Holland’s Theorem for Idempotent Semirings and Applications to Residuated Lattices

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Ordered Groups and Lattices in Algebraic Logic
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1. Motivation
Outline

1 Motivation
2 Idempotent semirings
Outline

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2. Idempotent semirings
3. Idempotent semimodules
Outline

1. Motivation
2. Idempotent semirings
3. Idempotent semimodules
4. Holland’s theorem for idempotent semirings
Outline

1. Motivation
2. Idempotent semirings
3. Idempotent semimodules
4. Holland’s theorem for idempotent semirings
5. FEP for integral idempotent semirings
Outline

1 Motivation
2 Idempotent semirings
3 Idempotent semimodules
4 Holland’s theorem for idempotent semirings
5 FEP for integral idempotent semirings
6 Applications to residuated lattices
   - Conuclei
   - Cayley’s and Holland’s theorem
   - FEP
Outline

1. Motivation
2. Idempotent semirings
3. Idempotent semimodules
4. Holland’s theorem for idempotent semirings
5. FEP for integral idempotent semirings
6. Applications to residuated lattices
   - Conuclei
   - Cayley’s and Holland’s theorem
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Motivation

Theorem (Cayley 1854)

*Every group* $G$ *is embeddable into* $\text{Sym}(G)$.
Motivation

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Every group $G$ is embeddable into $\text{Sym}(G)$.

Theorem (Holland 1963)
Every $\ell$-group can be embedded in the $\ell$-group $\text{Aut}(C)$ of the order-automorphisms on a chain $C$.

Theorem (Anderson-Edwards 1984)
Every distributive $\ell$-monoid can be embedded in the $\ell$-monoid $\text{End}(C)$ of the order-preserving maps on a chain $C$.

Theorem (Paoli-Tsinakis 2010)
Every distributive residuated lattice in which multiplication distributes over meets can be embedded as $\ell$-monoid into $\text{Res}(C)$ for a complete chain $C$. 
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1. Motivation
2. Idempotent semirings
3. Idempotent semimodules
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   - Conuclei
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Idempotent semirings

Definition
A structure $\mathbb{R} = \langle R, +, \cdot, 1 \rangle$ is called a (unital) semiring if
- $\langle R, + \rangle$ is a commutative semigroup,
- $\langle R, \cdot, 1 \rangle$ is a monoid,
- $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$. 

$R_{op} = \langle R, +, \circ, 1 \rangle$ denotes an opposite semiring where $x \circ y = y \cdot x$, $R^+ = \langle R, + \rangle$, $R$ is called idempotent if $a + a = a$. In that case $R^+$ forms a (join)-semilattice.
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- $R^{op} = \langle R, +, \circ, 1 \rangle$ denotes an opposite semiring where $x \circ y = y \cdot x$,
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Examples

Definition
Let \( P \) be a poset. A map \( f : P \rightarrow P \) is said to be residuated iff it has a (left) residual \( f^\dagger : P \rightarrow P \), i.e.

\[
f(x) \leq y \quad \text{iff} \quad x \leq f^\dagger(y).
\]

\( \text{Res}(P) \) denotes the set of all residuated maps on \( P \).
Examples

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Res$(\mathbf{P})$ denotes the set of all residuated maps on $\mathbf{P}$.

Example
Let $\mathbf{L} = \langle L, \lor \rangle$ be a join-semilattice.
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Res$(P)$ denotes the set of all residuated maps on $P$.

Example

Let $L = \langle L, \lor \rangle$ be a join-semilattice.

- $\text{End}(L) = \langle \text{End}(L), \lor, \circ, \text{id} \rangle$ is an idempotent semiring,
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Let $L = \langle L, \lor \rangle$ be a join-semilattice.

- $\text{End}(L) = \langle \text{End}(L), \lor, \circ, \text{id} \rangle$ is an idempotent semiring,
- Res($L$) forms a subsemiring Res($L$) of End($L$) since residuated maps are closed under composition and pointwise join.
Relational endomorphisms

Let $L$ be a join-semilattice. Recall that ideals on $L$ forms an algebraic lattice $\mathcal{I}(L) = \langle \mathcal{I}(L), \cap, \lor \rangle$. We identify binary relations on $L$ with functions from $L$ to $\mathcal{P}(L)$.
Relational endomorphisms

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**Definition**

A binary relation \( R \subseteq L \times L \) is called **compatible** if

1. \( R(x) \in \mathcal{I}(L) \),
2. \( R(x \lor y) = R(x) \lor R(y) \).

**Example**

The set of all compatible relations on \( L \) forms an idempotent semiring \( R_{\text{End}}(L) = \langle R_{\text{End}}(L), \lor, \circ, \text{Id} \rangle \), where \( \text{Id}(x) = \downarrow x \).

**Lemma**

\( R_{\text{End}}(L) \sim = \text{Res}(\mathcal{I}(L)) \).
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The set of all compatible relations on $L$ forms an idempotent semiring $\mathbf{REnd}(L) = \langle \mathbf{REnd}(L), \lor, \circ, \text{Id} \rangle$, where $\text{Id}(x) = \downarrow x$. 

Lemma

$\mathbf{REnd}(L) \cong \mathbf{Res}(\mathcal{I}(L))$. 

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1. Motivation
2. Idempotent semirings
3. Idempotent semimodules
4. Holland’s theorem for idempotent semirings
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Idempotent semimodules

Definition

Let $\mathbf{R} = \langle R, +, \cdot, 1 \rangle$ be a semiring. A left $\mathbf{R}$-semimodule $\mathbf{M}$ is a commutative semigroup $\langle M, + \rangle$ together with a map $\star : R \times M \rightarrow M$ such that:

1. $r \star (m + n) = r \star m + r \star n$,
2. $(r + s) \star m = r \star m + s \star m$,
3. $r \star (s \star m) = (r \cdot s) \star m$,
4. $1 \star m = m$.

$\mathbf{M}^+ = \langle M, + \rangle$ is the scalar-free reduct.

$\mathbf{M}$ is called idempotent if $m + m = m$. In that case $\mathbf{M}^+$ forms a (join)-semilattice.

A right $\mathbf{R}$-semimodule is defined analogously.
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- $M^+ = \langle M, + \rangle$ is the scalar-free reduct.
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Let $R = \langle R, +, \cdot, 1 \rangle$ be a semiring. A **left $R$-semimodule** $M$ is a commutative semigroup $\langle M, + \rangle$ together with a map $\star : R \times M \to M$ such that:

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**Definition**

- $M^+ = \langle M, + \rangle$ is the scalar-free reduct.
- $M$ is called **idempotent** if $m + m = m$. In that case $M^+$ forms a (join)-semilattice.
- A right $R$-semimodule is defined analogously.
Examples

Example
Every semiring $\mathbf{R} = \langle R, +, \cdot, 1 \rangle$ can be turned into a left $\mathbf{R}$-semimodule $\langle R, + \rangle$ using its multiplication as the left action.
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Example

Every commutative semigroup (join-semilattice) $L = \langle L, + \rangle$ can be turned into an (idempotent) $\text{End}(L)$-semimodule where the left action $\star: \text{End}(L) \times L \to L$ is defined by $f \star m = f(m)$. 
Separation set

Let $\mathbf{M}$ be a left $\mathbf{R}$-semimodule over a semiring $\mathbf{R}$.

**Definition**

A subset $E \subseteq M$ is called a **separating set** in $\mathbf{M}$ if for all $r, s \in R$ we have the following implication:

$$r \neq s \implies \exists e \in E \ (r \star e \neq s \star e).$$
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- The map $\phi : R \to \text{End}(M^+) \text{ sending } r \in R \text{ to } f_r(m) = r \star m$ is a semiring homomorphism.
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- If $M$ has a separating set $E$, then $\phi$ is an embedding.
Separation set

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Observations

- The map $\phi : R \to \text{End}(M^+) \text{ sending } r \in R \text{ to } f_r(m) = r \star m \text{ is a semiring homomorphism.}$
- If $M$ has a separating set $E$, then $\phi$ is an embedding.
- The above holds also for a right $R$-semimodule if we replace $\text{End}(M^+)$ by $\text{End}(M^+)^{op}$. 
Cayley’s theorem for idempotent semirings

Corollary

Every idempotent semiring $R$ embeds into $\text{End}(R^+)$. 

Proof.

Every idempotent semiring $R$ can be viewed as an idempotent $R$-semimodule whose left action is just the multiplication in $R$. Moreover, $\{1\}$ is a separating set in the semimodule $R$. 

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Residuated semimodules

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An $\mathbf{R}$-semimodule $\mathbf{M}$ is **residuated** if $\mathbf{M}$ is idempotent and there is a map $\presimo{\cdot}: \mathbf{R} \times \mathbf{M} \to \mathbf{M}$ such that

$$r \ast m \leq n \text{ iff } m \leq r \presimo{n}.$$
Residuated semimodules

Let $\mathbf{R}$ be an idempotent semiring.

**Definition**

An $\mathbf{R}$-semimodule $\mathbf{M}$ is **residuated** if $\mathbf{M}$ is idempotent and there is a map $\setminus : \mathbf{R} \times \mathbf{M} \to \mathbf{M}$ such that

$$r \star m \leq n \text{ iff } m \leq r \setminus n.$$ 

Then $\setminus$ is a right action since $1 \setminus m = m$ and $s \setminus (r \setminus m) = rs \setminus m.$
Residuated semimodules

Let $R$ be an idempotent semiring.

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An $R$-semimodule $M$ is **residuated** if $M$ is idempotent and there is a map $\setminus : R \times M \to M$ such that

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Then $\setminus$ is a right action since $1 \setminus m = m$ and $s \setminus (r \setminus m) = rs \setminus m$.

**Lemma**

*Let $M$ be a residuated $R$-semimodule such that $M^+$ forms a lattice.*

Then

- $r \setminus (m \land n) = r \setminus m \land r \setminus n$,
- $(r \lor s) \setminus m = r \setminus m \land s \setminus m$. 
Idempotent semimodules

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Let $R$ be an idempotent semiring.

**Definition**

An $R$-semimodule $M$ is **residuated** if $M$ is idempotent and there is a map $\setminus: R \times M \rightarrow M$ such that

$$r \ast m \leq n \iff m \leq r \setminus n.$$

Then $\setminus$ is a right action since $1 \setminus m = m$ and $s \setminus (r \setminus m) = rs \setminus m$.

**Lemma**

Let $M$ be a residuated $R$-semimodule such that $M^+$ forms a lattice. Then

- $r \setminus (m \wedge n) = r \setminus m \wedge r \setminus n$,
- $(r \vee s) \setminus m = r \setminus m \wedge s \setminus m$.

Thus $(M^+)^\partial$ together with $\setminus$ forms a right idempotent $R$-semimodule.
Relational Cayley’s theorem

Theorem

Let $R$ be an id. semiring and $M$ a left id. $R$-semimodule. Then

$$I(M)$$

is a complete residuated $R$-semimodule whose left action is given by

$$r \star I = \{ m \in M \mid (\exists n \in I) (m \leq r \star n) \}$$

and

$$r \setminus I = \{ m \in M \mid r \star m \in I \}.$$
**Relational Cayley’s theorem**

**Theorem**

Let $R$ be an id. semiring and $M$ a left id. $R$-semimodule. Then

1. $I(M)$ is a complete residuated $R$-semimodule whose left action is given by

\[ r \star l = \{ m \in M \mid (\exists n \in l)(m \leq r \star n) \} , \quad r \setminus J = \{ m \in M \mid r \star m \in J \} . \]

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Relational Cayley’s theorem

**Theorem**

Let $R$ be an id. semiring and $M$ a left id. $R$-semimodule. Then

1. $\mathcal{I}(M)$ is a complete residuated $R$-semimodule whose left action is given by

\[ r \star I = \{ m \in M \mid (\exists n \in I)(m \leq r \star n) \} , \quad r \setminus J = \{ m \in M \mid r \star m \in J \} . \]

2. $M$ embeds into $\mathcal{I}(M)$ as an $R$-semimodule via the map $m \mapsto \downarrow m$. 

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2. $M$ embeds into $\mathcal{I}(M)$ as an $R$-semimodule via the map $m \mapsto \downarrow m$.

3. If $E$ is a separating set in $M$ then $\{ \downarrow e \mid e \in E \}$ is a separating set in $\mathcal{I}(M)$. 

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   \[ r \star l = \{ m \in M \mid (\exists n \in l)(m \leq r \star n) \} , \quad r \backslash J = \{ m \in M \mid r \star m \in J \} . \]

2. $\mathbf{M}$ embeds into $\mathcal{I}(\mathbf{M})$ as an $\mathbf{R}$-semimodule via the map $m \mapsto \downarrow m$.

3. If $E$ is a separating set in $\mathbf{M}$ then \{\downarrow e \mid e \in E\} is a separating set in $\mathcal{I}(\mathbf{M})$.

Theorem

Any idempotent semiring $\mathbf{R}$ is embeddable into $\text{REnd}(\mathbf{R}^+)$ which is isomorphic to $\text{Res}(\mathcal{I}(\mathbf{R}^+))$. 
Outline

1. Motivation
2. Idempotent semirings
3. Idempotent semimodules
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Linear ideals

Let $M$ be a left idempotent $R$-semimodule over an idempotent semiring $R$. 

Lemma

An ideal $I \in I(M +)$ is linear iff $M/\sim I$ is linearly ordered.
Linear ideals

- Let $M$ be a left idempotent $R$-semimodule over an idempotent semiring $R$.
- Then every ideal $I \in \mathcal{I}(M^+)$ induces a congruence $\sim_I$ on $M$ defined as follows:

$$m \sim_I m' \iff (\forall r \in R)(r \star m \in I \iff r \star m' \in I).$$
Linear ideals

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$$m \sim_I m' \quad \text{iff} \quad (\forall r \in R)(r \star m \in I \iff r \star m' \in I).$$

An ideal $I \in \mathcal{I}(M^+)$ is called linear if $r \star m \in I$ and $s \star n \in I$ implies $r \star n \in I$ or $s \star m \in I$. 

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- An ideal $I \in \mathcal{I}(M^+)$ is called linear if $r \star m \in I$ and $s \star n \in I$ implies $r \star n \in I$ or $s \star m \in I$.

Lemma

An ideal $I \in \mathcal{I}(M^+)$ is linear iff $M/\sim_I$ is linearly ordered.
Consider the following quasi-identity in the language of semimodules:

\[ u \leq h \vee c \star a \quad \& \quad u \leq h \vee d \star b \implies u \leq h \vee c \star b \vee d \star a. \quad (EC) \]
(EC) condition

Consider the following quasi-identity in the language of semimodules:

\[ u \leq h \lor c \ast a \quad \& \quad u \leq h \lor d \ast b \quad \implies \quad u \leq h \lor c \ast b \lor d \ast a. \quad \text{(EC)} \]

The same quasi-identity can be considered also in the language of semirings:

\[ u \leq h \lor ca \quad \& \quad u \leq h \lor db \quad \implies \quad u \leq h \lor cb \lor da. \quad \text{(EC)} \]
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If an idempotent semiring \( R \) satisfies (EC) as a semiring then it satisfies (EC) when viewed as an \( R \)-semimodule and also vice versa.
Quasivariety generated by chains

Let $R$ be an idempotent semiring.

**Lemma**

1. Every linearly ordered left idempotent $R$-semimodule $M$ satisfies (EC).

2. Conversely, if $M$ satisfies (EC) then every ideal $I \in \mathcal{I}(M^+)$ maximal with respect to not containing an element $u$ is linear.
Quasivariety generated by chains

Let $\mathbf{R}$ be an idempotent semiring.

**Lemma**

1. Every linearly ordered left idempotent $\mathbf{R}$-semimodule $\mathbf{M}$ satisfies (EC).
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**Theorem**

Let $\mathbf{M}$ be a left idempotent $\mathbf{R}$-semimodule. T.F.A.E:

1. $\mathbf{M}$ satisfies (EC).
2. $\mathbf{M}$ is embeddable into $\prod_{i \in K} \mathbf{N}_i$ for some family $\{\mathbf{N}_i \mid i \in K\}$ of linearly ordered left idempotent $\mathbf{R}$-semimodules.
Ordinal sum of $R$-semimodules

Let $R$ be an idempotent semiring.

**Definition**
Ordinal sum of $\mathbb{R}$-semimodules

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Ordinal sum of $\mathbb{R}$-semimodules

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- Let $\langle K, \leq \rangle$ be a linearly ordered set and
- $\{M_i \mid i \in K\}$ a family of left idempotent $\mathbb{R}$-semimodules whose left actions are denoted $\star_i$. 
Ordinal sum of $\mathbf{R}$-semimodules

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**Definition**

- Let $\langle K, \leq \rangle$ be a linearly ordered set and
- \{\(M_i \mid i \in K\}\) a family of left idempotent $\mathbf{R}$-semimodules whose left actions are denoted $\star_i$.
- Then the ordinal sum $\bigoplus_{i \in K} M_i$ is a left idempotent $\mathbf{R}$-semimodule, whose underlying join-semilattice is the ordinal sum of $\{M_i^+ \mid i \in K\}$ and its left action is given by

\[
 r \star m = r \star_i m \quad \text{if } m \in M_i.
\]
Holland’s theorem for idempotent semirings

Let \( R \) be an idempotent semiring.

**Theorem**

Let \( M \) be a left idempotent \( R \)-semimodule satisfying (EC). Then

- There is a linearly ordered left id. \( R \)-semimodule \( N \) which is an ordinal sum of its s.i. factors.
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Let \( M \) be a left idempotent \( R \)-semimodule satisfying (EC). Then

- There is a linearly ordered left id. \( R \)-semimodule \( N \) which is an ordinal sum of its s.i. factors.
- If \( M \) has a one-element separating set \( \{ e \} \) then \( N \) has a separating set \( E \) which is dually well ordered.
Holland’s theorem for idempotent semirings

Let $R$ be an idempotent semiring.

**Theorem**

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**Theorem**

The following are equivalent:

1. $R$ satisfies (EC).
2. $R$ is embeddable into $\text{End}(C)$ for some chain $C$.
3. $R$ is embeddable into $R\text{End}(C) \cong \text{Res}(I(C))$ for some chain $C$. 
Outline

1 Motivation
2 Idempotent semirings
3 Idempotent semimodules
4 Holland’s theorem for idempotent semirings
5 FEP for integral idempotent semirings
6 Applications to residuated lattices
   - Conuclei
   - Cayley’s and Holland’s theorem
   - FEP
Finite embeddability property

- Recall that a class $K$ of algebras in the same language has the **FEP** if every finite partial subalgebra is embeddable into a finite member of $K$. 

An idempotent semiring $R$ is said to be integral if $1$ is a top element with respect to the join-semilattice order on $R$. 

We denote the variety of all integral idempotent semirings by $ISR$ and $Q$ its sub-quasivariety axiomatized by $(EC)$. 

Finite embeddability property

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- We denote the variety of all integral idempotent semirings by $\text{ISR}$ and $Q$ its sub-quasivariety axiomatized by (EC).
Theorem

Let $K$ be a subvariety of ISR and $R \in K$ generated by a finite set $C$. Then there is a finite $S \in K$ and a surjective homomorphism $\phi : R \to S$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for all $r \in R$ and $c \in C$. In addition, if $R \in K \cap Q$ then $S \in K \cap Q$ as well.
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Sketch of the proof for $R \in K \cap Q$
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Sketch of the proof for $R \in K \cap Q$

- There is a linearly ordered left id. $R$-semimodule $M$ with a d.w.o. separating set $E$.
- Take a $R$-subsemimodule $N$ of $M$ generated by $E$, i.e., $N = R \star E$. 
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- Then $N$ has is residuated and has ACC (Higman’s lemma).
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- The dual $N^\partial$ is a right idempotent R-semimodule having DCC.
- Take an $R$-subsemimodule $K$ of $N^\partial$ generated by $C \star E$.
- Then $K$ is finite residuated right $R$-semimodule.
Sketch of the proof (cont.)

Hence there is a semiring homomorphism \( \phi : R \rightarrow \text{Res}(K^+) \).

Let \( S = \phi[R] \).

Then \( S \in K \) because varieties are closed under homomorphic images.

Further \( S \in Q \) because \( K^+ \) is linearly ordered.

Moreover \( S \) is finite because \( K \) is finite.

Finally, if \( \phi(r) \leq \partial \phi(c) \) then \( r \leq c \).

Corollary

Let \( K \) be a subvariety of \( \text{ISR} \) and \( Q \) the quasivariety of idempotent semirings axiomatized by \( (EC) \). Then \( K \) and \( K \cap Q \) have the finite embeddability property.
Sketch of the proof (cont.)

- Hence there is a semiring homomorphism \( \phi : R \rightarrow \text{Res}(K^+)^{op} \).
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- Finally, if $\phi(r) \leq^d \phi(c)$ then $r \leq c$.

Corollary

*Let $K$ be a subvariety of ISR and $Q$ the quasivariety of idempotent semirings axiomatized by (EC). Then $K$ and $K \cap Q$ have the finite embeddability property.*
Outline

1. Motivation
2. Idempotent semirings
3. Idempotent semimodules
4. Holland’s theorem for idempotent semirings
5. FEP for integral idempotent semirings
6. Applications to residuated lattices
   - Conuclei
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   - FEP
Residuated lattices

Definition

A **residuated lattice** is an algebra \( A = \langle A, \wedge, \vee, \cdot, /, \backslash, 1 \rangle \), where

- \( \langle A, \wedge, \vee \rangle \) is a lattice,
- \( \langle A, \cdot, 1 \rangle \) is a monoid and
- \( x \cdot y \leq z \) iff \( x \leq z/y \) iff \( y \leq x \backslash z \).
Residuated lattices

Definition

A residuated lattice is an algebra $A = \langle A, \land, \lor, \cdot, /, \setminus, 1 \rangle$, where

- $\langle A, \land, \lor \rangle$ is a lattice,
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Every residuated lattice forms an idempotent semiring.
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Example

- If $L$ is a complete lattice then $\text{Res}(L)$ is a complete residuated lattice.
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Example

- If $L$ is a complete lattice then $\text{Res}(L)$ is a complete residuated lattice.
- If $L$ is a lattice then $\text{REnd}(L) \cong \text{Res}(\mathcal{I}(L))$ is a complete residuated lattice.
Interior operators

Definition
Let $P$ be a poset. A map $\sigma : P \to P$ is called an interior operator if

- $\sigma(x) \leq x$,
- $x \leq y$ implies $\sigma(x) \leq \sigma(y)$,
- $\sigma(\sigma(x)) = \sigma(x)$.
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Observation
Let $L$ be a complete lattice and $S \subseteq L$. Then $S$ induces an interior operator on $L$:

$$\sigma_S(x) = \bigvee \{ s \in S \mid s \leq x \}.$$
Conuclei

Definition

A conucleus $\sigma$ on a residuated lattice $L$ is an interior operator such that $\sigma(x)\sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$. 
Conuclei

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A conucleus $\sigma$ on a residuated lattice $L$ is an interior operator such that $\sigma(x)\sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$.

Theorem

Let $\sigma$ be a conucleus on a residuated lattice $L = \langle L, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$. Then $L_\sigma = \langle \sigma[L], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1 \rangle$ is a residuated lattice called conuclear contraction, where $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x / y = \sigma(x / y)$.
Conuclei

Definition

A conucleus $\sigma$ on a residuated lattice $L$ is an interior operator such that $\sigma(x)\sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$.

Theorem

Let $\sigma$ be a conucleus on a residuated lattice $L = \langle L, \land, \lor, \cdot, \backslash, /, 1 \rangle$. Then $L_\sigma = \langle \sigma[L], \land_\sigma, \lor, \cdot, \backslash_\sigma, /_\sigma, 1 \rangle$ is a residuated lattice called conuclear contraction, where $x \land_\sigma y = \sigma(x \land y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x / y = \sigma(x / y)$.

Lemma

Let $A$ be a complete residuated lattice and $S$ a submonoid of $A$. Then the interior operator $\sigma_S$ on $A$ is a conucleus.
Applications to residuated lattices
Cayley’s and Holland’s theorem

Key lemma

Lemma

Assumptions:

- Let $A, B$ be residuated lattices such that $B$ is complete and $C$ a partial subalgebra of $A$. 
Key lemma

Lemma

Assumptions:

- Let $\mathbf{A}, \mathbf{B}$ be residuated lattices such that $\mathbf{B}$ is complete and $\mathbf{C}$ a partial subalgebra of $\mathbf{A}$.
- Further, let $\mathbf{D}$ be the idempotent subsemiring of $\mathbf{A}$ generated by $\mathbf{C}$.
Key lemma

Lemma

Assumptions:

Let $A, B$ be residuated lattices such that $B$ is complete and $C$ a partial subalgebra of $A$.

Further, let $D$ be the idempotent subsemiring of $A$ generated by $C$.

Suppose that there is a semiring homomorphism $\phi: D \to B$ such that for all $d \in D$ and $c \in C$

$$\phi(d) \leq \phi(c) \implies d \leq c.$$
Key lemma

Lemma

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- Let $A, B$ be residuated lattices such that $B$ is complete and $C$ a partial subalgebra of $A$.
- Further, let $D$ be the idempotent subsemiring of $A$ generated by $C$.
- Suppose that there is a semiring homomorphism $\phi: D \to B$ such that for all $d \in D$ and $c \in C$

$$\phi(d) \leq \phi(c) \implies d \leq c.$$

Conclusions:
- Then $\sigma_{\phi[D]}$ is a conucleus and
- $\phi: C \to B^{\sigma_{\phi[D]}}$ is an embedding of residuated lattices.
Cayley’s theorem for residuated lattices

Theorem

Let $A, B$ be residuated lattices such that $B$ is complete. If $A$ embeds into $B$ via $\phi$ as an idempotent semiring, then $A$ embeds into $B^{\sigma_{\phi}[A]}$ as a residuated lattice.
Cayley’s theorem for residuated lattices

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Corollary (Cayley’s theorem for residuated lattices)

Let $A$ be a residuated lattice and $A^+$ its join-semilattice reduct. Then $A$ embeds into a conuclear contraction of $\text{REnd}(A^+) \cong \text{Res}(\mathcal{I}(A^+))$. In addition, if $A$ is complete then $A$ embeds into a conuclear contraction of $\text{Res}(A^+)$. 
Cayley’s theorem for residuated lattices

**Theorem**

Let $A, B$ be residuated lattices such that $B$ is complete. If $A$ embeds into $B$ via $\phi$ as an idempotent semiring, then $A$ embeds into $B^{\sigma_\phi[A]}$ as a residuated lattice.

**Corollary (Cayley’s theorem for residuated lattices)**

Let $A$ be a residuated lattice and $A^+$ its join-semilattice reduct. Then $A$ embeds into a conuclear contraction of $\text{REnd}(A^+) \cong \text{Res}(\mathcal{I}(A^+))$. In addition, if $A$ is complete then $A$ embeds into a conuclear contraction of $\text{Res}(A^+)$. 

**Theorem (Blount-Tsinakis)**

Every residuated lattice embeds into a nuclear retraction of a powerset monoid.
Holland’s theorem for residuated lattices

Theorem (Holland’s theorem for residuated lattices)

Let $A$ be a residuated lattice. The following are equivalent:

1. $A$ satisfies (EC) (equivalently, $(h \lor ca) \land (h \lor db) \leq h \lor cb \lor da$).
2. $A$ embeds into a conuclear contraction of $\text{REnd}(C)$ for a chain $C$.
3. $A$ embeds into a conuclear contraction of $\text{Res}(C')$ for a complete chain $C'$. 
Classes where Holland’s theorem (does not) applies

- **Prelinear** residuated lattices, i.e., those where
  \[ 1 = (x \setminus y \wedge 1) \lor (y \setminus x \wedge 1) \]
  holds. This class includes all semilinear varieties and \( \ell \)-groups.
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  \[x(y \land z) = xy \land xz\] and
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- There are also **non-distributive integral** residuated lattices satisfying (EC).
Classes where Holland’s theorem (does not) applies

- **Prelinear** residuated lattices, i.e., those where $1 = (x \setminus y \land 1) \lor (y \setminus x \land 1)$ holds. This class includes all semilinear varieties and $\ell$-groups.

- **Commutative cancellative** residuated lattices.

- **Distributive** residuated lattices satisfying $x(y \land z) = xy \land xz$ and $(y \land z)x = yx \land zx$.

- There are also **non-distributive integral** residuated lattices satisfying (EC).

- Let $\mathbb{Z}_2 = \langle \{0, 1\}, +, 0 \rangle$ be the two-element group (ordered discretely). Consider its extension by a top and bottom element $\top, \bot$. Then its lattice reduct is distributive and (EC) does not hold in this extension.
Let IRL be the variety of integral residuated lattices (i.e., $x \leq 1$).

**Theorem**

Let $V_1$ be a subvariety of IRL axiomatized by the set $E$ of identities using only $\lor$, $\cdot$, $1$. Further, let $V_2$ be the subvariety of $V_1$ relatively axiomatized by $(h \lor ca) \land (h \lor db) \leq h \lor cb \lor da$ (i.e., by $(EC)$). Then $V_1$ and $V_2$ have the finite embeddability property.

**Sketch of the proof for** $V_2$

Let $K$ be a subvariety of ISR axiomatized by $E$ and $Q$ the quasivariety of idempotent semirings axiomatized by $(EC)$. Suppose $A \in V_2$. Then its semiring reduct belongs to $K \cap Q$. 

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FEP for integral residuated lattices

Let IRL be the variety of integral residuated lattices (i.e., $x \leq 1$).

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FEP for integral residuated lattices

Let IRL be the variety of integral residuated lattices (i.e., $x \leq 1$).

**Theorem**

1. *Let $V_1$ be a subvariety of IRL axiomatized by the set $\mathcal{E}$ of identities using only $\lor, \cdot, 1$.\*

2. *Further, let $V_2$ be the subvariety of $V_1$ relatively axiomatized by $\left( h \lor ca \right) \land \left( h \lor db \right) \leq h \lor cb \lor da$ (i.e., by $\mathcal{EC}$).\*

3. *Then $V_1$ and $V_2$ have the finite embeddability property.*
FEP for integral residuated lattices

Let IRL be the variety of integral residuated lattices (i.e., $x \leq 1$).

**Theorem**

- Let $V_1$ be a subvariety of IRL axiomatized by the set $\mathcal{E}$ of identities using only $\lor$, $\cdot$, $1$.
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FEP for integral residuated lattices

Let IRL be the variety of integral residuated lattices (i.e., $x \leq 1$).

**Theorem**

- Let $V_1$ be a subvariety of IRL axiomatized by the set $\mathcal{E}$ of identities using only $\lor, \cdot, 1$.
- Further, let $V_2$ be the subvariety of $V_1$ relatively axiomatized by $(h \lor ca) \land (h \lor db) \leq h \lor cb \lor da$ (i.e., by (EC)).
- Then $V_1$ and $V_2$ have the finite embeddability property.

**Sketch of the proof for $V_2$**

- Let $K$ be a subvariety of ISR axiomatized by $\mathcal{E}$ and
FEP for integral residuated lattices

Let IRL be the variety of integral residuated lattices (i.e., \( x \leq 1 \)).

**Theorem**

- Let \( V_1 \) be a subvariety of IRL axiomatized by the set \( \mathcal{E} \) of identities using only \( \lor, \cdot, 1 \).
- Further, let \( V_2 \) be the subvariety of \( V_1 \) relatively axiomatized by \((h \lor ca) \land (h \lor db) \leq h \lor cb \lor da \) (i.e., by \( (EC) \)).
- Then \( V_1 \) and \( V_2 \) have the finite embeddability property.

**Sketch of the proof for \( V_2 \)**

- Let \( K \) be a subvariety of ISR axiomatized by \( \mathcal{E} \) and
- \( Q \) the quasivariety of idempotent semirings axiomatized by \( (EC) \).
FEP for integral residuated lattices

Let IRL be the variety of integral residuated lattices (i.e., $x \leq 1$).

**Theorem**

- Let $V_1$ be a subvariety of IRL axiomatized by the set $\mathcal{E}$ of identities using only $\lor, \cdot, 1$.
- Further, let $V_2$ be the subvariety of $V_1$ relatively axiomatized by $(h \lor ca) \land (h \lor db) \leq h \lor cb \lor da$ (i.e., by (EC)).
- Then $V_1$ and $V_2$ have the finite embeddability property.

**Sketch of the proof for $V_2$**

- Let $K$ be a subvariety of ISR axiomatized by $\mathcal{E}$ and $Q$ the quasivariety of idempotent semirings axiomatized by (EC).
- Suppose $A \in V_2$. Then its semiring reduct belongs to $K \cap Q$. 
Let $C$ be a finite partial subalgebra of $A$. 

Consider the subsemiring $R$ of $A$ generated by $C$. Then $R \in K \cap Q$. By FEP for $K \cap Q$ there is a finite semiring $S \in K \cap Q$ and there is a surjective semiring homomorphism $\phi: R \rightarrow S$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for all $r \in R$ and $c \in C$. Since $A$ is integral, it is possible to show that $S$ is in fact a finite residuated lattice. Thus $C$ embeds as a residuated lattice into $S_\sigma$ for some conucleus $\sigma$. Since $S_\sigma$ is a subsemiring of $S$, $S_\sigma$ is finite and belongs to $K \cap Q$. 

Sketch of the proof (cont.)
Let $C$ be a finite partial subalgebra of $A$.
Consider the subsemiring $R$ of $A$ generated by $C$. 
Sketch of the proof (cont.)

- Let $\mathbf{C}$ be a finite partial subalgebra of $\mathbf{A}$.
- Consider the subsemiring $\mathbf{R}$ of $\mathbf{A}$ generated by $\mathbf{C}$.
- Then $\mathbf{R} \in \mathbb{K} \cap \mathbb{Q}$.

Since $\mathbf{A}$ is integral, it is possible to show that $\mathbf{S}$ is in fact a finite residuated lattice. Thus $\mathbf{C}$ embeds as a residuated lattice into $\mathbf{S}_\sigma$ for some conucleus $\sigma$. Since $\mathbf{S}_\sigma$ is a subsemiring of $\mathbf{S}$, $\mathbf{S}_\sigma$ is finite and belongs to $\mathbb{K} \cap \mathbb{Q}$.
Applications to residuated lattices

Sketch of the proof (cont.)

- Let $C$ be a finite partial subalgebra of $A$.
- Consider the subsemiring $R$ of $A$ generated by $C$.
- Then $R \in K \cap Q$.
- By FEP for $K \cap Q$ there is a finite semiring $S \in K \cap Q$ and
Let $\mathbf{C}$ be a finite partial subalgebra of $\mathbf{A}$.
Consider the subsemiring $\mathbf{R}$ of $\mathbf{A}$ generated by $\mathbf{C}$.
Then $\mathbf{R} \in K \cap \mathbb{Q}$.
By FEP for $K \cap \mathbb{Q}$ there is a finite semiring $\mathbf{S} \in K \cap \mathbb{Q}$ and there is a surjective semiring homomorphism $\phi: \mathbf{R} \to \mathbf{S}$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for all $r \in R$ and $c \in C$. 

Since $\mathbf{A}$ is integral, it is possible to show that $\mathbf{S}$ is in fact a finite residuated lattice.
Thus $\mathbf{C}$ embeds as a residuated lattice into $\mathbf{S}$ via some conucleus $\sigma$.
Let $\mathbf{C}$ be a finite partial subalgebra of $\mathbf{A}$.

Consider the subsemiring $\mathbf{R}$ of $\mathbf{A}$ generated by $\mathbf{C}$.

Then $\mathbf{R} \in \mathbf{K} \cap \mathbf{Q}$.

By FEP for $\mathbf{K} \cap \mathbf{Q}$ there is a finite semiring $\mathbf{S} \in \mathbf{K} \cap \mathbf{Q}$ and there is a surjective semiring homomorphism $\phi: \mathbf{R} \to \mathbf{S}$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for all $r \in \mathbf{R}$ and $c \in \mathbf{C}$.

Since $\mathbf{A}$ is integral, it is possible to show that $\mathbf{S}$ is in fact a finite residuated lattice.
Let $C$ be a finite partial subalgebra of $A$.
Consider the subsemiring $R$ of $A$ generated by $C$.
Then $R \in K \cap Q$.
By FEP for $K \cap Q$ there is a finite semiring $S \in K \cap Q$ and
there is a surjective semiring homomorphism $\phi: R \rightarrow S$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for all $r \in R$ and $c \in C$.
Since $A$ is integral, it is possible to show that $S$ is in fact a finite residuated lattice.
Thus $C$ embeds as a residuated lattice into $S_\sigma$ for some conucleus $\sigma$. 
Let \( C \) be a finite partial subalgebra of \( A \).

Consider the subsemiring \( R \) of \( A \) generated by \( C \).

Then \( R \in K \cap Q \).

By FEP for \( K \cap Q \) there is a finite semiring \( S \in K \cap Q \) and there is a surjective semiring homomorphism \( \phi: R \to S \) such that \( \phi(r) \leq \phi(c) \) implies \( r \leq c \) for all \( r \in R \) and \( c \in C \).

Since \( A \) is integral, it is possible to show that \( S \) is in fact a finite residuated lattice.

Thus \( C \) embeds as a residuated lattice into \( S_\sigma \) for some conucleus \( \sigma \)

Since \( S_\sigma \) is a subsemiring of \( S \), \( S_\sigma \) is finite and belongs to \( K \cap Q \).
Thank you!