Distributive substructural logics as coalgebraic logics over posets

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Abstract

We show how to understand frame semantics of distributive substructural logics coalgebraically, thus opening a possibility to study them as coalgebraic logics. As an application of this approach we prove a general version of Goldblatt-Thomason theorem that characterizes definability of classes of frames for logics extending the distributive Full Lambek logic, as e.g. relevance logics, many-valued logics or intuitionistic logic. The paper is rather conceptual and does not claim to contain significant new results.

We consider a category of frames as posets equipped with monotone relations, and show that they can be understood as coalgebras for an endofunctor of the category of posets. In fact, we adopt a more general definition of frames that allows to cover a wider class of distributive modal logics. Goldblatt-Thomason theorem for classes of resulting coalgebras for instance shows that frames for axiomatic extensions of distributive Full Lambek logic are modally definable classes of certain coalgebras, the respective modal algebras being precisely the corresponding subvarieties of distributive residuated lattices.

Keywords: Substructural logics, frame semantics, coalgebras, coalgebraic logic, Goldblatt-Thomason theorem.

1 Introduction

Modal logics are coalgebraic, the relational frames of classical modal logics can be seen as Set coalgebras for the powerset functor. Given an endofunctor $T$ on Set, a conceptually clear setting of classical coalgebraic logic of $T$-coalgebras can be based on an adjunction called logical connection, linking categories Set and BA of sets and Boolean algebras [5,6] and capturing syntax and semantics of the propositional part of the language. Such connection can be “lifted” to a connection between categories of $T$-coalgebras and Boolean algebras with

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operators, which is in general "almost" an adjunction, capturing syntax and semantics of the modal part of the language. From certain properties of the lifted connection one automatically obtains soundness, completeness and expressivity of the modal language. One can also explore the connection to obtain the Goldblatt-Thomason definability theorem for classes of $T$-coalgebras for a reasonable class of $\text{Set}$ functors [24].

In this paper, lead by a motivation to approach distributive substructural logics in a coalgebraic way, we use an (enriched) logical connection [25,29] between categories $\text{Pos}$ of posets and $\text{DL}$ of distributive lattices. We consider a general language of distributive lattices with operators, including the usual language of substructural logics as an instance. We start with requiring no additional axioms the operators should satisfy (not even the residuation laws), obtaining coalgebras for a certain endofunctor $T$ on posets as semantics of this language. As an application of this setting we prove Goldblatt-Thomason definability theorem for classes of $T$-coalgebras. Classes of $T$-coalgebras definable by additional axioms of distributive substructural logics then precisely correspond to frames for these logics as surveyed and studied in [30]. Distributive modal logics have been treated coalgebraically before [7,27]. We see the main novelty of this paper in the fact that we use a weaker assumption than a duality of the category of algebras and certain topological spaces, thus resulting in non-topological coalgebras as semantics of distributive modal or substructural logics.

A leading example of a logic, semantics of which we want to cover, is the distributive full Lambek calculus $\text{dFL}$ [15] in the following language

$$\varphi ::= p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \otimes \varphi \mid \varphi \rightarrow \varphi \mid \varphi \leftarrow \varphi \mid e$$

where $p$ ranges through a given poset of atomic propositions, $\land$ and $\lor$ are tied together by a distributive law, and the remaining four connectives $\otimes$, $\leftarrow$, $\rightarrow$, $e$ satisfy additional equational axioms as, for example, the residuation laws. The algebraic semantics of $\text{dFL}$ are residuated lattices.

We want to take the stance that $\land$ and $\lor$ are the only propositional connectives of the language, while the remaining four constructions $\otimes$, $\leftarrow$, $\rightarrow$, $e$ are modalities. To prove that the study of relational models of the above language falls into the realm of coalgebraic modal logic it will be essential to start with a weaker setting, with no additional requirements on the modalities, apart from being monotone and preserving $\land$ or $\lor$, i.e. being operators over distributive lattices.

As it turns out, the natural environment for giving models of the above language is the one of posets and monotone relations. Namely, a relational model will consist of a poset $\mathcal{W}$ and four monotone relations $P_\otimes$, $P_\leftarrow$, $P_\rightarrow$ and $P_e$ on $\mathcal{W}$. For example, $P_\otimes$ will be a monotone relation (i.e., a monotone map $P_\otimes : \mathcal{W}^{\text{op}} \times \mathcal{W}^{\text{op}} \times \mathcal{W} \rightarrow 2$, where 2 is the two-element chain) that we will denote by $P_\otimes : \mathcal{W} \times \mathcal{W} \rightarrow 2$. Hence the “arity” of $P_\otimes$ mirrors the arity of the “modality” $\otimes$. Analogously, $P_\rightarrow$ will be a monotone relation of the form $P_\rightarrow : \mathcal{W} \rightarrow 2$ where $\mathcal{W}$ denotes the one-element preorder. Hence $P_\rightarrow$ will appear
as a “nullary” monotone relation, mirroring the fact that the “modality” \(e\) is nullary. We prove that the above quintuple \(W = (\mathcal{W}, P_\otimes, P_\leftarrow, P_\rightarrow, P_e)\) can be seen as a coalgebra for an endofunctor \(T\) of the category \(\textbf{Pos}\) of posets and monotone maps.

The reasoning does not change much if we incorporate slightly more general languages of the form

\[
\varphi ::= p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \bigtriangledown(\varphi_0, \ldots, \varphi_{n-1}) \mid (\varphi_0, \ldots, \varphi_{l-1}) \rightarrow \psi \mid \sim \psi \tag{2}
\]

where \(p\) ranges through a poset \(A\) of atomic propositions, the connectives \(\land, \lor\) are tied together by the distributive law, \(\bigtriangledown\) is an \(n\)-ary fusion-like connective, \(\rightarrow\) is an \(l\)-ary implication-like connective, and \(\sim\) is a negation-like connective. These connectives are required to interact with \(\land\) and \(\lor\) in the sense that the following equalities are valid for each \(0 \leq i \leq n\):

\[
\begin{align*}
\bigtriangledown(\ldots, \varphi_i \lor \varphi'_i, \ldots) &= \bigtriangledown(\ldots, \varphi_i, \ldots) \lor \bigtriangledown(\ldots, \varphi'_i, \ldots) \\
(\ldots, \varphi_i \lor \varphi'_i, \ldots) \rightarrow \psi &= ((\ldots, \varphi_i, \ldots) \rightarrow \psi) \land ((\ldots, \varphi'_i, \ldots) \rightarrow \psi) \\
(\varphi_0, \ldots, \varphi_{l-1}) \rightarrow (\psi \land \psi') &= ((\varphi_0, \ldots, \varphi_{l-1}) \rightarrow \psi) \land ((\varphi_0, \ldots, \varphi_{l-1}) \rightarrow \psi') \\
\sim(\varphi_1 \lor \varphi_2) &= \sim \varphi_1 \land \sim \varphi_2
\end{align*}
\]

In slogans: \(\bigtriangledown\) should preserve \(\lor\) pointwise, \(\rightarrow\) should pointwise transform \(\lor\) in its premises to \(\land\), and it should preserve \(\land\) in its conclusion, \(\sim\) should transform \(\lor\) into \(\land\).\(^2\)

We will prove that:

(i) Relational models of the language (2) are precisely the coalgebras for an endofunctor \(T : \textbf{Pos} \rightarrow \textbf{Pos}\). Moreover, the construction of \(T\) copies the syntax of the “modalities” \(\bigtriangledown, \rightarrow, \sim\) in (2).

(ii) The algebraic semantics of (2) will be given by a variety \(\text{DL}_{\bigtriangledown, \rightarrow, \sim}\) of distributive lattices with operators \(\bigtriangledown, \rightarrow\) and \(\sim\).

(iii) It is essential to start with no requirements on the modalities in order to obtain a coalgebraic description. Any additional equational requirements on the modalities \(\bigtriangledown, \rightarrow\) and \(\sim\) will result in a modally definable class of \(T\)-coalgebras. We characterize modally definable classes in the spirit of Goldblatt-Thomason Theorem known from the classical modal logic.

As an illustration, we explain how various classes of frames for languages of the type (2) can be perceived as modally definable. In particular, we cover all the frames for the distributive substructural logics as studied in [30], namely:

- The class of frames modelling the distributive full Lambek calculus is modally definable by the equations for residuated distributive lattices. The modalities are \(\otimes, \rightarrow, \leftarrow\) and \(e\).
- The class of frames modelling the intuitionistic logic is classically definable by the equations for Heyting algebras. The modalities are \(\otimes\) (coinciding with \(\land\)) and \(\rightarrow\).

\(^2\) The language above, in its greatest generality, allows for finitely many connectives of each kind, all of various arities. In order not to make the notation too heavy, we will assume that there is just one connective of each kind in our signature. The results for the general case are straightforward generalisations of results for our simplification.
• The class of frames modelling relevance logic is modally definable. The modalities are $\otimes, \rightarrow, \leftarrow, e$ and $\sim$.

**Related work:** Using relational models on posets for modelling semantics of various nonclassical logics goes back at least to the work of Routley and Meyer [31], and Dunn, see [10,11,12] or [30] for an overview. We see the novelty of our approach in the fact that we can systematically work with such frames as coalgebras, hence one has a canonical notion of a frame morphism as morphism of corresponding coalgebras. 

The original Goldblatt-Thomason theorem for modal logics [21] characterizes modally definable classes of Kripke frames. For positive modal logic it was proved in [9]. Our version of the theorem is an analog of coalgebraic Goldblatt-Thomason theorem for Set coalgebras [24, Theorem 3.15(2.)]. Possibilities to generalize the theorem to coalgebras over measurable spaces have been explored in [28]. Coalgebraic Goldblatt-Thomason theorem for classes of models can be found in [20] and [24, Theorem 3.15(1.)].

Our approach relates to, but significantly differs from extensive work relating algebraic and frame (or topological) semantics of modal and substructural logics, using dualities and discrete dualities for distributive lattices [17,18,19], distributive lattices with operators [32,33,23,27], or posets [13], most of it using canonical extensions: in contrast to this approach we do not use a dual equivalence of distributive lattices and certain topological spaces, a weaker kind of adjunction between DL and posets, called logical connection, is enough. The frames, and thus the coalgebras we consider are not topological as those obtained in [27], [7] or [1], they can however be seen as non-topological analogues of those.

**Organisation of the paper:** Section 2 is devoted to fixing the terminology and notation for monotone relations. In Section 3 we briefly recall how the semantics of the propositional part of coalgebraic logic is captured by an adjunction of a special kind, called logical connection. Relational frames as coalgebras are introduced in Section 4. Complex algebras and canonical frames are studied in Sections 5 and 6. Our main result: the modal definability theorem is proved in Section 7. We illustrate this result by examples of distributive full Lambek calculus, relevance logic, etc. We hint at possible generalizations of our approach in Section 8.

**Remark on the notation we use:** We work with posets and monotone relations as with categories enriched over the two-element chain $\mathbf{2}$, see Section 2. Therefore our formulas for manipulation monotone relations use the structure of the complete Boolean algebra $\mathbf{2}$ and are to be computed there. We think that the notation will become convenient in future generalizations to enriched categories, see Section 8.

Due to space limitations we have omitted most of the proofs.

## 2 Preliminaries

Recall that a poset $\mathcal{W}$ is a set $W$ equipped with a reflexive, transitive and antisymmetric relation $\leq$. Instead of writing $x \leq x'$ we will often write $\mathcal{W}(x,x') = 1$ (and writing $\mathcal{W}(x,x') = 0$, if $x \leq x'$ does not hold). This is in compliance with the fact that a poset $\mathcal{W}$ can be seen as a small category enriched in the two-element chain $\mathbf{2}$.

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3 The usual notion of morphism for substructural frames is different — it requires equalities $a = f(x), b = f(y)$ in the back condition in 4.6. The same notion of a frame morphism as ours in the special case of frames for fuzzy logics is given in [8].

5 For the purposes of the refereeing process, all the proofs are in the Appendix.
2. Although we will not use any machinery of enriched category theory explicitly, we find the above notation convenient in the view of further generalizations, see Section 8 below.

An opposite \( \mathcal{W}^{\text{op}} \) of the poset \( \mathcal{W} \) has the same set of elements as \( \mathcal{W} \), but we put \( \mathcal{W}^{\text{op}}(x,x') = \mathcal{W}(x',x) \).

Recall further that a monotone map \( f : \mathcal{W}_1 \to \mathcal{W}_2 \) consists of an assignment \( x \mapsto fx \) such that, for any \( x \) and \( x' \), the inequality \( \mathcal{W}_1(x,x') \leq \mathcal{W}_2(fx,fx') \) holds in 2. The poset of all monotone maps from \( \mathcal{W}_1 \) to \( \mathcal{W}_2 \), with the order defined pointwise, is denoted by \( \mathcal{W}_1 \to \mathcal{W}_2 \). A product \( \mathcal{W}_1 \times \mathcal{W}_2 \) of posets \( \mathcal{W}_1, \mathcal{W}_2 \) is an order on the pairs of elements, defined pointwise. We denote by \( \mathcal{W}^n \) the product of \( n \)-many copies of \( \mathcal{W} \) with itself, writing \( \mathcal{W}^0 = 1 \) — the one-element poset.

Given posets \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \), a monotone relation from \( \mathcal{W}_1 \to \mathcal{W}_2 \), denoted by \( R : \mathcal{W}_1 \downarrow \to \mathcal{W}_2 \), is a monotone map of the form \( R : \mathcal{W}^{\text{op}}_1 \times \mathcal{W}_2 \to 2 \). We write \( R(x,x') = 1 \) to denote that \( x \) is related to \( x' \). In what follows we will omit the adjective 'monotone' and speak just of relations. A relation of the form \( R : \mathcal{W}^n \to \mathcal{W} \) is called an \( n \)-ary relation on \( \mathcal{W} \), where \( n \geq 0 \). For \( n = 0 \) we obtain \( R : \mathcal{W} \to \mathcal{W} \) and it is easy to see that such a relation corresponds to an upper set of \( \mathcal{W} \), i.e., the set \( U = \{ x \mid Rx = 1 \} \) has the property: if \( x \in U \) and \( x \leq x' \), then \( x' \in U \).

Relations compose in the usual way: the composite of the relations \( R : \mathcal{W}_1 \to \mathcal{W}_2 \) and \( S : \mathcal{W}_2 \to \mathcal{W}_3 \) is a relation \( S \circ R : \mathcal{W}_1 \to \mathcal{W}_3 \) given by the formula

\[
S \circ R(x,z) = \bigvee_y S(y,z) \land R(x,y)
\]

For every poset \( \mathcal{W} \), the identity relation \( \text{id}_\mathcal{W} : \mathcal{W} \to \mathcal{W} \) is defined by putting \( \text{id}_\mathcal{W}(x,x') = 1 \) iff \( x \leq x' \). Hence it is consistent to write \( \mathcal{W} \) instead of \( \text{id}_\mathcal{W} \).

It is easy to see that the above composition is associative and that it has identity relations as units, hence we obtain a category (enriched in posets) of posets and relations. The above definitions are specializations of the theory of profunctors (also distributors, or, modules), known from enriched category theory. See, for example, [34] for more details.

3 The logical connection

The semantics of the propositional part of the language, i.e., of the language

\[
\varphi ::= p \mid \varphi \land \varphi \mid \varphi \lor \varphi
\]  

where \( p \) ranges through a poset \( \mathcal{A} \) of atomic propositions and \( \land \) and \( \lor \) are tied by the distributive law, will be given by a logical connection of the category \( \text{Pos} \) of posets and monotone maps and the category \( \text{DL} \) of distributive lattices and lattice morphisms.

The logical connection

\[
\text{Stone} \dashv \text{Pred} : \text{Pos}^{\text{op}} \to \text{DL}
\]

is given by the two-element chain 2 as a schizophrenic object. Recall how the above connection works (we refer to [29] or [25] for more details on logical connections):
(i) \( \text{Pred} \) sends a poset \( \mathcal{W} \) to the distributive lattice \( ([\mathcal{W}, 2], \cap, \cup) \) of uppersets on \( \mathcal{W} \). A monotone map \( f \) is sent to \( [f, 2] : U \to U \cdot f \).

For a poset \( \mathcal{W} \), the distributive lattice \( \text{Pred}(\mathcal{W}) \) is to be considered as the “distributive lattice of truth-distributions on \( \mathcal{W} \)”.

(ii) For a distributive lattice \( \mathcal{A} \), \( \text{Stone}(\mathcal{A}) \) is the poset \( \text{DL}(\mathcal{A}, 2) \) of prime filters on \( \mathcal{A} \). The mapping \( \text{Stone}(h) \) is given by composition: a prime filter \( F \) is sent to the prime filter \( F \cdot h \).

The poset \( \text{Stone}(\mathcal{A}) \) is the “Stone space” of the distributive lattice \( \mathcal{A} \).

(iii) The unit \( \eta : \mathcal{A} \to [\text{DL}(\mathcal{A}, 2), 2] \) is the lattice homomorphism sending \( x \) in \( \mathcal{A} \) to the upperset of all prime filters on \( \mathcal{A} \) that contain \( x \).

(iv) The counit \( \varepsilon : \mathcal{W} \to \text{DL}(\mathcal{W}, 2) \) is the monotone map sending \( x \) in \( \mathcal{W} \) to the prime filter of those uppersets on \( \mathcal{W} \) that contain \( x \).

The semantics of the propositional language (3) is given by the logical connection (4), together with another adjunction

\[
\mathcal{F} : \text{DL} \to \text{Pos}
\]

where \( U \) denotes the obvious forgetful functor and \( \mathcal{F} \) sends a poset \( \mathcal{A} \) to the free distributive lattice on \( \mathcal{A} \). More in detail, the semantics is given as follows:

(i) Fix a poset \( \text{At} \) of atomic propositions. The distributive lattice \( \mathcal{F} \text{At} \) is then the \( \text{Lindenbaum-Tarski} \) algebra of formulas.

(ii) Observe that \( U(\text{Pred}(\mathcal{W})) = [\mathcal{W}, 2] \), for every poset \( \mathcal{W} \). Hence, due to the adjunction \( \mathcal{F} : 1 \to U \), monotone maps of the form \( \text{val} : \text{At} \to [\mathcal{W}, 2] \) are in bijective correspondence with lattice morphisms \( [\text{-}] \to \text{val} : \text{Pred}(\mathcal{W}) \to \text{Pred}(\mathcal{W}) \).

Of course, as the notation suggests, the monotone map \( \text{val} \) is the \text{valuation} of atomic propositions, assigning to every \( p \) the upperset \( \text{val}(p) \) of those \( x \)'s in \( \mathcal{W} \), where \( p \) is valid. The lattice homomorphism \( [\text{-}] \to \text{val} \) is then the free extension of the valuation \( \text{val} \). It can be described inductively as follows:

\[
\| p \|_{\text{val}} = \text{val}(p), \quad \| \varphi_1 \land \varphi_2 \|_{\text{val}} = \| \varphi_1 \|_{\text{val}} \cap \| \varphi_2 \|_{\text{val}}, \quad \| \varphi_1 \lor \varphi_2 \|_{\text{val}} = \| \varphi_1 \|_{\text{val}} \cup \| \varphi_2 \|_{\text{val}}
\]

We will later add more connectives (fusion-like, implication-like and negation-like) but we are going to consider them as \text{modal operators} on distributive lattices. In fact, as we will see, such extension of the language will yield extensions of the above two functors \( \text{Pred} \) and \( \text{Stone} \).

4 Relational frames as coalgebras

We define structures that we call (relational) frames for the language of the type (2). Frames will consist of a poset of states and various relations reflecting the syntax of “modalities” of the language, compare to frames in [30]. We prove that frames are exactly the coalgebras for a certain endofunctor of the category of posets.

\textbf{Notation 4.1} We will introduce the following “vector” conventions: for a relation \( P : \mathcal{W} \to \mathcal{W} \) we will write \( P(\vec{x}; x) \) instead of \( P(x_0, \ldots, x_{n-1}; x) \). For \( P : \mathcal{W} \times (\mathcal{W}^{\times n}) 	imes \mathcal{W} \) we will write \( P(x; \vec{y}, z) \) instead of \( P(x; y_0, \ldots, y_{n-1}, z) \). Analogously we will write \( \mathcal{W}_2(\vec{a}, f \vec{x}) \) instead of \( \mathcal{W}_2(a_0, f x_0) \land \cdots \land \mathcal{W}_2(a_{n-1}, f x_{n-1}) \), etc.
Definition 4.2 A relational frame for the language (2) is a quadruple \( \mathbb{W} = (\mathcal{W}, P_\land, P_\lor, P_\to) \), consisting of a poset \( \mathcal{W} \), and relations

\[
P_\land : \mathcal{W} \to \mathcal{W}, \quad P_\lor : \mathcal{W} \to (\mathcal{W}^{\text{op}})^{\text{op}} \times \mathcal{W}, \quad P_\to : \mathcal{W} \to \mathcal{W}^{\text{op}}
\]

A morphism from \( \mathbb{W}_1 = (\mathcal{W}_1, P^1_\land, P^1_\lor, P^1_\to) \) to \( \mathbb{W}_2 = (\mathcal{W}_2, P^2_\land, P^2_\lor, P^2_\to) \) is a monotone map \( f : \mathcal{W}_1 \to \mathcal{W}_2 \) such that the following three equations hold:

\[
P^2_\land(\vec{a}; f\vec{y}) = \bigvee_{\vec{x}} \mathcal{W}_2(\vec{a}, f\vec{x}) \land P^2_\land(\vec{x}; y) \tag{6}
\]

\[
P^2_\lor(f\vec{x}; b, c) = \bigvee_{\vec{y}, z} \mathcal{W}_2(b, f\vec{y}) \land \mathcal{W}_2(f\vec{x}, c) \land P^1_\lor(x; \vec{y}, z) \tag{7}
\]

\[
P^2_\to(f\vec{x}; b) = \bigvee_{y} \mathcal{W}_2(b, f\vec{y}) \land P^1_\to(x; y) \tag{8}
\]

We write \( f : \mathbb{W}_1 \to \mathbb{W}_2 \) to indicate that \( f \) is a morphism of relational frames.

Remark 4.3 We have not defined semantics in a relational frame yet, but the following intuitions about the “meaning” of the individual relations \( P_\land, P_\lor \) and \( P_\to \) on \( \mathcal{W} \) might be useful (see Notation 4.1).

(i) \( P_\land(\vec{x}, y) = 1 \) holds, if \( \vec{x} \models \varphi \) implies \( y \models \varphi \).

(ii) \( P_\lor(x; \vec{y}, z) = 1 \) holds, if \( x \models \varphi \rightarrow \psi \) and \( \vec{y} \models \varphi \) imply \( z \models \psi \).

(iii) \( P_\to(x; y) = 1 \) holds, if \( y \models \varphi \) implies \( x \nmodels \varphi \).

See Remark 5.5 below for precising the above intuitions.

Example 4.4 A relational frame \( \mathbb{W} \) for the language (1) consists of a poset \( \mathcal{W} \), together with fusion-like relations \( P_\land : \mathcal{W} \times \mathcal{W} \to \mathcal{W} \), \( P_\lor : \mathcal{W} \to \mathcal{W}^{\text{op}} \times \mathcal{W} \), and implication-like relations \( P_\to : \mathcal{W} \to \mathcal{W}^{\text{op}} \times \mathcal{W} \).

Let us stress that the relations \( P_\land, P_\lor, P_\to \) and \( P_\to \) are (as of yet) arbitrary. When one needs special properties as, for example, the frame to be the model of a distributive full Lambek calculus (for such frames see [30]), one needs to invoke modal definability theorem. This is shown in Example 7.7 below.

Example 4.5 Relational frames for the language \( \land, \lor, \otimes, \rightarrow, \land, \land \) and \( \sim \) of relevance logic, see [12], are posets \( \mathcal{W} \) equipped with relations \( P_\land : \mathcal{W} \times \mathcal{W} \to \mathcal{W} \), \( P_\lor : \mathcal{W} \to \mathcal{W}^{\text{op}} \times \mathcal{W} \), \( P_\to : \mathcal{W} \to \mathcal{W}^{\text{op}} \times \mathcal{W} \). The above relations are as of yet arbitrary. Frames for various classes of relevance logic are modally definable, see Remark 7.8 below.

Remark 4.6 It is very easy to prove that the above equations (6)–(8) can be “split” into six inequalities, giving us the back \& forth description of morphisms for fusion-like, implication-like and negation-like connectives. More precisely:

(i) The equation (6) is equivalent to the conjunction of the following two inequalities

\[
P^2_\land(\vec{x}; y) \leq P^2_\land(f\vec{x}; f\vec{y}) \tag{9}
\]

\[
P^2_\land(\vec{a}; f\vec{y}) \leq \bigvee_{\vec{x}} \mathcal{W}_2(\vec{a}, f\vec{x}) \land P^2_\land(\vec{x}; y) \tag{10}
\]

(ii) The equation (7) is equivalent to the conjunction of the following two inequalities

\[
P^1_\lor(x; \vec{y}, z) \leq P^2_\lor(f\vec{x}; f\vec{y}, f\vec{z}) \tag{11}
\]

\[
P^2_\lor(f\vec{x}; b, c) \leq \bigvee_{\vec{y}, z} \mathcal{W}_2(b, f\vec{y}) \land \mathcal{W}_2(f\vec{x}, c) \land P^1_\lor(x; \vec{y}, z) \tag{12}
\]
(iii) The equation (8) is equivalent to the conjunction of the following two inequalities

\[ P^1(x; y) \leq P^2(fx; fy) \]  
\[ P^2(fx; b) \leq \bigvee_y \mathcal{W}_2(b, fy) \land P^1(x; y) \]  

We define now three functors

\[ T_{\bowtie} : \text{Pos} \to \text{Pos}, \quad T_{\triangleright} : \text{Pos} \to \text{Pos}, \quad T_{\sim} : \text{Pos} \to \text{Pos} \]

and prove that their product \( T = T_{\bowtie} \times T_{\triangleright} \times T_{\sim} \) gives rise to relational frames and their morphisms. Namely: frames are \( T \)-coalgebras and frame morphisms are \( T \)-coalgebra morphisms.

**Definition 4.7**

(i) The functor \( T_{\bowtie} \) sends \( \mathcal{W} \) to the poset \( \left( (\mathcal{W}^n)^{op}, 2 \right) \) of lowersets on \( \mathcal{W}^n \). For a monotone map \( f : \mathcal{W}_1 \to \mathcal{W}_2 \), the map \( T_{\bowtie}(f) \) sends \( \vec{b} \mapsto \bigvee \mathcal{W}_2(\vec{b}, f\vec{x}) \land \vec{x} \)

(ii) The functor \( T_{\triangleright} \) sends \( \mathcal{W} \) to the poset \( \left( (\mathcal{W}^1)^{op} \times \mathcal{W}, 2 \right)^{op} \). For a monotone map \( f : \mathcal{W}_1 \to \mathcal{W}_2 \), the map \( T_{\triangleright}(f) \) sends \( X : (\mathcal{W}_1)^{op} \times \mathcal{W}_1 \to 2 \) to

\[ \vec{b}, c \mapsto \bigvee \mathcal{W}_2(\vec{b}, f\vec{y}) \land \mathcal{W}_2(fz, c) \land X(\vec{y}, z) \]

(iii) The functor \( T_{\sim} \) sends \( \mathcal{W} \) to the poset \( \left( \mathcal{W}^{op}, 2 \right)^{op} \). For a monotone map \( f : \mathcal{W}_1 \to \mathcal{W}_2 \), the map \( T_{\sim}(f) \) sends \( X : \mathcal{W}_1^{op} \to 2 \) to

\[ b \mapsto \bigvee \mathcal{W}_2(b, fy) \land X y \]

**Proposition 4.8** Put \( T = T_{\bowtie} \times T_{\triangleright} \times T_{\sim} \). The category of relational frames and their morphisms is isomorphic to the category \( \text{Pos}_T \) of \( T \)-coalgebras and their morphisms.

**Proof.**

(i) To give a monotone map \( \gamma : \mathcal{W} \to T(\mathcal{W}) \) is to give three monotone maps \( \gamma_{\bowtie} : \mathcal{W} \to T_{\bowtie}(\mathcal{W}), \gamma_{\triangleright} : \mathcal{W} \to T_{\triangleright}(\mathcal{W}) \) and \( \gamma_{\sim} : \mathcal{W} \to T_{\sim}(\mathcal{W}) \). Each of the three maps, however, can be uncurried to produce monotone maps \( P_{\bowtie} : (\mathcal{W}^n)^{op} \times \mathcal{W} \to 2, P_{\triangleright} : \mathcal{W}^{op} \times (\mathcal{W}^1)^{op} \times \mathcal{W} \to 2 \) and \( P_{\sim} : \mathcal{W}^{op} \times \mathcal{W}^{op} \to 2 \).

To conclude: \( T \)-coalgebras are exactly the relational frames.

(ii) To give a monotone map \( f : \mathcal{W}_1 \to \mathcal{W}_2 \) such that the square

\[ \begin{array}{ccc} \mathcal{W}_1 & \xrightarrow{\gamma_1} & T(\mathcal{W}_1) \\ f \downarrow & & \downarrow T(f) \\ \mathcal{W}_2 & \xrightarrow{\gamma_2} & T(\mathcal{W}_2) \end{array} \]

commutes, is, by Definition 4.7, to give a monotone map \( f \) such that equations (6)–(8) hold. To conclude: coalgebra homomorphisms are exactly the morphisms of relational frames.

\[ \square \]
5 Complex algebras

The complex algebra $\text{Pred}^F(\mathcal{W})$ of the frame $\mathcal{W}$ will be a distributive lattice $\text{Pred}(\mathcal{W})$, equipped with extra operators $\forall$, $\neg$ and $\sim$.

We prove that taking a complex algebra defines a functor $\text{Pred}^F$ from the (opposite of the) category of relational frames and their morphisms to the category $\text{DL}_{\forall,-,\sim}$ of distributive lattices equipped with extra operations. Moreover, this construction extends the predicate functor $\text{Pred} : \text{Pos}^{op} \rightarrow \text{DL}$ in the sense that the square

$$
\begin{array}{ccc}
\text{Pos}^{op} & \xrightarrow{\text{Pred}} & \text{DL}_{\forall,-,\sim} \\
\downarrow V^\ast & & \downarrow V^\ast_{\forall,-,\sim} \\
\text{Pos}^{op} & \xrightarrow{\text{Pred}} & \text{DL}
\end{array}
$$

commutes. Above, $V^\ast : \text{Pos}^{op} \rightarrow \text{Pos}$ is the forgetful functor sending a coalgebra $(\mathcal{W}, \gamma)$ to the poset $\mathcal{W}$.

**Definition 5.1** The category $\text{DL}_{\forall,-,\sim}$ is defined as follows:

(i) Objects are distributive lattices $\mathcal{A} = (\mathcal{A}_0, \wedge, \vee)$ (where $\mathcal{A}_0$ denotes the underlying poset), together with monotone maps

$$[\forall] : \mathcal{A}_0 \rightarrow \mathcal{A}_0, \quad [\neg] : (\mathcal{A}_0^{op})^{op} 	imes \mathcal{A}_0 \rightarrow \mathcal{A}_0, \quad [\sim] : \mathcal{A}_0^{op} \rightarrow \mathcal{A}_0$$

called the interpretations of $\forall$, $\neg$ and $\sim$. We will usually omit the brackets $[\cdot]_\mathcal{A}$ and denote $(\mathcal{A}, \forall, \neg, \sim)$ by $\mathcal{A}$.

The operations are required to satisfy the following axioms, for each $0 \leq i \leq n$:

$$\forall(x_0, \ldots, x_i \vee x'_i, \ldots) = \forall(x_0, \ldots, x_i, \ldots) \vee \forall(x_0, \ldots, x'_i, \ldots)$$
$$\forall(x_i \vee x'_i, \ldots) \rightarrow y = ((\ldots, x_i, \ldots) \rightarrow y) \wedge ((\ldots, x'_i, \ldots) \rightarrow y)$$
$$\neg \forall(x \rightarrow y_1 \cap y_2) = (\neg \forall x \rightarrow y_1) \wedge (\neg \forall x \rightarrow y_2)$$
$$\sim(x_1 \cup x_2) = \sim x_1 \wedge \sim x_2$$

(ii) A morphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ is a lattice morphism $h : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ preserving the additional operations $\forall$, $\rightarrow$ and $\sim$ on the nose.

The obvious underlying functor will be denoted by $U_{\forall,-,\sim} : \text{DL}_{\forall,-,\sim} \rightarrow \text{DL}$.

**Remark 5.2** It is clear that $\text{DL}_{\forall,-,\sim}$ is a finitary variety over $\text{Pos}$ in the sense of categorical universal algebra. More precisely: the composite $U : U_{\forall,-,\sim} : \text{DL}_{\forall,-,\sim} \rightarrow \text{Pos}$ of the obvious forgetful functors is a monadic functor. In particular, the forgetful functor $U : U_{\forall,-,\sim} : \text{DL}_{\forall,-,\sim} \rightarrow \text{Pos}$ has a left adjoint, hence there also exists a left adjoint $F_{\forall,-,\sim} : \text{DL} \rightarrow \text{DL}_{\forall,-,\sim}$ to $U_{\forall,-,\sim}$. Thus, given a poset $\mathcal{A}$, we can form $F_{\forall,-,\sim}(F(\mathcal{A}))$. This is the Lindenbaum-Tarski algebra of formulas for the language (2) and we denote it by $\mathcal{L}(\mathcal{A})$.

**Definition 5.3** The complex algebra $\text{Pred}^F(\mathcal{W}) = (\{\mathcal{W}, \mathcal{W}, \cap, \cup, \forall, \neg, \sim\})$ is defined as follows:

(i) Given a vector $\vec{U}$ of uppersets $U_0, \ldots, U_{n-1}$, the upperset $\forall \vec{U}$ is defined by the formula

$$y \mapsto \bigvee_{\vec{x}} \vec{U} \vec{x} \wedge P_{\forall}(\vec{x}; y)$$
(ii) Given a vector $\vec{U}$ of uppersets $U_0, \ldots, U_{i-1}$, and an upperset $W$, the upperset $\vec{U} \rightarrow W$ is defined by the formula

$$x \mapsto \bigwedge_{\vec{g},x} \hat{U} \hat{g} \land P_{\neg}(x; \hat{g}, z) \Rightarrow Wz$$

(iii) Given an upperset $U$, the upperset $\sim U$ is defined by the formula

$$x \mapsto \bigwedge_{y} P_{\neg}(x; y) \Rightarrow -Uy$$

where the $\sim$ sign is negation in 2.

The following result is easy to prove, when one uses the back & forth description of morphism of frames, see Remark 4.6.

**Proposition 5.4** The assignment $\mathcal{W} \mapsto \text{Pred}^2(\mathcal{W})$ can be extended to a functor from $(\text{Pos}^+)^{op}$ to $\text{DL}_{\top, \rightarrow, \sim}$. Moreover, the square (15) commutes.

**Remark 5.5** The square (15) allows us to give semantics of the language. More precisely, we saw in Section 3 that the adjunction $F \dashv U : \text{DL} \rightarrow \text{Pos}$, together with Stone $\dashv \text{Pred} : \text{Pos}^{op} \rightarrow \text{DL}$, takes care of the semantics $\mathcal{W}, x \mapsto \mathbb{L}(x)$ of the propositional part of the logic.

The adjunction $F_{\top, \rightarrow, \sim} \dashv U_{\top, \rightarrow, \sim} : \text{DL}_{\top, \rightarrow, \sim} \rightarrow \text{DL}$, together with square (15), allows us to define, for every frame $\mathcal{W}$, a semantics morphism

$$\mathcal{W}, x \mapsto \mathbb{L}(x)$$

in $\text{DL}_{\top, \rightarrow, \sim}$ as the transpose under the composite adjunction

$$\text{DL}_{\top, \rightarrow, \sim} \xrightarrow{F_{\top, \rightarrow, \sim}} \text{DL} \xrightarrow{F} \text{Pos}$$

of a valuation $\mathcal{W} \mapsto \text{val} : \text{At} \rightarrow [2]$. It is possible to give an inductive description of $\mathcal{W}, x \mapsto \text{val}$ of the language. More precisely, we saw in Section 3 that the adjunction $F \dashv U : \text{DL} \rightarrow \text{Pos}$, together with Stone $\dashv \text{Pred} : \text{Pos}^{op} \rightarrow \text{DL}$, takes care of the semantics $\mathcal{W}, x \mapsto \mathbb{L}(x)$ of the propositional part of the logic.

The adjunction $F_{\top, \rightarrow, \sim} \dashv U_{\top, \rightarrow, \sim} : \text{DL}_{\top, \rightarrow, \sim} \rightarrow \text{DL}$, together with square (15), allows us to define, for every frame $\mathcal{W}$, a semantics morphism

$$\mathcal{W}, x \mapsto \mathbb{L}(x)$$

in $\text{DL}_{\top, \rightarrow, \sim}$ as the transpose under the composite adjunction

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The adjunction $F_{\top, \rightarrow, \sim} \dashv U_{\top, \rightarrow, \sim} : \text{DL}_{\top, \rightarrow, \sim} \rightarrow \text{DL}$, together with square (15), allows us to define, for every frame $\mathcal{W}$, a semantics morphism

$$\mathcal{W}, x \mapsto \mathbb{L}(x)$$

in $\text{DL}_{\top, \rightarrow, \sim}$ as the transpose under the composite adjunction

$$\text{DL}_{\top, \rightarrow, \sim} \xrightarrow{F_{\top, \rightarrow, \sim}} \text{DL} \xrightarrow{F} \text{Pos}$$

of a valuation $\mathcal{W} \mapsto \text{val} : \text{At} \rightarrow [2]$.
6 Canonical relational frames

The assignment of the canonical frame \( \text{Stone}^\sharp(\mathcal{A}) \) to an object \( \mathcal{A} \) of \( \text{DL}_\preceq,\rightarrow,\sim \) is, in a way, dual to the formation of complex algebras. We prove below that \( \mathcal{A} \mapsto \text{Stone}^\sharp(\mathcal{A}) \) is functorial and that the square

\[
\begin{array}{ccc}
\text{DL}_\preceq,\rightarrow,\sim & \xrightarrow{\text{Stone}^\sharp} & (\text{Pos}_T)^{\text{op}} \\
\downarrow \text{U}_\preceq,\rightarrow,\sim & & \downarrow (\text{V}_T)^{\text{op}} \\
\text{DL} & \xrightarrow{\text{Stone}} & \text{Pos}^{\text{op}}
\end{array}
\]  

(16)

commutes.

Definition 6.1 Suppose \( \mathcal{A} = (\mathcal{A}, \preceq, \rightarrow, \sim) \) is in \( \text{DL}_\preceq,\rightarrow,\sim \). Define \( \text{Stone}^\sharp(\mathcal{A}) \) as follows:

(i) The underlying poset of \( \text{Stone}^\sharp(\mathcal{A}) \) is the poset \( \text{DL}(\mathcal{A}, 2) \) of prime filters on the distributive lattice \( \mathcal{A} \).

(ii) The relation \( P_\preceq \) is defined as follows:

\[
P_\preceq(F; G) = \bigwedge F \preceq G(\preceq x)
\]

(iii) The relation \( P_\rightarrow \) is defined as follows:

\[
P_\rightarrow(F; \vec{G}, H) = \bigwedge_{x,y} F(\rightarrow x) \wedge \vec{G} x \Rightarrow H y
\]

(iv) The relation \( P_\sim \) is defined as follows:

\[
P_\sim(F; G) = \bigwedge_{x} G x \Rightarrow \neg F(\sim x)
\]

where the \( \neg \) sign is the negation in 2.

The above definitions clearly make sense if we work with mere uppersets in lieu of prime filters. We will need the following three technical results that slightly generalize the results originating in the work on relevance logic, see Section 6 of [11].

Lemma 6.2 (Squeeze Lemma for \( \preceq \)) Suppose \( P_\preceq(\vec{F}; G) = 1 \) holds, where \( \vec{F} \) is a vector of filters and \( G \) a prime filter. Then there is a vector \( \vec{F} \) of prime filters that extends \( \vec{F} \) and \( P_\preceq(\vec{F}; G) = 1 \).

Lemma 6.3 (Squeeze Lemma for \( \rightarrow \)) Suppose \( P_\rightarrow(F; \vec{G}, H) = 1 \), where \( F \) is a prime filter, \( \vec{G} \) is a vector of filters and \( \vec{T} \) is a complement of an ideal \( I \). Then there exists a vector \( \vec{G} \) of prime filters such that \( \vec{G} \) extends \( \vec{G} \) and a prime ideal \( I \) that extends \( I \) and \( P_\rightarrow(F; \vec{G}, I) = 1 \), where \( \vec{I} \) denotes the complement of \( I \).

Lemma 6.4 (Squeeze Lemma for \( \sim \)) Suppose \( P_\sim(F; G') = 1 \), where \( F \) is a prime filter and \( G' \) is a filter. Then there exists a prime filter \( G \) extending \( G' \) such that \( P_\sim(F; G) = 1 \).

The above three lemmata allow us to prove that the computation of a canonical frame is a functorial process.

Proposition 6.5 The assignment \( \mathcal{A} \mapsto \text{Stone}^\sharp(\mathcal{A}) \) can be extended to a functor from \( \text{DL}_\preceq,\rightarrow,\sim \) to \( (\text{Pos}_T)^{\text{op}} \). Moreover, the square (16) commutes.
7 Modal definability

Our modal definability theorem (Theorem 7.6 below) will identify classes $C$ of frames such that the image of $C$ under $\text{Pred}^4$ is an “HSP” class in $\text{DL}_{\vee,\wedge,\neg}$, i.e., it is a variety (compare with the version of Goldblatt-Thomason theorem for modal logics [3, Theorem 5.54] and [24, Theorem 3.15/2]). Since we work over posets, the notion of HSP-closedness has to take this fact under consideration. Namely, we will use the factorization system $(\mathcal{E},\mathcal{M})$ on $\text{Pos}$ where $\mathcal{E}$ consists of surjective monotone maps and $\mathcal{M}$ of monotone maps reflecting order, i.e., $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is in $\mathcal{M}$ if $\mathcal{W}_1(x,x') = \mathcal{W}_2(fx,fx')$ holds for every $x$ and $x'$. That $(\mathcal{E},\mathcal{M})$ is indeed a factorization system on $\text{Pos}$ is proved in [4]. We will use the HSP Theorem w.r.t. a factorization system, see [26]:

A class $A$ of algebras in a variety $\mathcal{V}$ over $\text{Pos}$ is definable by equations in $\mathcal{V}$ iff $A$ satisfies the following conditions ($U : \mathcal{V} \rightarrow \text{Pos}$ denotes the underlying functor):

(i) If $e : A_1 \rightarrow A_2$ is such that $U(e)$ is a split epi in $\text{Pos}$ and $A_1$ is in $\mathcal{A}$, then $A_2$ is in $A$.

(ii) If $m : A_1 \rightarrow A_2$ is such that $U(m)$ is in $\mathcal{M}$ and $A_2$ is in $\mathcal{A}$, then $A_1$ is in $A$.

(P) If $A_i, i \in I$, are in $\mathcal{A}$, then $\prod_{i \in I} A_i$ is in $A$.

In fact, since the algebraic semantics of our logic takes place in (distributive) lattices, we may as well replace equationally defined classes by inequationally defined. We prefer to introduce the inequational description, since it is often more useful in applications.

Definition 7.1 Suppose $\mathcal{W}$ is a relational frame. We say that $\alpha$ entails $\beta$, and denote this fact by $\alpha \models_{\mathcal{W}} \beta$, provided that $||\alpha||_{\mathcal{W}} \leq ||\beta||_{\mathcal{W}}$ holds, for every valuation $\text{val} : At \rightarrow [\mathcal{W}, \mathcal{2}]$.

Given a class $\Sigma$ of pairs of formulas, we denote by $\text{Mod}(\Sigma)$ the class of frames $\mathcal{W}$ such that $\alpha \models_{\mathcal{W}} \beta$, for all $(\alpha, \beta) \in \Sigma$.

The following result is trivial.

Lemma 7.2 $\alpha \models_{\mathcal{W}} \beta$ holds iff $\text{Pred}^4(\mathcal{W}) \models \alpha \land \beta = \alpha$, where the $\models$ sign on the right denotes validity in the sense of universal algebra.

Although the notation might suggest it, it is not the case that the logical connection $\text{Stone} \dashv \text{Pred}$ lifts to an adjunction $\text{Stone}^2 \dashv \text{Pred}^4$. The unit of $\text{Stone} \dashv \text{Pred}$ does lift, however, and we will need this technicality in the proof of Theorem 7.6.

Lemma 7.3 The unit $\eta$ of $\text{Stone} \dashv \text{Pred}$ is a morphism in $\text{DL}_{\vee,\wedge,\neg}$, i.e., $\eta$ lifts along the functor $\text{U}_{\vee,\wedge,\neg} : \text{DL}_{\vee,\wedge,\neg} \rightarrow \text{DL}$ to a natural transformation $\eta^* : Id_{\text{DL}_{\vee,\wedge,\neg}} \rightarrow \text{Pred}^4 \text{Stone}^2$.

Another technical result that we need for Theorem 7.6 is the following one.

Lemma 7.4 The functor $\text{Stone}$ sends maps reflecting order to surjective monotone maps.

Finally, before stating Theorem 7.6, we need to introduce the concept of a prime extension of a frame.

Definition 7.5 The frame $\mathcal{W}^* = \text{Stone}^2 \text{Pred}^4(\mathcal{W})$ is called the prime extension of $\mathcal{W}$.

Theorem 7.6 Suppose $C$ is a class of relational frames that is closed under prime extensions (if $\mathcal{W}$ is in $C$, then $\mathcal{W}^*$ is in $C$). Then the following are equivalent:

(i) There is $\Sigma$ such that $C = \text{Mod}(\Sigma)$.
(ii) C satisfies the following four conditions:
   (a) C is closed under “surjective coalgebraic quotients”, i.e., if e : \( \nu_1 \to \nu_2 \) is surjective and \( \nu_1 \) is in C, so is \( \nu_2 \).
   (b) C is closed under “subcoalgebras”, i.e., if \( m : \nu_1 \to \nu_2 \) reflects order and \( \nu_2 \) is in C, so is \( \nu_1 \).
   (c) C is closed under coproducts.
   (d) C reflects prime extensions: if \( \nu_1 \) is in C, so is \( \nu_2 \).

Proof. For proof see Appendix.

Example 7.7 The distributive and associative full Lambek calculus (denoted by dFL) is given by the grammar (1), where \( \otimes \) is required to be associative, to have \( e \) as a unit and to satisfy the residuation laws \( \varphi \otimes \psi \leq \chi \) iff \( \psi \leq \varphi \) iff \( \varphi \leq \chi \). Thus, the subvariety of \( \mathsf{dL}_{\otimes,\to,\neg,\vdash} \) that we want to deal with is exactly that of distributive residuated lattices.

The frames that are definable by the above (in)equations are precisely the quintuples \( (\mathcal{W}, P_\otimes, P_\to, P_\neg, P_\vdash) \) that satisfy the following conditions (for details see [30, Chapter 11]):

(i) \( P_\otimes \) is associative: \( \bigvee_x (P_\otimes (x, y; z) \land P_\otimes (z, w; v)) = \bigvee_w (P_\otimes (y, u; w) \land P_\otimes (x, w; v)) \)

(ii) and has \( P_e \) as a (left and right) unit:

\[
W(x, y) = \bigvee_z (P_\otimes (z, x; y)) = \bigvee_z (P_\otimes (y, z; x))
\]

(iii) The equalities \( P_\otimes (x_0, x_1; y) = P_\to (x_1; x_0, y) = P_\to (x_0; x_1, y) \) hold.

Class C of frames satisfying the above conditions is easily seen to verify the conditions in Theorem 7.6.

Example 7.8 Many interesting examples can be found among the extensions of (associative) dFL with, e.g., the structural rules, or when expanding the language by negation. Instances of the first possibility are: dFL extended with any combination of exchange, weakening, contraction. See [30] for details on what follows.

(i) The exchange rule corresponds to the commutativity of \( P_\otimes \), i.e. to the equality \( P_\otimes (x, y; z) = P_\otimes (y, x; z) \).

(ii) Weakening corresponds to: \( P_\otimes (x_0, x_1; y) \) implies \( x_0 \leq y \) and \( x_1 \leq y \).

(iii) Contraction corresponds to the equality \( P_\otimes (x, x; x) = 1 \).

This includes, for example, intuitionistic logic, obtained as an extension of dFL with all the three structural rules.\(^4\) Instances of the second possibility include, e.g., the relevance logic R, see [12] or [30]. Here the language \( \otimes, \to, \neg, \vdash, e \) is extended by a negation connective \( \sim \).

The frames \( (\mathcal{W}, P_\otimes, P_\to, P_\neg, P_\vdash) \) for the relevance logic R are the frames for dFL satisfying, in addition, the contraction equality together with the following three axioms ([30]):

(a) \( P_\neg (x; y) = P_\neg (y; x) \),

(b) \( \bigvee_y P_\neg (x_0, x_1; y) \land P_\neg (y; u) \leq \bigvee_u P_\neg (u, x_0; s) \land P_\neg (x_1; s) \),

(c) \( \bigvee_y (P_\neg (x; y) \land \bigwedge_u (P_\neg (y; z) \Rightarrow \mathcal{W}(z, x))) = 1 \).

The class C of frames satisfying these axioms is easily seen to verify the conditions of Theorem 7.6. It is modally definable by corresponding axioms of R.

\(^4\) A usual frame \( (X, \leq) \) for intuitionistic logic can be perceived as a relational frame defining \( P(x, y; z) = x \leq z \land y \leq z \). Then coalgebraic morphisms correspond precisely to bounded morphisms.
8 Conclusions and further work

We have shown that frames for various kinds of distributive substructural logic can be perceived naturally as modally definable classes of poset coalgebras. It seems natural to construct first frames for logics that have minimal necessary restrictions on the modalities — these frames are exactly the coalgebras for a certain endofunctor of the category of posets. Such an approach yields the notion of frame morphisms for free: the morphisms of frames are exactly the coalgebra morphisms. Any (in)equational requirement on the modalities results in singling out a subclass of frames that is modally definable in the sense of Goldblatt-Thomason Theorem. Hence any subvariety of modal algebras (= distributive lattices with operators) defines a Goldblatt-Thomason subclass of frames, and vice versa, which has been illustrated by well-known examples of frames for distributive full Lambek calculus, relevance logic, etc.

The limitation of our result lies certainly in the presence of the distributive law for the propositional part of the logic since it leaves out nondistributive substructural logics. We believe that this can be easily overcome by passing to general lattices and using a two-sorted representation of lattices in the sense of [22]. The underlying logical connection will be two-sorted, hence the “state space” will consist of two posets connected with a monotone relation. This is in compliance with various notions of generalized frames, as studied, e.g., in [16] and [14]. Furthermore, this approach will also allow to pass naturally from posets to categories enriched in a general commutative quantale. In the latter framework, we believe to be able to study, e.g., many-valued modal and substructural logics in a rather conceptual way.

A natural further direction would be to prove a more general Goldblatt-Thomason theorem for coalgebras over posets or categories enriched in a general commutative quantale, obtaining an analogue of [24, Theorem 3.15]. Another line of research explores the fact that the coalgebraic functor we obtained is easily seen to satisfy the Beck-Chevalley Condition in the sense of [2]. Hence it will be possible to develop the theory of cover modalities over coalgebras for distributive substructural logics.

References


Appendix

The verification of the “back & forth” description of frame morphisms — Remark 4.6

(i) Suppose (6) holds. Then the computations

\[ P_2^2(f \bar{x}; f y) = \bigvee_{x'} \mathcal{W}_2(f \bar{x}, f x') \land P_2^1(x'; y) \]

\[ \geq \bigvee_{x'} \mathcal{W}_1(\bar{x}, x') \land P_1^1(x'; y) \]

\[ = P_2^1(\bar{x}; y) \]

verify the “forth” condition (9) for \( \sqcup \) and the “back” condition (10) for \( \sqcup \) is trivial.

Conversely, suppose inequalities (9) and (10) hold. Then the inequalities

\[ \bigvee_x \mathcal{W}_2(\bar{a}, f \bar{x}) \land P_2^1(\bar{x}; y) \leq \bigvee_x \mathcal{W}_2(\bar{a}, f \bar{x}) \land P_2^2(f \bar{x}; f y) \]

\[ \leq P_2^2(\bar{a}; f y) \]

hold by (9) and monotonicity of \( P_2^2 \). This proves (6).

(ii) Suppose (7) holds. We only need to verify the inequality (11):

\[ P_2^2(f x; f \bar{y}, f z) = \bigvee_{\bar{y}', \bar{z}'} \mathcal{W}_2(f \bar{y}, f \bar{y}') \land \mathcal{W}_2(f \bar{z}, f \bar{z}') \land P_1^1(x; \bar{y}', \bar{z}') \]

\[ \geq \bigvee_{\bar{y}', \bar{z}'} \mathcal{W}_1(\bar{y}, \bar{y}') \land \mathcal{W}_1(\bar{z}, \bar{z}') \land P_1^1(x; \bar{y}', \bar{z}') \]

\[ = P_1^1(x; \bar{y}, \bar{z}) \]

Conversely, suppose inequalities (11) and (12) hold. Then the inequalities

\[ \bigvee_{\bar{y}, \bar{z}} \mathcal{W}_2(\bar{b}, f \bar{y}) \land \mathcal{W}_2(f z, c) \land P_1^1(x; \bar{y}, \bar{z}) \leq \bigvee_{\bar{y}, \bar{z}} \mathcal{W}_2(\bar{b}, f \bar{y}) \land \mathcal{W}_2(f z, c) \land P_2^2(f x; f \bar{y}, f z) \]

\[ \leq P_2^2(f x; b, c) \]

prove (7).

(iii) Suppose (8) holds. Then we have inequalities

\[ P_2^1(f x; f y) = \bigvee_{y'} \mathcal{W}_2(f y, f y') \land P_1^1(x; y') \]

\[ \geq \bigvee_{y'} \mathcal{W}_1(y, y') \land P_1^1(x; y') \]

\[ = P_1^1(x; y) \]

and inequality (13) hold.

Conversely, suppose inequalities (13) and (14) hold. Then we have inequalities

\[ \bigvee_y \mathcal{W}_2(b, f y) \land P_1^1(x; y) \leq \bigvee_y \mathcal{W}_2(b, f y) \land P_2^2(f x; f y) \]

\[ \leq P_2^2(f x; b) \]

proving (8).
Proof of Proposition 5.4

It is easy to verify that, given a frame \( \mathcal{W} \), the algebra \( \text{Pred}^\sharp(\mathcal{W}) \) is an object of \( \text{DL}\). For a frame morphism \( f : \mathcal{W}_1 \rightarrow \mathcal{W}_2 \), put \( \text{Pred}^\sharp(f) \) to be the mapping \( [f, 2] : [\mathcal{W}_2, 2] \rightarrow [\mathcal{W}_1, 2] \). We verify that the three operations are preserved on the nose:

(i) The commutativity of the square

\[
\begin{array}{ccc}
[\mathcal{W}_2, 2]^n & \xrightarrow{[f, 2]^n} & [\mathcal{W}_1, 2]^n \\
\triangleright & & \\
[\mathcal{W}_2, 2] & \xrightarrow{[f, 2]} & [\mathcal{W}_1, 2]
\end{array}
\]

is the requirement that the equality

\[
\bigvee \bar{a} \wedge P_2^\#(\bar{a}; f y) = \bigvee \bar{f} \bar{x} \wedge P_1^\#(\bar{x}; y)
\]

holds for every \( y \). The inequality \( \geq \) is obvious: put \( \bar{a} = f \bar{x} \) and use that \( P_1^\#(\bar{x}; y) \leq P_2^\#(f \bar{x}; f y) \) holds, see (9). The converse inequality follows from the inequality (10).

(ii) The commutativity of the square

\[
\begin{array}{ccc}
([\mathcal{W}_2, 2]^n)^{\text{op}} \times [\mathcal{W}_2, 2] & \xrightarrow{([f, 2]^n)^{\text{op}} \times [f, 2]} & ([\mathcal{W}_1, 2]^n)^{\text{op}} \times [\mathcal{W}_1, 2] \\
\triangleright & & \\
[\mathcal{W}_2, 2] & \xrightarrow{[f, 2]} & [\mathcal{W}_1, 2]
\end{array}
\]

is the requirement that the equality

\[
\bigwedge \bar{b} \wedge P_2^\#(f x; \bar{b}, c) \Rightarrow W c = \bigwedge \bar{y} \bar{g} \wedge P_1^\#(x; \bar{y}, z) \Rightarrow W f z
\]

holds for every \( x \). The inequality \( \leq \) follows from \( P_1^\#(x; \bar{y}, z) \leq P_2^\#(f x; f \bar{y}, f z) \), see (11). For the converse inequality, use inequality (12).

(iii) The commutativity of the square

\[
\begin{array}{ccc}
[\mathcal{W}_2, 2]^n & \xrightarrow{[f, 2]^n} & [\mathcal{W}_1, 2]^n \\
\sim & & \\
[\mathcal{W}_2, 2] & \xrightarrow{[f, 2]} & [\mathcal{W}_1, 2]
\end{array}
\]

is the requirement that the equality

\[
\bigwedge b P_2^\#(f x; b) \Rightarrow \neg U b = \bigwedge y P_1^\#(x; y) \Rightarrow \neg U f y
\]

holds for every \( x \). The inequality \( \leq \) follows from inequality (13). For the converse inequality, use inequality (14).
Proof of Lemma 6.2
Consider the following system
\[ E = \{ \vec{P} \mid \vec{P} \text{ extends } \vec{P}^{\prime} \text{ and } P_0(\vec{P}; G) = 1 \} \]
of vectors of filters, ordered by inclusion. The set \( E \) is nonempty by assumption and every nonempty chain in \( E \) has clearly a supremum. By Zorn’s Lemma, there exists a maximal element \( \vec{F} = (F_0, \ldots, F_{n-1}) \) of \( E \). We prove that it is a vector of prime filters. We only prove that \( F_0 \) is a prime filter, the reasoning about the remaining cases is the same.
Suppose \( a \notin F_0 \) and \( b \notin F_0 \) and denote by \( F_a \) the filter generated by \( F_0 \cup \{a\} \) and by \( F_b \) the filter generated by \( F_0 \cup \{b\} \). We can write
\[ F_a = \{ y \mid \text{there exists } x \in F_0 \text{ such that } a \wedge x \leq y \} \]
\[ F_b = \{ y \mid \text{there exists } x \in F_0 \text{ such that } b \wedge x \leq y \} \]
By maximality of \( \vec{F} \), neither \( \vec{F}^a = (F_a, F_1, \ldots, F_{n-1}) \) nor \( \vec{F}^b = (F_b, F_1, \ldots, F_{n-1}) \) is in \( E \). It only holds if both vectors violate the conditions \( P_0(\vec{F}^a; G) = 1 \) and \( P_0(\vec{F}^b; G) = 1 \). Thus we have witnesses \( y^a = (y^a_1, y^a_2, \ldots, y^a_{n-1}) \) in \( \vec{F}^a \), and \( y^b = (y^b_1, y^b_2, \ldots, y^b_{n-1}) \) in \( \vec{F}^b \) such that \( \triangledown y^a \notin G \) and \( \triangledown y^b \notin G \). In particular, there are \( x^a \in F_0, x^b \in F_0 \) such that \( a \wedge x^a \leq y^a \) and \( b \wedge x^b \leq y^b \). Put \( x = x^a \wedge x^b \) and \( y = y^a \wedge y^b \). Use that \( F_a, F_b \) and \( G \) are filters to obtain \( a \wedge x \leq y^a \) and \( \triangledown (y^a, y_1, \ldots, y_{n-1}) \notin G \), and \( b \wedge x \leq y^b \) and \( \triangledown (y^b, y_1, \ldots, y_{n-1}) \notin G \). Thus \( \triangledown (a \wedge x, y_1, \ldots, y_{n-1}) \notin G \) and \( \triangledown (b \wedge x, y_1, \ldots, y_{n-1}) \notin G \). Since \( G \) is assumed to be prime, we have proved
\[ \triangledown (a \wedge x, y_1, \ldots, y_{n-1}) \vee \triangledown (b \wedge x, y_1, \ldots, y_{n-1}) \notin G \]
Since \( \triangledown \) preserves joins by Definition 5.1, and the lattice is distributive, we obtain
\[ \triangledown (a \wedge x, y_1, \ldots, y_{n-1}) \vee \triangledown (b \wedge x, y_1, \ldots, y_{n-1}) = \triangledown ((a \wedge x) \vee (b \wedge x), y_1, \ldots, y_{n-1}) \]
\[ = \triangledown ((a \lor b) \wedge x, y_1, \ldots, y_{n-1}) \notin G \]
If we assume that \( a \lor b \notin F_0 \), then \( (a \lor b) \wedge x \in F_0 \), since \( x \in F_0 \) and \( F_0 \) is a filter. But then \( (a \lor b) \wedge x, y_1, \ldots, y_{n-1} \) in \( \vec{F} \), yielding \( \triangledown ((a \lor b) \wedge x, y_1, \ldots, y_{n-1}) \in G \), a contradiction.

Proof of Lemma 6.3
Consider the set
\[ E = \{ (\vec{Q}, J) \mid \vec{Q} \text{ extends } \vec{Q}^{\prime}, J \text{ extends } I, \text{ and } P_0(\vec{Q}; \vec{Q}, J, I) = 1 \} \]
of pairs (vector of filters, ideal), ordered by inclusion. The set is nonempty by assumption and it clearly has suprema of nonempty chains. By Zorn’s Lemma, there exists a maximal element \( (\vec{G}, I) \) in \( E \).
We prove that \( \vec{G} \) is a vector of prime filters and that \( I \) is a prime ideal.
(i) To prove that \( G_i \) in \( \vec{G} = (G_0, \ldots, G_{l-1}) \) is a prime filter is analogous to previous lemma.
We prove that \( G_0 \) in \( \vec{G} = (G_0, \ldots, G_{l-1}) \) is a prime filter, the reasoning for the remaining cases is the same.
Suppose \( a \notin G_0 \) and \( b \notin G_0 \). Denote by \( G_a \) the filter generated by \( G_0 \cup \{a\} \) and by \( G_b \) the filter generated by \( G_0 \cup \{b\} \). Moreover, we can write
chains, and by Zorn’s Lemma it therefore possesses a maximal element $G$ of filters, ordered by inclusion. The set $G$ is nonempty, has suprema of nonempty chains, and by Zorn’s Lemma it therefore possesses a maximal element $G$. To prove that $G$ is a prime filter is analogous to previous cases.

Proof of Lemma 6.4

Consider the set

$$E = \{P \mid P \text{ extends } G' \text{ and } P_\square(F; P) = 1\}$$

of filters, ordered by inclusion. The set $E$ is nonempty, has suprema of nonempty chains, and by Zorn’s Lemma it therefore possesses a maximal element $G$. To prove that $G$ is a prime filter is analogous to previous cases.
Suppose a \( \notin G \) and b \( \notin G \). Denote by \( G_a \) the filter generated by \( G \cup \{a\} \) and by \( G_b \) the filter generated by \( G \cup \{b\} \). In formulas:

\[
G_a = \{ y \mid \text{there exists } x \in G \text{ such that } a \wedge x \leq y \}\]

\[
G_b = \{ y \mid \text{there exists } x \in G \text{ such that } b \wedge x \leq y \}\]

By maximality, neither \( P_a(\bar{F};G_a) = 1 \) nor \( P_b(\bar{F};G_b) = 1 \) holds. Thus we have some \( y_1 \in G_a \) and \( y_2 \in G_b \) with \( y_1 \sim y_2 \in \bar{F} \) and \( \sim y_2 \in \bar{F} \). Since \( \bar{F} \) is a filter and by Definition 5.1, \( \sim y_1 \wedge \sim y_2 = \sim(y_1 \vee y_2) \in \bar{F} \). As before, there is \( x \in G \) such that \( a \wedge x \leq y_1 \) and \( b \wedge x \leq y_2 \). By distributivity and Definition 5.1 again, \( (a \vee b) \wedge x \leq y_1 \vee y_2 \) and \( \sim(y_1 \vee y_2) \leq \sim((a \vee b) \wedge x) \). Thus, \( \sim((a \vee b) \wedge x) \in \bar{F} \).

Now suppose for contradiction that \( a \vee b \in G \). Then \( (a \vee b) \wedge x \in G \), contradicting \( G \in \mathbf{E} \).

**Proof of Proposition 6.5**

Given \( h : \Delta_1 \rightarrow \Delta_2 \), we define \( \text{Stone}^h(h) \) as \( \text{DL}(h, 2) : \text{DL}(\omega^h_2, 2) \rightarrow \text{DL}(\omega^h_1, 2) \). We only need to prove that equations (6)–(8) are satisfied. For the purposes of better readability we denote \([h, 2]\) by \( h^1 \) in what follows.

(i) The required equality

\[
P^1_\sim(\bar{K}; h^1 G) = \bigvee_{\bar{F}} \text{DL}(\omega^h_1, 2)(\bar{K}, h^1 \bar{F}) \wedge P^2_\sim(\bar{F}; G)
\]

can be rewritten, using the definition of \( h^1 \), to the equation

\[
P^1_\sim(\bar{K}; Gh) = \bigvee_{\bar{F}} \text{DL}(\omega^h_1, 2)(\bar{K}, \bar{F}h) \wedge P^2_\sim(\bar{F}; G)
\]

We prove inequalities (9) and (10):

(a) To prove \( P^2_\sim(\bar{K}; G) \leq P^2_\sim(\bar{F}; Gh) \), suppose \( \bar{F} h x = 1 \). Then \( G(\bar{F} h x) = G h(\bar{F} x) = 1 \) and we are done.

(b) We prove \( P^1_\sim(\bar{K}; Gh) \leq P^2_\sim(\bar{K}; \bar{F}h) \wedge P^2_\sim(\bar{F}; G) \).

Define a vector \( \bar{K}' \) of filters on \( \omega^h_2 \) by putting

\[
\bar{K}' x = \bigvee_{\bar{x}} \omega^h_2(\bar{h} x, \bar{a}) \wedge \bar{K} \bar{x}
\]

We prove \( P^2_\sim(\bar{K}'; G) = 1 \), supposing \( P^1_\sim(\bar{K}; Gh) = 1 \). To that end, suppose \( \bar{K}' x = 1 \) and choose \( \bar{x} \) such that \( \omega^h_2(\bar{h} x, \bar{a}) \wedge \bar{K} \bar{x} = 1 \). Then \( G h(\bar{K} \bar{x}) = G(\bar{F} h x) = 1 \), hence \( G \bar{K} \bar{x} = 1 \), since \( \bar{F} \) is monotone.

Now use Lemma 6.2 to find a vector \( \bar{F} \) of prime filters such that \( \bar{F} \) extends \( \bar{K}' \); if \( \bar{K}' x = 1 \), then \( \bar{K}'(h \bar{x}) = 1 \), hence \( \bar{F} h \bar{x} = 1 \).

(ii) The required equality

\[
P^1_\sim(h^1 F; \bar{L}, M) = \bigvee_{\bar{a}, \bar{G}} \text{DL}(\omega^h_1, 2)(\bar{L}, h^1 \bar{G}) \wedge \text{DL}(\omega^h_2, 2)(h^1 H, M) \wedge P^2_\sim(F; \bar{G}, H)
\]

can be rewritten to the equality

\[
P^1_\sim(Fh; \bar{L}, M) = \bigvee_{\bar{a}, \bar{G}} \text{DL}(\omega^h_1, 2)(\bar{L}, \bar{G}h) \wedge \text{DL}(\omega^h_2, 2)(h h, M) \wedge P^2_\sim(F; \bar{G}, H)
\]
We prove inequalities (11) and (12):

(a) For proving the inequality $P^i_\omega(F; \vec{G}, H) \leq P^i_\omega(Fh; \vec{G}h, Hh)$, assume $P^i_\omega(F; \vec{G}, H) = 1$. If $Fh(x \leadsto y) \land \vec{G}h x = F(hx \leadsto hy) \land \vec{G}h x = 1$, then $Hy = 1$, which was to be proved.

(b) We prove the inequality

$$P^i_\omega(Fh; \vec{L}, M) \leq \bigvee_{\vec{G}, H} \mathcal{D}L(\vec{a}_1, 2)(\vec{L}, \vec{G}h) \land \mathcal{D}L(\vec{a}_1, 2)(Hh, M) \land P^i_\omega(F; \vec{G}, H)$$

Define

$$\vec{G}' = \bigvee_y \vec{a}_2(hy, b) \land \vec{L}y,$$

and observe that $\vec{G}'$ is a vector of filters and $I'$ is an ideal. Moreover, the complement $\overline{\mathcal{T}}$ of $I'$ is given by the formula

$$\overline{\mathcal{T}}c = \bigwedge_z \vec{a}_2(c, hz) \Rightarrow Mz$$

We will prove that $P^i_\omega(F; \vec{G}', \overline{\mathcal{T}}) = 1$, if we suppose $P^i_\omega(Fh; \vec{L}, M) = 1$.

To that end, suppose $F(\vec{b} \leadsto c) \land \vec{G}' \vec{b} = 1$ and suppose $z$ is such that $\vec{a}_2(c, hz) = 1$. We need to prove $Mz = 1$.

Pick $\vec{y}$ witnessing $\vec{G}' \vec{b} = 1$. Then $F(hy \leadsto hz) = Fh(\vec{y} \leadsto z) = 1$ and $\vec{L}y = 1$. Therefore $Mz = 1$, since we assumed $P^i_\omega(Fh; \vec{L}, M) = 1$.

By Lemma 6.3 there exist $\vec{G}$ and $I$ such that $\vec{G}$ is a vector of prime filters extending $\vec{G}'$, $I$ is a prime ideal extending $I'$, and $P^i_\omega(F, \vec{G}, \overline{\mathcal{T}}) = 1$ holds.

Since a complement of a prime ideal is a prime filter, we can put $H = \overline{\mathcal{T}}$.

It remains to show that $\vec{G}h$ extends $\vec{L}$ and $Hh$ is extended by $M$.

Since $\vec{G}'h$ clearly extends $\vec{L}$, so does $\vec{G}h$ (use that $\vec{G}$ extends $\vec{G}'$).

Since $\overline{\mathcal{T}}h$ is extended by $M$, so is $\overline{\mathcal{T}}h$. This follows from the fact that $I$ extends $I'$.

(iii) The required equality

$$P^i_\omega(h^1 F; L) = \bigvee_{\vec{G}} \mathcal{D}L(\vec{a}_1, 2)(L, h^1 G) \land P^i_\omega(F; G)$$

can be rewritten to the equality

$$P^i_\omega(Fh; L) = \bigvee_{\vec{G}} \mathcal{D}L(\vec{a}_1, 2)(L, Gh) \land P^i_\omega(F; G)$$

We prove inequalities in (13) and (14):

(a) To prove the inequality $P^i_\omega(F; G) \leq P^i_\omega(Fh; Gh)$, suppose that $P^i_\omega(F; G) = 1$ and $Fh x = 1$. Then $\neg G(\neg hx) = \neg Gh(\neg x) = 1$, which had to be proved.

(b) We prove the inequality $P^i_\omega(Fh; L) \leq \bigvee_{\vec{G}} \mathcal{D}L(\vec{a}_1, 2)(L, Gh) \land P^i_\omega(F; G)$.

Define the filter $\vec{G}'$ by the formula

$$\vec{G}' \vec{b} = \bigvee_y \vec{a}_2(hy, b) \land Ly$$

and observe that $P^i_\omega(F; \vec{G}') = 1$ holds, if we assume $P^i_\omega(Fh; L) = 1$. 

Indeed: suppose \( G' b = 1 \) and let \( y \) witness this equality. We need to prove \( \neg F(\sim b) = 1 \). But we know \( \neg F h(\sim y) = \neg F(\sim (hy)) = 1 \). Therefore \( \neg F(\sim b) = 1 \), since \( \mathcal{A}_2(hy, b) = 1 \) and \( F \) is an upperset.

By Lemma 6.4 we can find a prime filter \( G \) extending \( G' \) such that \( P^2(F; G) = 1 \) holds. Moreover, \( Gh \) extends \( L \), since \( G'h \) does.

**Proof of Lemma 7.3**

Recall from Section 3 that \( \eta_{\mathcal{A}} \) is the lattice homomorphism that maps an element \( x \) to the set of all prime filters on \( \mathcal{A} \) that contain \( x \).

We want to prove that, for any \( \mathcal{A} = ((\mathcal{A}, \land, \lor), \sqcap, \sim, \sim) \), the underlying monotone mapping \( \iota_{\mathcal{A}} \) of \( \eta_{\mathcal{A}} \) preserves the operations \( \land, \lor \) and \( \sim, \sim \). That is, \( \iota_{\mathcal{A}} \) works exactly like \( \eta_{\mathcal{A}} \), we only “forget” that it is a lattice homomorphism.

For better readability, we denote the poset \( [DL(\mathcal{A}, 2), 2] \) by \( \mathcal{A}^* \) and the operations thereon by \( \land^*, \lor^*, \sim^* \).

(i) The commutativity of the square

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota_{\mathcal{A}}} & \mathcal{A}^* \\
\downarrow{\land} & & \downarrow{\lor^*} \\
\mathcal{A} & \xrightarrow{\iota_{\mathcal{A}}} & \mathcal{A}^* \\
\end{array}
\]

means, when chasing the elements, the requirement

\[
\begin{array}{c}
\mathcal{A} \\
\downarrow{\land} \\
\mathcal{A}
\end{array} \rightarrow \{ F | F \mathcal{A} = 1 \} \\
\begin{array}{c}
\mathcal{A} \\
\downarrow{\lor^*} \\
\mathcal{A}
\end{array} \rightarrow \{ G | G(\mathcal{Y} \mathcal{A}) = 1 \} = \{ G | \lor \mathcal{A} \mathcal{Y} \mathcal{A} \land P_{\lor \mathcal{A}}(F; G) = 1 \}
\]

i.e., the requirement on the equality

\[
G(\mathcal{Y} \mathcal{A}) = \lor \mathcal{A} \land P_{\lor \mathcal{A}}(F; G)
\]

to hold, where \( P_{\lor \mathcal{A}}(F; G) = \land \mathcal{A} \mathcal{Y} \mathcal{A} \Rightarrow G(\mathcal{Y} \mathcal{A}) \).

The inequality \( \geq \) holds: suppose \( \lor \mathcal{A} \land P_{\lor \mathcal{A}}(F; G) = 1 \). Then \( G(\mathcal{Y} \mathcal{A}) = 1 \) by the definition of \( P_{\lor \mathcal{A}} \).

The inequality \( \leq \) holds: define a “vector”

\[
F^h = \mathcal{A} \land (\mathcal{Y}, -)
\]

of uppersets and observe that \( F^h \) is a vector of filters on \( \mathcal{A} \). Observe further that \( P_{\lor \mathcal{A}}(F^h; G) \leq 1 \) holds. Use Lemma 6.2 to produce a vector of prime filters such that \( F^h \leq F^h \) and \( P_{\lor \mathcal{A}}(F^h; G) = 1 \). This finishes the proof of \( \leq \).

(ii) The commutativity of the square

\[
\begin{array}{ccc}
(\mathcal{A}^*)^\lor & \xrightarrow{(t_{\mathcal{A}})^* \times \mathcal{A}} & (\mathcal{A}^*)^\lor \times \mathcal{A} \\
\downarrow{\sim} & & \downarrow{\sim^*} \\
\mathcal{A} & \xrightarrow{t_{\mathcal{A}}} & \mathcal{A}^*
\end{array}
\]


means, when chasing the elements, the requirement
\[
(\vec{y}, z) \to \{(\vec{G}, H) \mid \vec{G}\vec{y} \land Hz = 1\}
\]
\[
\vec{y} \circ z \to \{F \mid F(\vec{y} \circ z) = 1\} = \{F \mid \bigwedge_{\vec{G}, H} \vec{G}\vec{y} \land P_{\circ}(F; \vec{G}, H) \Rightarrow Hz = 1\}
\]
i.e., the requirement on the equality
\[
F(\vec{y} \circ z) = \bigwedge_{\vec{G}, H} \vec{G}\vec{y} \land P_{\circ}(F; \vec{G}, H) \Rightarrow Hz
\]
to hold, where \(P_{\circ}(F; \vec{G}, H) = \bigwedge_{\vec{g}, z} F(\vec{g} \circ z) \land \vec{G}\vec{y} \Rightarrow Hz\).
The inequality \(\leq\) holds: suppose \(F(\vec{y} \circ z) = 1\) and \(\vec{G}\vec{y} \land P_{\circ}(F; \vec{G}, H) = 1\). Then \(Hz = 1\) holds by the definition of \(P_{\circ}\).
The inequality \(\geq\) holds: define a “vector”
\[
\vec{G}' = \mathfrak{A}^*(\vec{y}, -)
\]
of uppersets, and an upperset
\[
H' = F(\vec{y} \circ -)
\]
Then \(\vec{G}'\) is a vector of filters and \(H'\) is a filter. Observe further that \(P_{\circ}(F; \vec{G}', H') = 1\). Now use Lemma 6.3 to produce prime filters \(\vec{G}, H\) such that \(\vec{G}' \leq \vec{G}\) and \(H \leq H'\) and \(P_{\circ}(F; \vec{G}, H) = 1\). Hence \(\vec{G}\vec{y} \land P_{\circ}(F; \vec{G}, H) = 1\), therefore \(Hz = 1\). Finally \(H'z = 1\), which we were supposed to prove.

(iii) The commutativity of the square
\[
\mathfrak{A}^{op} \circ \sim \to (\mathfrak{A}^*)^{op} \sim\]
\[
\mathfrak{A} \circ \sim \to \mathfrak{A}^*
\]
means, when chasing the elements, the requirement
\[
\sim x \to \{G \mid Gx = 1\}
\]
\[
\sim x \to \{F \mid F(\sim x) = 1\} = \{F \mid \bigwedge_{G} P_{\circ}(F; G) \Rightarrow \neg Gx\}
\]
i.e., the requirement on the equality
\[
F(\sim x) = \bigwedge_{G} P_{\circ}(F; G) \Rightarrow \neg Gx
\]
to hold, where \(P_{\circ}(F; G) = \bigwedge_{G} Gx \Rightarrow F(\sim x)\).
The inequality \(\leq\) holds: suppose \(F(\sim x) = 1\) and \(P_{\circ}(F; G) = 1\). Suppose further that \(\neg Gx = 0\), or, equivalently \(Gx = 1\) then \(F(\sim x) = 1\), which contradicts \(F(\sim x) = 1\).
The inequality $\geq$ holds: suppose $\bigvee_u P_u(F;G) \Rightarrow \neg Gx = 1$ and $F(\neg x) = 0$. Then $G' = \mathcal{A}(x; \neg)$ is a filter and $P_u(F;G') = 1$ holds: if $x \leq x'$ in $\mathcal{A}$, then $\neg F(\neg x') = 1$ holds since $F$ is an upperset. Use Lemma 6.4 to produce a prime filter $G$ extending $G'$ such that $P_u(F;G) = 1$. Then $\neg Gx = 1$, i.e., $Gx = 0$. This is a contradiction.

**Proof of Lemma 7.4**

Suppose $m : \mathcal{A} \rightarrow \mathcal{B}$ is a lattice homomorphism that reflects order. We need to prove that the monotone map $\text{Stone}(m) : \text{Stone}(\mathcal{B}) \rightarrow \text{Stone}(\mathcal{A})$ is surjective. To that end, fix a prime filter $F$ on $\mathcal{A}$. Define the set

$$E = \{ G \mid G : m = F \}$$

of filters on $\mathcal{B}$, ordered by inclusion. The set $E$ is nonempty, since $m$ reflects order: put $Gb = \bigvee_a \mathcal{B}(ma, b) \wedge Fa$ and observe that $G$ is in $E$. Furthermore, the union of a nonempty chain of elements of $E$ is an element of $E$. By Zorn’s Lemma, $E$ has a maximal element $G_0$. It is easy to prove that it is a prime filter.

Choose $b_1$ and $b_2$ such that $b_1 \lor b_2 \in G_0$ and $b_1 \notin G_0$, $b_2 \notin G_0$. Define

$$G_{b_1} = \{ x \mid \text{there exists } z \in G_0 \text{ such that } b_1 \land z \leq x \}$$

$$G_{b_2} = \{ x \mid \text{there exists } z \in G_0 \text{ such that } b_2 \land z \leq x \}$$

By maximality of $G_0$, neither $G_{b_1} \cdot m = F$, nor $G_{b_2} \cdot m = F$ holds. Hence there are $z^{b_1}$, $z^{b_2}$ in $G_0$ and $a_1$, $a_2$ in $\mathcal{A}$, both not in $F$, such that $b_1 \land z^{b_1} \leq ma_1$ and $b_2 \land z^{b_2} \leq ma_2$ hold. Since $G_0$ is a filter, $z = z^{b_1} \land z^{b_2}$ is in $G_0$. Moreover, $b_1 \land z \leq ma_1$ and $b_2 \land z \leq ma_2$. Since $m$ is monotone, the inequalities $b_1 \land z \leq m(a_1 \lor a_2)$ and $b_2 \land z \leq m(a_1 \lor a_2)$ hold. Hence, using distributivity, the inequality $(b_1 \land z) \lor (b_2 \land z) = (b_1 \lor b_2) \land z \leq m(a_1 \lor a_2)$ holds. This proves that $a_1 \lor a_2$ is in $F$, hence $a_1$ or $a_2$ is in $F$, since $F$ is supposed to be prime. This is a contradiction.

**Proof of Theorem 7.6**

1 implies 2. Suppose $C = \text{Mod}(\Sigma)$. We will verify the four conditions for $C$.

(a) Suppose $\nu : W_1 \rightarrow W_2$ is a surjective coalgebra morphism. We prove that if $\alpha \models_{W_1} \beta$, then $\alpha \models_{W_2} \beta$.

Consider $y \in W_2$ and a valuation $\text{val} : A \rightarrow [W_2, 2]$. We can define a new valuation $\text{val}' : A \rightarrow [W_1, 2]$ by the composition

$$A \xrightarrow{\text{val}} [W_2, 2] \xrightarrow{\langle \; \rangle} [W_1, 2]$$

Then the diagram

$$\xymatrix{ X(A) \ar[r]^{\dashv}_{\text{val}} \ar[dr]_{\dashv}^\dashv & \text{Pred}^f(W_2) \ar[r]^{\text{Pred}^f(\langle \; \rangle)} & \text{Pred}^f(W_1) \ar[l]_{\dashv}^\dashv}_{\dashv}$$

commutes in $\text{DL}_{\sim, \dots, \sim}$.

Let $x$ be such that $ex = y$. Then, by assumption, $x \models_{\text{val}'} \alpha \leq \beta$, hence

$$\| \alpha \land \beta \|_{\text{val}'} x = [e, 2](\| \alpha \land \beta \|_{\text{val}} x) = \| \alpha \land \beta \|_{\text{val}'} x = \| \alpha \|_{\text{val}'} x = [e, 2](\| \alpha \|_{\text{val}} x) = \| \alpha \|_{\text{val}'} x$$

Therefore $ex \models_{\text{val}} \alpha \leq \beta$, i.e., $y \models_{\text{val}} \alpha \leq \beta$. 

(b) Suppose \( m : \mathcal{W}_1 \to \mathcal{W}_2 \) is a coalgebra morphism with \( m \) reflecting order. We prove that if \( \alpha \models_{\mathcal{W}_2} \beta \), then \( \alpha \models_{\mathcal{W}_1} \beta \).

Observe that \( [m, 2] : \mathcal{W}_2 \to \mathcal{W}_1, 2 \) is a split epimorphism in \( \text{Pos} \). Indeed: there exists a monotone map \( z : \mathcal{W}_1, 2 \to \mathcal{W}_2, 2 \) such that \( [m, 2] \circ z = \text{id} \).

Given \( u : \mathcal{W}_1 \to 2 \), define \( v : \mathcal{W}_2 \to 2 \) by the formula

\[
vy = \bigvee_x m(mx, y) \land ux
\]

Then \( z : u \to v \) is monotone and the equalities

\[
vmx' = \bigvee_x m(mx, mx') \land ux = \bigvee_x m(x, x') \land ux = ux'
\]

prove \( [m, 2] \circ z = \text{id} \) (above, we have used that \( m \) reflects order).

Suppose \( \text{val} : \text{At} \to \mathcal{W}_1, 2 \) is given. To prove \( x \in \| \alpha \|_{\text{val}} \), consider

\[
\text{val} \equiv \text{At} \xrightarrow{\text{val}} \mathcal{W}_1, 2 \xrightarrow{z} \mathcal{W}_2, 2
\]

By assumption, \( \| \alpha \land \beta \|_{\text{val}} mx = \| \alpha \|_{\text{val}} mx. \) But the diagram

\[
\begin{array}{c}
\mathcal{L}(\text{At}) \xrightarrow{\| \alpha \|_{\text{val}}} \text{Pred}^f(\mathcal{W}_2) \xrightarrow{\text{Pred}^f(m)} \text{Pred}^f(\mathcal{W}_1) \\
\end{array}
\]

commutes in \( \text{DL}_{\text{L}, \text{R}, \sim} \) due to \( [m, 2] \circ z = \text{id} \). Hence \( \| \alpha \land \beta \|_{\text{val}} x = \| \alpha \|_{\text{val}} x. \)

(c) Suppose \( \alpha \models_{\mathcal{W}_i} \beta \), for all \( i \in I \). We prove that \( \alpha \models_{\prod_{i \in I} \mathcal{W}_i} \beta \).

The functor \( \text{Pred}^f \) preserves products (in fact, it preserves all limits). Products in \( (\text{Pos}_T)^{op} \) are, of course, coproducts in \( \text{Pos}_T \).

Consider \( x \in \prod_{i \in I} \mathcal{W}_i \). Since coproducts of frames are formed on the level of posets, there is \( i \in I \) such that \( x \in \mathcal{W}_i \). Let \( \text{val} : \text{At} \to \prod_{i \in I} \mathcal{W}_i, 2 \) be any valuation. Then, by assumption, \( x \models_{\text{val}_{\mathcal{W}_i}} \alpha \land \beta = \alpha \), where

\[
\text{val}_{\mathcal{W}_i} \equiv \text{At} \xrightarrow{\text{val}} \prod_{i \in I} \mathcal{W}_i, 2 \xrightarrow{p_i} \mathcal{W}_i, 2
\]

and where \( p_i \) denotes the \( i \)-th projection.

This proves \( \| \alpha \land \beta \|_{\text{val}} x = \| \alpha \|_{\text{val}} x. \)

(d) Suppose \( \alpha \models_{\mathcal{W}_i} \beta \). We prove that \( \alpha \models_{\mathcal{W}} \beta \).

Take \( x \in \mathcal{W} \) and \( \text{val} : \text{At} \to \mathcal{W}, 2 \). Recall that, by Lemma 7.3, \( \eta \) lifts to \( \eta^f \), hence we can consider the valuation

\[
\text{val} \equiv \text{At} \xrightarrow{\text{val}} \mathcal{W}, 2 \xrightarrow{\eta^f} \text{StonePred}(\mathcal{W}), 2
\]

and therefore the diagram

\[
\begin{array}{c}
\mathcal{L}(\text{At}) \xrightarrow{\| \alpha \|_{\text{val}}} \text{Pred}^f(\mathcal{W}) \xrightarrow{\text{Pred}^f(\eta^f)} \text{StonePred}^f(\mathcal{W})
\end{array}
\]

(i)
commutes in $\mathbf{DL}_{\sim \to \sim}$. Thus, we obtain a commutative diagram

$$
\begin{array}{ccc}
U_{\sim \to \sim}(\mathbf{At}) & \xrightarrow{U_{\sim \to \sim}(\sim |\sim)} & \text{Pred}(\mathcal{W}) \\
& \downarrow_{U_{\sim \to \sim}(\sim |\sim)} & \downarrow_{\text{Pred}(\mathcal{W})}
\end{array}
\xrightarrow{\eta_{\text{Pred}(\mathcal{W})}}
\begin{array}{c}
\text{PredStonePred}(\mathcal{W})
\end{array}
$$

in $\mathbf{DL}$ (apply $U_{\sim \to \sim}$ to diagram (i) and use that $U_{\sim \to \sim}(\eta^*_{\text{Pred}(\mathcal{W})}) = \eta_{\text{Pred}(\mathcal{W})}$).

Hence also the diagram

$$
\begin{array}{ccc}
U_{\sim \to \sim}(\mathbf{At}) & \xrightarrow{U_{\sim \to \sim}(\sim |\sim)} & \text{Pred}(\mathcal{W}) \\
& \downarrow_{U_{\sim \to \sim}(\sim |\sim)} & \downarrow_{\eta_{\text{Pred}(\mathcal{W})}}
\end{array}
\xrightarrow{(\text{ii})}
\begin{array}{c}
\text{PredStonePred}(\mathcal{W})
\end{array}
$$

commutes in $\mathbf{DL}$. In fact, the area $(\ast)$ in the above diagram is just one of the triangle equalities for $\text{Stone} \to \text{Pred}$.

By assumption, $\models_{\mathcal{W}} \alpha \land \beta = \alpha$. From the lower triangle in (ii) it follows that $x \models_{\mathcal{W}} \alpha \land \beta = \alpha$:

$$
\begin{align*}
\|\alpha \land \beta\|_{\mathcal{W}} x &= \left[\models_{\mathcal{W}}(\|\alpha \land \beta\|_{\mathcal{W}} x)\right] = \left[\|\alpha \land \beta\|_{\mathcal{W}}(\models_{\mathcal{W}} x)\right] = \|\models_{\mathcal{W}} x\|_{\mathcal{W}} x \\
&= \left[\models_{\mathcal{W}}(\|\models_{\mathcal{W}} x\|_{\mathcal{W}} x)\right] = \|\models_{\mathcal{W}} x\|_{\mathcal{W}} x
\end{align*}
$$

2 implies 1. Denote by $\Sigma$ the set of pairs $(\alpha, \beta)$ such that $\alpha \models_{\mathcal{W}} \beta$, for all $\mathcal{W}$ in $\mathbf{C}$. Hence $\mathbf{C} \subseteq \text{Mod}(\Sigma)$ by definition.

Suppose $\mathcal{W}_0$ is in $\text{Mod}(\Sigma)$, we want to prove that $\mathcal{W}_0$ is in $\mathbf{C}$.

Define $\mathbf{A}$ to be the closure of $\{\text{Pred}(\mathcal{W}) \mid \mathcal{W} \in \mathbf{C}\}$ under products, subalgebras along monotone maps reflecting order and images along split epis in $\mathbf{Pos}$. Therefore $\text{Pred}(\mathcal{W}_0)$ is in $\mathbf{A}$ and there is a diagram

$$
\begin{array}{c}
\text{Pred}(\mathcal{W}_0)
\end{array}
\xleftarrow{\epsilon}
\begin{array}{c}
\mathbf{A}
\end{array}
\xrightarrow{m}
\begin{array}{c}
\prod_{i \in I} \text{Pred}(\mathcal{W}_i)
\end{array}
$$

in $\mathbf{DL}_{\sim \to \sim}$, where $\mathbf{A}$ is in $\mathbf{A}$, $\mathcal{W}_i$ are in $\mathbf{C}$, for all $i \in I$, and $m$ reflects orders, and $\epsilon$ is split epi in $\mathbf{Pos}$.

Consider the image of the above diagram

$$
\text{Stone}^2 \text{Pred}^2(\mathcal{W}_0) \xleftarrow{\text{Stone}^2(\epsilon)} \text{Stone}^2(\mathbf{A}) \xrightarrow{\text{Stone}^2(m)} \text{Stone}^2(\prod_{i \in I} \text{Pred}^2(\mathcal{W}_i))
$$

under $\text{Stone}^2 : \mathbf{DL}_{\sim \to \sim} \to (\mathbf{Pos}_T)^\text{op}$. When reading the above diagram in $\mathbf{Pos}_T$, i.e., when reversing the arrows, we obtain a diagram

$$
\text{Stone}^2 \text{Pred}^2(\mathcal{W}_0) \xrightarrow{\text{Stone}^2(\epsilon)} \text{Stone}^2(\mathbf{A}) \xleftarrow{\text{Stone}^2(m)} \text{Stone}^2(\prod_{i \in I} \text{Pred}^2(\mathcal{W}_i))
$$

Then:
(i) \( \text{Stone}^\sharp(\text{Pred}^\sharp(\bigsqcup_{i \in I} \mathcal{N}_i))) \) is in \( \mathcal{C} \), since it is a prime extension of a coproduct of elements of \( \mathcal{C} \).

(ii) \( \text{Stone}^\sharp(\mathcal{A}) \) is in \( \mathcal{C} \).

(a) By Lemma 7.4, \( \text{Stone}^\sharp(m) \) is a surjective coalgebra homomorphism. Indeed, the underlying map of \( \text{Stone}^\sharp(m) \) is \( \text{Stone}(m) \) by (16).

(b) Since \( \text{Stone}^\sharp(\text{Pred}^\sharp(\bigsqcup_{i \in I} \mathcal{N}_i))) \) is in \( \mathcal{C} \), so is \( \text{Stone}^\sharp(\mathcal{A}) \). Use properties of \( \mathcal{C} \).

(iii) \( \text{Stone}^\sharp \text{Pred}^\sharp(\mathcal{U}_0) \) is in \( \mathcal{C} \).

This will follow after we prove that \( \text{Stone}^\sharp(e) \) reflects orders. Its underlying map is restriction along \( e \) from the poset of prime filters on \( \mathcal{A} \) to the poset of prime filters on \( \text{Pred}(\mathcal{U}_0) \). Recall that \( e \) is a split epimorphism, denote by \( z \) the monotone map satisfying \( e \cdot z = \text{id} \). Consider two prime filters \( u, u' \) on \( \mathcal{A} \) such that \( u \cdot e \leq u' \cdot e \) holds. Then \( u = u \cdot e \cdot z \leq u' \cdot e \cdot z = u' \) holds.

Since we proved that \( \text{Stone}^\sharp \text{Pred}^\sharp(\mathcal{U}_0) \) is in \( \mathcal{C} \), we know that \( \mathcal{U}_0 \) is in \( \mathcal{C} \), since \( \mathcal{C} \) reflects ultrafilter extensions.