Residuated Lattices, Regular Languages, and Burnside Problem

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Topology, Algebra, and Categories in Logic
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Outline

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Residuated lattices

Definition

Let $M = \langle M, \cdot, 1 \rangle$ be a monoid. A quasi-order $\leq$ on $M$ is called compatible if for all $x, y, u, v \in M$:

$$x \leq y \implies uxv \leq uyv.$$
Residuated lattices

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Definition
A residuated lattice $A = \langle A, \wedge, \vee, \cdot, \setminus, /, 1 \rangle$ is a monoid such that $\langle A, \wedge, \vee \rangle$ is a lattice and for all $a, b, c \in A$:

$$a \cdot b \leq c \iff b \leq a \setminus c \iff a \leq c / b.$$
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Let \( M = \langle M, \cdot, 1 \rangle \) be a monoid. A quasi-order \( \leq \) on \( M \) is called compatible if for all \( x, y, u, v \in M \):

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Definition

A residuated lattice \( A = \langle A, \land, \lor, \cdot, \backslash, /, 1 \rangle \) is a monoid such that \( \langle A, \land, \lor \rangle \) is a lattice and for all \( a, b, c \in A \):

\[
a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c / b.
\]

Logic = a substructural logic, i.e., an axiomatic extension of FL.
Powerset monoid

Example
Let $M = \langle M, \cdot, 1 \rangle$ be a monoid. Then

$$\mathcal{P}(M) = \langle \mathcal{P}(M), \cap, \cup, \cdot, \setminus, \cup, \{1\} \rangle$$

is a residuated lattice, where

- $X \cdot Y = \{ xy \in M \mid x \in X, y \in Y \}$,
- $X \setminus Z = \{ y \in M \mid X \cdot \{y\} \subseteq Z \}$,
- $Z / Y = \{ x \in M \mid \{x\} \cdot Y \subseteq Z \}$.
Powerset monoid

Example

Let $M = \langle M, \cdot, 1 \rangle$ be a monoid. Then

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Other examples can be obtained by introducing a suitable closure operator on $\mathcal{P}(M)$. 
Nuclei

Definition

Let \( M \) be a monoid and \( \gamma \) a closure operator on \( \mathcal{P}(M) \). The collection of \( \gamma \)-closed sets is denoted \( \mathcal{P}(M)_\gamma \). Then \( \gamma \) is called a nucleus if for every \( u, v \in M \) we have

\[
X \in \mathcal{P}(M)_\gamma \implies \{u\} \setminus X/\{v\} \in \mathcal{P}(M)_\gamma.
\]
Nuclei

Definition
Let $M$ be a monoid and $\gamma$ a closure operator on $\mathcal{P}(M)$. The collection of $\gamma$-closed sets is denoted $\mathcal{P}(M)_{\gamma}$. Then $\gamma$ is called a nucleus if for every $u, v \in M$ we have

$$X \in \mathcal{P}(M)_{\gamma} \implies \{u\} \setminus X/\{v\} \in \mathcal{P}(M)_{\gamma}.$$ 

Example
Let $M$ be a monoid and $\gamma$ a nucleus on $\mathcal{P}(M)$. Then $\mathcal{P}(M)_{\gamma} = \langle \mathcal{P}(M)_{\gamma}, \cap, \cup, \cdot, \setminus, \cup, \gamma \{1\} \rangle$ is a residuated lattice, where

$$X \cup_{\gamma} Y = \gamma(X \cup Y),$$

$$X \cdot_{\gamma} Y = \gamma(X \cdot Y).$$
### Regular languages

#### Definition

A language $L \subseteq \Sigma^*$ is called **regular** iff it is accepted by a finite automaton.
Regular languages

Definition

A language $L \subseteq \Sigma^*$ is called regular iff it is accepted by a finite automaton.
**Syntactic monoid**

**Definition**
Given an alphabet $\Sigma$ and a language $L \subseteq \Sigma^*$, we define

1. **syntactic congruence:** $x \sim_L y$ iff $(\forall u, v \in \Sigma^*) (uxv \in L \iff uyv \in L)$,
2. **syntactic monoid:** $M(L) = \Sigma^*/\sim_L$.

**Theorem**
1. The syntactic congruence $\sim_L$ is the largest congruence saturating $L$, i.e., $L = \bigcup_{w \in L} w/\sim_L$.
2. $M(L)$ is finite iff $L$ is regular (Myhill-Nerode Theorem).

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Definition
Given a logic $L$, we define
Lindenbaum–Tarski algebra

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1. Leibniz congruence:

   $\alpha \sim_L \beta \iff (\forall \varphi \in Fm)(\vdash_L \varphi(\alpha) \iff \vdash_L \varphi(\beta))$, 

Theorem

Leibniz congruence $\sim_L$ is the largest congruence saturating the set of theorems of $L$. 

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Leibniz congruence \( \sim_L \) is the largest congruence saturating the set of theorems of \( L \).
Eilenberg variety theorem

The assignment $L \mapsto M(L)$ induces a correspondence between varieties of regular languages and pseudovarieties of finite monoids.

- $\mathcal{L} \mapsto$ the pseudovariety generated by $\{M(L) \mid L \in \mathcal{L}\}$.
- $\mathcal{V} \mapsto$ the variety $\mathcal{L}$ of regular languages $L$ s.t. $M(L) \in \mathcal{V}$. 

Theorem (Eilenberg 1976)

The above maps are mutually inverse, order-preserving bijections.

Theorem

Let $L$ be a logic. The map $L \mapsto Fm/\sim_L$ induces a dual-isomorphism between the lattice of axiomatic extensions of $L$ and the subvariety lattice of the variety generated by $Fm/\sim_L$. 

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Eilenberg variety theorem

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## Analogy table

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- Lindenbaum-Tarski algebra is used to prove the completeness theorem for a logic $L$.
- Nevertheless, there is also another construction used in order to prove it.
- Does it have its analogy on the language side?
Another way of proving completeness

- Let $L$ be a logic presented by a single-conclusion sequent calculus.
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- Look for the pointwise largest nucleus $\gamma$ on $\mathcal{P}(Fm^*)$ making the following set $\gamma$-closed for every $\varphi \in Fm$:

$$S_\varphi = \{ \Gamma \in Fm^* \mid \vdash_L \Gamma \Rightarrow \varphi \}.$$
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- Then $\mathcal{P}(Fm^*)_\gamma$ is the algebra used to prove the completeness theorem.
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- Then $\mathcal{P}(Fm^*)_\gamma$ is the algebra used to prove the completeness theorem.

- If $\delta$ is a nucleus on $\mathcal{P}(Fm^*)$ making all $S_\varphi$’s $\delta$-closed then $\delta(X) \subseteq \gamma(X)$ for all $X \subseteq Fm^*$. 
Definition

Let \( L \subseteq \Sigma^* \) be a language. The pointwise largest nucleus \( \gamma_L \) making \( L \) a closed set is called syntactic nucleus. Then \( R(L) = \mathcal{P}(\Sigma^*)_{\gamma_L} \) is called a syntactic residuated lattice.
Syntactic residuated lattice

**Definition**

Let $L \subseteq \Sigma^*$ be a language. The pointwise largest nucleus $\gamma_L$ making $L$ a closed set is called syntactic nucleus. Then $R(L) = \mathcal{P}(\Sigma^*)_{\gamma_L}$ is called a syntactic residuated lattice.

**Theorem**

Theorem 1: $\{\gamma_{\{x\}} | x \in \Sigma^*\}$ forms a submonoid isomorphic to the syntactic monoid $M(L)$.

Theorem 2: $R(L)$ is finite iff $L$ is regular.
Syntactic residuated lattice

Definition
Let \( L \subseteq \Sigma^* \) be a language. The pointwise largest nucleus \( \gamma_L \) making \( L \) a closed set is called syntactic nucleus. Then \( R(L) = \mathcal{P}(\Sigma^*)\gamma_L \) is called a syntactic residuated lattice.

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1. \( \{\gamma\{x\} \mid x \in \Sigma^*\} \) forms a submonoid isomorphic to the syntactic monoid \( M(L) \).
**Definition**

Let $L \subseteq \Sigma^*$ be a language. The pointwise largest nucleus $\gamma_L$ making $L$ a closed set is called **syntactic nucleus**. Then $R(L) = \mathcal{P}(\Sigma^*)_\gamma_L$ is called a **syntactic residuated lattice**.

**Theorem**

1. $\{\gamma\{x\} \mid x \in \Sigma^*\}$ forms a submonoid isomorphic to the syntactic monoid $M(L)$.

2. $R(L)$ is **finite** iff $L$ is **regular**.
Is it good for something?

Syntactic monoids were mainly applied in the realm of regular languages.

Beyond regular languages – they do not contain sufficiently enough information to distinguish very different languages.

---

Example (Sakarovitch)

Consider the following languages over \( \Sigma = \{0, 1\} \):

\[
L_1 = \{ww^R \mid w \in \Sigma^* \},
\]

\[
L_2 = \{w \in \Sigma^* \mid w \text{ is prime} \}.
\]

Then \( M(L_1) = M(L_2) = \Sigma^* \).
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Beyond regular languages

Consider the following rule over:

\[ u x v, u x^2 v \in L \implies u v \in L. \quad (r) \]

Then \( L_1 \) is closed under \((r)\) and \( L_2 \) not.
Beyond regular languages

Consider the following rule over:

\[ uxv, ux^2v \in L \implies uv \in L. \]  \hspace{1cm} (r)

Then \( L_1 \) is closed under (r) and \( L_2 \) not.

Theorem

A language \( L \) is closed under (r) iff \( R(L) \) satisfies

\[ 1 \leq x \lor x^2 \lor x \setminus y. \]

Thus the languages \( L_1, L_2 \) can be separated by a variety of residuated lattices.
How to construct the largest nucleus?

Let $M$ be a monoid and $B = \{ S_i \subseteq M \mid i \in I \}$. 

Use residuated frames (Galatos, Jipsen).
How to construct the largest nucleus?

- Let $M$ be a monoid and $B = \{S_i \subseteq M \mid i \in I\}$.
- How to find the largest nucleus on $\mathcal{P}(M)$ making all sets in $B$ closed?
How to construct the largest nucleus?

Let $M$ be a monoid and $B = \{S_i \subseteq M \mid i \in I\}$.

How to find the largest nucleus on $P(M)$ making all sets in $B$ closed?

Use residuated frames (Galatos, Jipsen).
Frames

A frame $\mathbf{W} = \langle M, B, N \rangle$:  

\[ M \quad \frac{N}{\ } \quad B \]
A frame $\mathbf{W} = \langle M, B, N \rangle$:

$$\begin{array}{ccc}
M & \text{N} & B \\
\uparrow & & \uparrow \\
\mathcal{P}(M) & \bowtie & \mathcal{P}(B)
\end{array}$$

$$X^\triangleright = \{ b \in B \mid (\forall a \in X)(a N b) \},$$

$$Y^\triangleleft = \{ a \in M \mid (\forall b \in Y)(a N b) \}.$$
Frames

A frame $\mathbf{W} = \langle M, B, N \rangle$:

$M \xrightarrow{N} B$

$\mathcal{P}(M) \xrightarrow{\triangle} \mathcal{P}(B)$

$X^{\triangleright} = \{ b \in B \mid (\forall a \in X)(a N b) \}$,

$Y^{\triangleleft} = \{ a \in M \mid (\forall b \in Y)(a N b) \}$.

$\gamma(X) = X^{\triangleright\triangleleft}$ is a closure operator on $\mathcal{P}(M)$. 

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Frames

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\text{frames} \\
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- $X^\triangledown = \{ b \in B \mid (\forall a \in X)(a \mathcal{N} b) \}$,
- $Y^\triangleleft = \{ a \in M \mid (\forall b \in Y)(a \mathcal{N} b) \}$.

- $\gamma(X) = X^{\triangledown\triangleleft}$ is a closure operator on $\mathcal{P}(M)$.
- It is the pointwise largest closure operator making all sets in its basis $\{ \{ b \}^\triangleleft \mid b \in B \}$ $\gamma$-closed.
A frame $\mathbf{W} = \langle M, B, N \rangle$:

$$\begin{align*}
\mathcal{P}(M) & \xymatrix{ & N \ar[dl] \ar[dr] & \\
M & & B}
\end{align*}$$

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- It is the pointwise largest closure operator making all sets in its basis $\{ \{ b \}^\triangledown \mid b \in B \}$ $\gamma$-closed.
- The collection of closed sets forms a complete lattice $\mathbf{W}^+ = \langle \mathcal{P}(M)_\gamma, \cap, \cup_\gamma \rangle$, where

$$X \cup_\gamma Y = \gamma(X \cup Y).$$
Residuated frames

- Given a monoid $M$ and an frame $W = \langle M, B, N \rangle$, the corresponding induced closure operator $\gamma$ need not be a nucleus.

\[
\hat{W} = \langle M, M^2 \times B, \hat{N} \rangle,
\]
where $x \hat{N} \langle u, v, b \rangle$ iff $uxv N b$.

The closure operator $\gamma$ induced by $\hat{N}$ is a nucleus.

Then $\hat{W}^+$ forms a complete residuated lattice.

Moreover, $\gamma$ is the pointwise largest nucleus making all $\{1, 1, b\} \triangleleft \gamma$'s.
Residuated frames

- Given a monoid $M$ and an frame $W = \langle M, B, N \rangle$, the corresponding induced closure operator $\gamma$ need not be a nucleus.

- Define an extended (residuated) frame $\hat{W} = \langle M, M^2 \times B, \hat{N} \rangle$, where $x \hat{N} \langle u, v, b \rangle$ iff $uxv \subseteq b$.

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Residuated frames

- Given a monoid $M$ and an frame $\mathcal{W} = \langle M, B, N \rangle$, the corresponding induced closure operator $\gamma$ need not be a nucleus.

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- The closure operator $\gamma$ induced by $\hat{N}$ is a nucleus.

- Then $\hat{\mathcal{W}}^+ = \mathcal{P}(M)_\gamma$ forms a complete residuated lattice.

- Moreover, $\gamma$ is the pointwise largest nucleus making all $\{1, 1, b\}^\downarrow$’s $\gamma$-closed.
Construction of $R(L)$

- Let $L$ be a logic and consider the frame $W = \langle Fm^*, Fm, N \rangle$ where

$$\Gamma N \varphi \iff \Gamma \in S\varphi \iff \vdash_L \Gamma \Rightarrow \varphi.$$ 

Then $\hat{W}^+$ is the algebra used to prove the completeness.
Construction of $R(L)$

- Let $L$ be a logic and consider the frame $\mathcal{W} = \langle Fm^*, Fm, N \rangle$ where
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  Then $\hat{\mathcal{W}}^+$ is the algebra used to prove the completeness.

- Let $L \subseteq \Sigma^*$ be a language. Define frame $\mathcal{W} = \langle \Sigma^*, \{L\}, N \rangle$, where $N \subseteq \Sigma^* \times \{L\}$ is defined by
  \[ x \in N L \text{ iff } x \in L. \]
  Then $R(L) = \hat{\mathcal{W}}^+$ is the syntactic residuated lattice of $L$. 
A class of algebras $\mathcal{K}$ of the same type has the **finite embeddability property** (FEP) if every finite partial subalgebra $\mathbf{B}$ of any algebra $\mathbf{A} \in \mathcal{K}$ is embeddable into a finite algebra $\mathbf{D} \in \mathcal{K}$.
FEP

Definition (Evans)

A class of algebras $\mathcal{K}$ of the same type has the finite embeddability property (FEP) if every finite partial subalgebra $B$ of any algebra $A \in \mathcal{K}$ is embeddable into a finite algebra $D \in \mathcal{K}$.

- Let $\mathcal{K}$ be a variety of residuated lattices.
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- Let $\mathcal{K}$ be a variety of residuated lattices.
- Start with $A \in \mathcal{K}$ and a finite $B \subseteq A$. 
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- Start with $A \in \mathcal{K}$ and a finite $B \subseteq A$.
- Let $M$ be the sub(po)monoid of $A$ generated by $B$. 

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- Let $\mathcal{K}$ be a variety of residuated lattices.
- Start with $A \in \mathcal{K}$ and a finite $B \subseteq A$.
- Let $M$ be the sub(po)monoid of $A$ generated by $B$.
- Consider the frame $\mathcal{W} = \langle M, B, N \rangle$ where
  \[ x \leq N b \iff x \leq^A b. \]
FEP

Definition (Evans)
A class of algebras $\mathcal{K}$ of the same type has the **finite embeddability property** (FEP) if every finite partial subalgebra $B$ of any algebra $A \in \mathcal{K}$ is embeddable into a finite algebra $D \in \mathcal{K}$.

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- Then $\hat{\mathcal{W}}^+$ is a residuated lattice and $B$ embeds to it.
- Is $\hat{\mathcal{W}}^+$ finite? Does $\hat{\mathcal{W}}^+$ belong to $\mathcal{K}$?
Generalized Myhill Theorem

**Theorem**

Let $\mathbf{M}$ be a monoid and $\mathbf{W} = \langle \mathbf{M}, B, N \rangle$ a frame where $B$ is finite. Then $\hat{\mathbf{W}}^+$ is finite iff there is a compatible dual well quasi-order $\sqsubseteq$ on $\mathbf{M}$ such that

$$x \sqsubseteq y, \ y \vdash_N b \implies x \vdash_N b.$$
**Generalized Myhill Theorem**

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**Corollary (Generalized Myhill Theorem – Ehrenfeucht, Rozenberg)**

A language $L \subseteq \Sigma^*$ is regular iff $L$ is downward closed w.r.t. a compatible dual well quasi-order on $\Sigma^*$. 
Generalized Myhill Theorem

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**Corollary (Generalized Myhill Theorem – Ehrenfeucht, Rozenberg)**

A language $L \subseteq \Sigma^*$ is regular iff $L$ is downward closed w.r.t. a compatible dual well quasi-order on $\Sigma^*$.

$\mathcal{V} = \text{finitely gen. subpomonoids of members from } \mathcal{K}$

Find a compatible dual well quasi-order $\sqsubseteq$ on $\Sigma^*$ s.t. all pomonoids from $\mathcal{V}$ are homomorphic images of $\Sigma^*/\sqsubseteq$. 
Weakening rule

Theorem (Blok, van Alten)

The variety of integral residuated lattices \((x \leq 1)\) has the FEP.

Proof.
Consider the least compatible quasi-order \(\sqsubseteq\) on \(\Sigma^*\) such that \(\Sigma^*/\sqsubseteq\) satisfies \(x \leq 1\). Show by Higman's lemma that \(\sqsubseteq\) is dually well.

Theorem
Every language \(L\) closed under the following rule is regular:

\[
uv \in L \quad \text{and} \quad uxv \in L.
\]
Weakening rule

**Theorem (Blok, van Alten)**

*The variety of integral residuated lattices* (*x* ≤ 1) *has the FEP.*

**Proof.**

Consider the least compatible quasi-order ⊑ on Σ* such that Σ*/⊑ satisfies *x* ≤ 1. Show by Higman’s lemma that ⊑ is dually well.

□
Weakening rule

Theorem (Blok, van Alten)

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Consider the least compatible quasi-order \(\sqsubseteq\) on \(\Sigma^*\) such that \(\Sigma^*/\sqsubseteq\) satisfies \(x \leq 1\). Show by Higman’s lemma that \(\sqsubseteq\) is dually well. \(\square\)

Theorem

Every language \(L\) closed under the following rule is regular:

\[
\frac{uv \in L}{uxv \in L}.
\]
Exchange and knotted rules

Let $m \geq 1$, $n \geq 0$ and $m \neq n$.

**Theorem (van Alten)**

The variety of commutative $(xy = yx)$ residuated lattices satisfying $x^m \leq x^n$ has the FEP.
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**Theorem**

Every language closed under the following rules is regular:

- $uxyv \in L$
- $ux^n v \in L$
- $uyxv \in L$
- $ux^m v \in L$.
Inspired by language theory

Theorem (de Luca, Varricchio)

Language \( L \) is regular iff \( L \) is permutable and quasi-periodic or co-quasi-periodic.

Let \( \sigma \in S_k \setminus \{id\} \) for \( k \geq 2 \) and \( m, n \in \mathbb{N} \) such that \( m > n \geq 1 \).

In particular, they prove that the least compatible quasi-order \( \sqsubseteq \) on \( \Sigma^+ \) such that \( \Sigma^+ / \sqsubseteq \) satisfies \( x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)} \) and \( x^m \leq x^n \) is dually well.
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In particular, they prove that the least compatible quasi-order $\sqsubseteq$ on $\Sigma^+$ such that $\Sigma^+ / \sqsubseteq$ satisfies $x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$ and $x^m \leq x^n$ is dually well.

**Theorem**

Let $\sigma \in S_k \setminus \{id\}$ for $k \geq 2$ and $m, n \in \mathbb{N}$ such that $m > n \geq 1$. Then the variety of resideduated lattice-ordered semigroups axiomatized by $x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$ and $x^m \leq x^n$ has the FEP.
Knotted axioms

Let $m, n \geq 1$ and $m \neq n$. The variety of res. lattices defined by $x^m \leq x^n$ is denoted $RL^n_m$. 

Theorem

The word problem for $RL^n_m$ is undecidable for $1 \leq m < n$ and $2 \leq n < m$.

Thus $RL^n_m$ does not have the FEP.

Theorem

There is an undecidable language $L$ closed under the following rule:

$ux^2v \in L \quad uxv \in L,$

$\{u^{\alpha\beta}v \in L\}$

$\alpha, \beta \in \{x, y, z\}$

$uxyzv \in L$.

The only remaining cases are

$x^m \leq x$ for $m \geq 2$. 

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Knotted axioms

Let $m, n \geq 1$ and $m \neq n$. The variety of res. lattices defined by $x^m \leq x^n$ is denoted $\mathcal{RL}^n_m$.

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The word problem for $\mathcal{RL}^n_m$ is undecidable for $1 \leq m < n$ and $2 \leq n < m$. Thus $\mathcal{RL}^n_m$ does not have the FEP.
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There is an undecidable language $L$ closed under the following rule:

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\begin{align*}
ux^2v \in L & \quad \Rightarrow \quad \exists \alpha, \beta \in \{x, y, z\} \quad \{u\alpha\beta v \in L\}
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\]

\[
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uxv \in L \quad & \quad \Rightarrow \quad \{u\alpha\beta v \in L\}_{\alpha, \beta \in \{x, y, z\}}
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Let $m, n \geq 1$ and $m \neq n$. The variety of res. lattices defined by $x^m \leq x^n$ is denoted $\mathcal{RL}_m^n$.

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The word problem for $\mathcal{RL}_m^n$ is undecidable for $1 \leq m < n$ and $2 \leq n < m$. Thus $\mathcal{RL}_m^n$ does not have the FEP.

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uxyzv & \in L
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The only remaining cases are $x^m \leq x$ for $m \geq 2$. 

Partial order

Let $m \geq 2$. The variety $RL^1_m$ can be axiomatized by

$$ux_1v \leq z \; \& \; \ldots \; \& \; ux_mv \leq z \implies ux_1 \cdots x_mv \leq z. \quad (q_m)$$
Partial order

Let $m \geq 2$. The variety $\mathcal{RL}_m^1$ can be axiomatized by

\[
ux_1 v \leq z \land \ldots \land ux_m v \leq z \quad \implies \quad ux_1 \cdots x_m v \leq z. 
\]

$(q_m)$

Consider subsets of $\Sigma^*$ closed under the following rule:

\[
\frac{ux_1 v \in L \land \ldots \land ux_m v \in L}{ux_1 \cdots x_m v \in L}.
\]

$(r_m)$
Partial order

Let \( m \geq 2 \). The variety \( \mathcal{RL}^1_m \) can be axiomatized by

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Consider subsets of \( \Sigma^* \) closed under the following rule:

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\frac{ux_1 v \in L \ldots ux_m v \in L}{ux_1 \cdots x_m v \in L}. \quad (r_m)
\]

This rule induces a nucleus \( \gamma_m \) on \( \mathcal{P}(\Sigma^*) \). Define the following binary relation on \( \Sigma^* \):

\[
x \leq_m y \quad \text{iff} \quad \gamma_m \{x\} \subseteq \gamma_m \{y\}.
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Partial order

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$$ux_1 v \leq z \& \ldots \& ux_m v \leq z \implies ux_1 \cdots x_m v \leq z. \quad (q_m)$$

Consider subsets of $\Sigma^*$ closed under the following rule:

$$\frac{ux_1 v \in L \ldots ux_m v \in L}{ux_1 \cdots x_m v \in L}. \quad (r_m)$$

This rule induces a nucleus $\gamma_m$ on $\mathcal{P}(\Sigma^*)$. Define the following binary relation on $\Sigma^*$:

$$x \leq_m y \text{ iff } \gamma_m \{x\} \subseteq \gamma_m \{y\}.$$ 

Lemma

The relation $\leq_m$ is the least compatible quasi-order on $\Sigma^*$ such that $\Sigma^*/\leq_m$ satisfies $(q_m)$.
Burnside problem

Let $G_m$ be the variety of groups satisfying $x^m = 1$. 
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Problem (Burnside)

Given $m \in \mathbb{N}$, is $G_m$ locally finite?
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Given $m \in \mathbb{N}$, is $\mathcal{G}_m$ locally finite?

Theorem (Burnside, Sanov, Hall)

The answer is affirmative for $m = 1, 2, 3, 4, 6$.

Theorem (Adian)

The answer is negative for odd $m \geq 665$.

Theorem (Ivanov)

The answer is negative for $m \geq 248$. 

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**Theorem (Ivanov)**

*The answer is negative for $m \geq 2^{48}$.***
Three implications

Theorem

Let $m > 1$. Suppose that $\leq_m$ is dual well partial order. Then

Corollary

The partial order $\leq_m$ is not dual well for even $m \geq 666$ and $m \geq 248$. 

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Three implications

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Let $m > 1$. Suppose that $\leq_m$ is dual well partial order. Then

1. Burnside problem for $m - 1$ has an affirmative answer.

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The partial order $\leq_m$ is not dual well for even $m \geq 666$ and $m \geq 248$. 
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Theorem

Let $m > 1$. Suppose that $\leq_m$ is dual well partial order. Then

1. Burnside problem for $m - 1$ has an affirmative answer.
2. The variety $\mathcal{RL}_m^n$ has the FEP.
Three implications

Theorem

Let $m > 1$. Suppose that $\leq_m$ is dual well partial order. Then

1. Burnside problem for $m - 1$ has an affirmative answer.
2. The variety $\mathcal{RL}_m^n$ has the FEP.
3. Every language closed under the following rule is regular:

$$
ux_1v \in L \cdots ux_mv \in L \\
\frac{u \cdot x_1 \cdots \cdot x_m v \in L}{u x_1 \cdots x_m v \in L}.
$$
Three implications

Theorem

Let $m > 1$. Suppose that $\leq_m$ is dual well partial order. Then

1. Burnside problem for $m - 1$ has an affirmative answer.
2. The variety $\mathcal{RL}_m^n$ has the FEP.
3. Every language closed under the following rule is regular:

$$ux_1v \in L \quad \ldots \quad ux_mv \in L \quad \Rightarrow \quad ux_1 \cdots x_mv \in L.$$ 

Corollary

The partial order $\leq_m$ is not dual well for even $m \geq 666$ and $m \geq 2^{48}$. 

Mingle rule

Theorem

*The variety $RL^1_2$ has the FEP.*
Mingle rule

Theorem

The variety $\mathcal{RL}_2^1$ has the FEP.

Theorem

Every language $L \subseteq \Sigma^*$ closed under the following rule is regular:

$$
\frac{uxv \in L \quad uyv \in L}{uxyv \in L}.
$$

$(r_2)$

Example

The language $a + (b(a + b + c))^* b + b c$ is closed under $(r_2)$. 

Lemma

Let $w \in \Sigma^*$ and $\text{Alph}(w) = \Gamma$. Then $ww \leq 2^w$ for every $u \in \Gamma^*$. 

Rostislav Horčík (ICS)
Mingle rule

Theorem

The variety $R\mathcal{L}_2^1$ has the FEP.

Theorem

Every language $L \subseteq \Sigma^*$ closed under the following rule is regular:

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\frac{uxv \in L \quad uyv \in L \quad uxyv \in L}{uxyv \in L}.
\]  

(r2)

Example

The language $a^+(b(a + b + c)^*b + b)c^+$ is closed under (r2).
Mingle rule

Theorem
The variety $RL_2^1$ has the FEP.

Theorem
Every language $L \subseteq \Sigma^*$ closed under the following rule is regular:

$$uxv \in L \quad uyv \in L \quad \frac{uxyv \in L}{u \in \Gamma}.$$ 

Example
The language $a^+(b(a + b + c)^*b + b)c^+$ is closed under $(r_2)$.

Lemma
Let $w \in \Sigma^*$ and $\text{Alph}(w) = \Gamma$. Then $wuw \leq_2 w$ for every $u \in \Gamma^*$.
Higman’s lemma

Definition

Let \( \langle Q, \leq \rangle \) be a quasi-ordered set. Define a binary relation \( \leq^* \) on \( Q^* \) by

\[
\begin{align*}
\quad a_1 \ldots a_n \leq^* b_1 \ldots b_m \text{ iff there is a strictly increasing map } \\
f : [1, n] \to [1, m] \text{ s.t. } a_i \leq b_{f(i)} \text{ for all } i \in [1, n].
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Lemma (Higman’s lemma)
If $\langle Q, \leq \rangle$ is a well quasi-ordered set then so is $\langle Q^*, \leq^* \rangle$. 

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Higman’s lemma

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Lemma (Higman’s lemma)

If \( \langle Q, \leq \rangle \) is a well quasi-ordered set then so is \( \langle Q^*, \leq^* \rangle \).
Modified Higman’s lemma

**Definition**

Let $\langle Q, \leq \rangle$ be a quasi-ordered set. Define a binary relation $\leq^+$ on $Q^+$ by

\[
a_1 \ldots a_n \leq^+ b_1 \ldots b_m \text{ iff there is a strictly increasing map } f : [1, n + 1] \to [1, m + 1] \text{ such that }
\]

- $f(1) = 1$ and $f(n + 1) = m + 1$,
- $a_i \leq b_{f(i)}$ and $a_i \leq b_{f(i+1)} - 1$ for all $i \in [1, n]$. 

Definition

Let \( \langle Q, \leq \rangle \) be a quasi-ordered set. Define a binary relation \( \leq^+ \) on \( Q^+ \) by

\[
a_1 \ldots a_n \leq^+ b_1 \ldots b_m \iff \text{there is a strictly increasing map } f : [1, n + 1] \rightarrow [1, m + 1] \text{ such that}
\]

- \( f(1) = 1 \) and \( f(n + 1) = m + 1 \),
- \( a_i \leq b_{f(i)} \) and \( a_i \leq b_{f(i+1)−1} \) for all \( i \in [1, n] \).
Modified Higman’s lemma

Definition

Let \( \langle Q, \leq \rangle \) be a quasi-ordered set. Define a binary relation \( \leq^+ \) on \( Q^+ \) by

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& f(1) = 1 \text{ and } f(n + 1) = m + 1, \\
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\end{align*}
\]
Modified Higman’s lemma (cont.)

Lemma

If \( \langle Q, \leq \rangle \) is a well quasi-ordered set then \( \langle Q^+, \leq^+ \rangle \) forms a well quasi-ordered set as well.
Conclusion

- Is it interesting for people working in substructural logics?
Conclusion

- Is it interesting for people working in substructural logics?
- Could it be interesting for people working in language theory?
**Conclusion**

- Is it interesting for people working in substructural logics?

- Could it be interesting for people working in language theory?

- Is the compatible quasi-order $\leq_m$ on $\Sigma^*$ dually well for $m = 3, 4, 5, \ldots, 665, 667, 669, \ldots, 2^{48} - 1$?
Thank you!