\heartsuit,\multimap,\sim}$, where $\mathbb{A}$ is in $\mathbf{A}$, $\mathbb{W}_i$ are in $\mathbf{C}$, for all $i \in I$, and m reflects orders, and e is split epi in Pos.

Consider the image of the above diagram

$$\mathit{Stone}^{\sharp}\mathit{Pred}^{\sharp}(\mathbb{W}_0) \xleftarrow{\mathit{Stone}^{\sharp}(e)} \mathit{Stone}^{\sharp}(\mathbb{A}) \xrightarrow{\mathit{Stone}^{\sharp}(m)} \mathit{Stone}^{\sharp}(\textstyle\prod_{i \in I} \mathit{Pred}^{\sharp}(\mathbb{W}_i))$$

under $\mathit{Stone}^{\sharp} : \mathsf{DL}_{\heartsuit,\multimap,\sim} \longrightarrow (\mathsf{Pos}_T)^{op}$.

When reading the above diagram in Pos_T, i.e., when reversing the arrows, we obtain a diagram

$$\mathit{Stone}^{\sharp}\mathit{Pred}^{\sharp}(\mathbb{W}_0) \xrightarrow{\mathit{Stone}^{\sharp}(e)} \mathit{Stone}^{\sharp}(\mathbb{A}) \xleftarrow{\mathit{Stone}^{\sharp}(m)} \mathit{Stone}^{\sharp}(\mathit{Pred}^{\sharp}(\textstyle\coprod_{i \in I} \mathbb{W}_i))$$

Then:

(i) $\mathit{Stone}^\sharp(\mathit{Pred}^\sharp(\coprod_{i\in I}\mathbb{W}_i))$ is in $\mathbf{C}$, since it is a prime extension of a coproduct of elements of $\mathbf{C}$.

(ii) $\mathit{Stone}^\sharp(\mathbb{A})$ is in $\mathbf{C}$.

  (a) By Lemma 7.4, $\mathit{Stone}^\sharp(m)$ is a surjective coalgebra homomorphism. Indeed, the underlying map of $\mathit{Stone}^\sharp(m)$ is $\mathit{Stone}(m)$ by (16).

  (b) Since $\mathit{Stone}^\sharp(\mathit{Pred}^\sharp(\coprod_{i\in I}\mathbb{W}_i))$ is in $\mathbf{C}$, so is $\mathit{Stone}^\sharp(\mathbb{A})$. Use properties of $\mathbf{C}$.

(iii) $\mathit{Stone}^\sharp\mathit{Pred}^\sharp(\mathbb{W}_0)$ is in $\mathbf{C}$.

This will follow after we prove that $\mathit{Stone}^\sharp(e)$ reflects orders. Its underlying map is restriction along e from the poset of prime filters on $\mathbb{A}$ to the poset of prime filters on $\mathit{Pred}(\mathscr{W}_0)$. Recall that e is a split epimorphism, denote by z the monotone map satisfying $e\cdot z = id$. Consider two prime filters u, u' on $\mathbb{A}$ such that $u\cdot e \leq u'\cdot e$ holds. Then $u = u\cdot e\cdot z \leq u'\cdot e\cdot z = u'$ holds.

Since we proved that $\mathit{Stone}^\sharp\mathit{Pred}^\sharp(\mathbb{W}_0)$ is in $\mathbf{C}$, we know that $\mathbb{W}_0$ is in $\mathbf{C}$, since $\mathbf{C}$ reflects ultrafilter extensions.