Positive Fragment of MTL with One Variable and Its Computational Complexity

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Motivation

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- For instance, we still do not known the complexity of one of the most prominent fuzzy logic MTL = FL_{ew} plus prelinearity.
- One way how to approach this problem is to look at various fragments and discuss their complexity.
- In this talk, we concentrate on positive fragment of MTL (MTL^+) with only one variable (MTL_1^+).

Algebraic semantics

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- An integral commutative residuated lattice (ICRL) is a lattice ordered algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, e\rangle$ where $\langle A, \cdot, e\rangle$ is a commutative monoid, e is a top element, and $xy \leq z$ iff $x \leq y \rightarrow z$.

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- A representable ICRL is an ICRL which is isomorphic to a subdirect product of totally ordered members.
- Thus SI-members in our variety are chains. We denote them shortly ICRCs.

Main result

Theorem

Each finitely generated ICRC can be embedded into a 1*-generated ICRC.*

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Corollary

The variety of representable integral commutative residuated lattices is generated (as a quasi-variety) by its 1-generated finite totally ordered members.

Lexicographic product

Lemma

Let $\mathbf{A}, \mathbf{B}$ *be ICRCs such that* $\mathbf{A}$ *is cancellative. Then the lexicographic product* $\mathbf{A} \overrightarrow{\times} \mathbf{B}$ *is an ICRC.*

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$$\langle a, x\rangle \rightarrow \langle b, y\rangle = \begin{cases} \langle a \rightarrow_A b, 1_B\rangle & \text{if } a \cdot_A (a \rightarrow_A b) <_A b, \\ \langle a \rightarrow_A b, x \rightarrow_B y\rangle & \text{otherwise.} \end{cases}$$

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In particular, if $\mathbf{A} = \mathbf{Z}^-$, then for $\langle a, x\rangle > \langle b, y\rangle$ we have

$$\langle a, x\rangle \rightarrow \langle b, y\rangle = \langle b - a, x \rightarrow_B y\rangle .$$

Nucleus and conucleus

Definition

- A closure operator γ on an ICRL $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \to, e\rangle$ is called a nucleus if $\gamma(x)\gamma(y) \leq \gamma(xy)$.

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Let $\gamma : L \to L$ be an operator on L. The image of γ is denoted L_γ.

Closure retraction and interior extraction

Lemma

- *An operator γ on $\mathbf{L}$ is nucleus iff L_γ satisfies*

$$\min\{a \in L_\gamma \mid x \leq a\} \text{ exists for all } x \in L.$$

and

$$x \to y \in L_\gamma \text{ for all } x \in L \text{ and } y \in L_\gamma.$$

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- *An operator σ on $\mathbf{L}$ is conucleus iff L_σ is a submonoid of $\mathbf{L}$ and*

  $$\max\{a \in L_\sigma \mid a \le x\} \text{ exists for all } x \in L.$$

  L_σ is called conuclear (interior) contraction.

Resulting ICRCs

Lemma

If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \rightarrow, e\rangle$ *is an ICRC and* γ *a nucleus on it, then* $\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee, \circ_\gamma, \rightarrow, e\rangle$ *is an ICRC, where* $x \circ_\gamma y = \gamma(x \cdot y)$.

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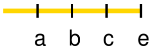
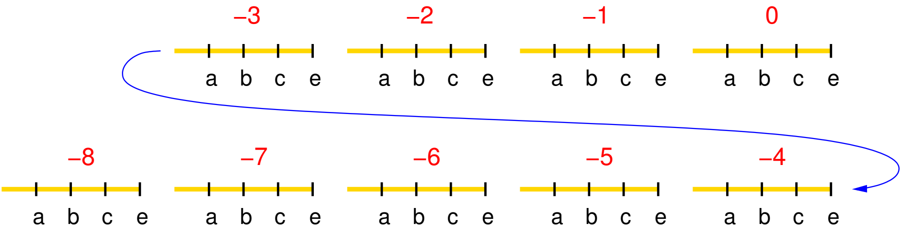
If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \to, e\rangle$ *is an ICRC and* σ *a conucleus on it, then* $\mathbf{L}_\sigma = \langle L_\sigma, \wedge, \vee, \cdot, \to_\sigma, e\rangle$ *is an ICRC, where* $x \to_\sigma y = \sigma(x \to y)$.

Sketch of the proof



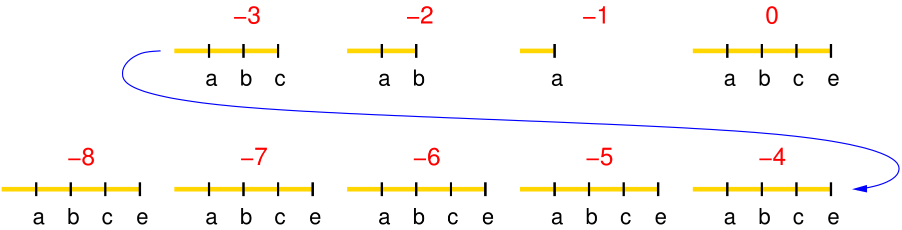
Let **A** be an ICRC generated by $\{a, b, c\}$.
We will construct a 1-generated ICRC in which **A** can be embedded.

Sketch of the proof



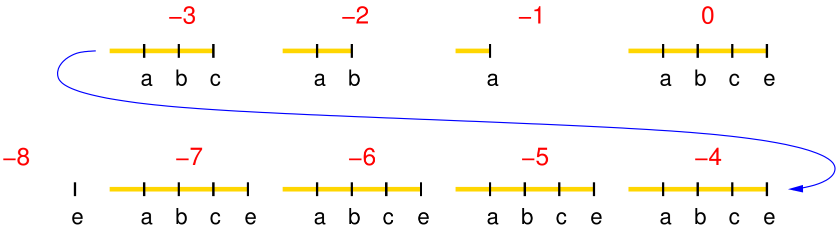
Consider the lexicographic product $\mathbf{Z}^{-} \overrightarrow{\times} \mathbf{A}$.
The elements are tuples $\langle x, y\rangle$ where $x \in \mathbf{Z}^{-}$ and $y \in \mathbf{A}$.

Sketch of the proof



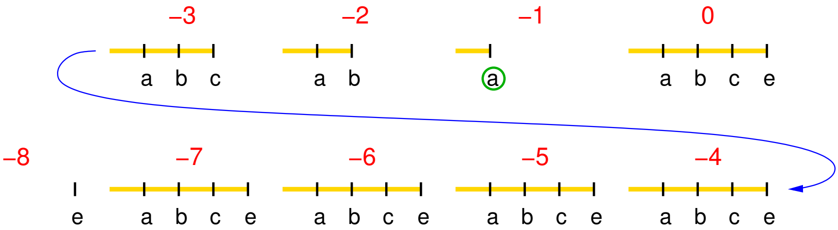
Take the conuclear contraction of $\mathbf{Z}^- \overrightarrow{\times} \mathbf{A}$ by deleting $\{\langle -1, y\rangle \mid y > a\} \cup \{\langle -2, y\rangle \mid y > b\} \cup \{\langle -3, y\rangle \mid y > c\}$.
Denote the corresponding conucleus σ.

Sketch of the proof



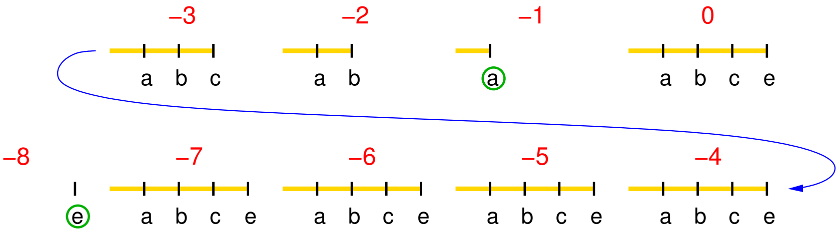
Consider the nucleus $\gamma(x) = x \vee \langle -8, e \rangle$ and its corresponding nuclear retraction.

Sketch of the proof



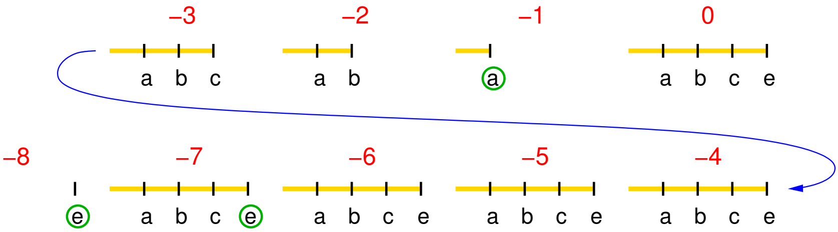
Finally, let **C** be the subalgebra generated by the element $g = \langle -1, a \rangle$. We will prove that **A** can be embedded into **C**.

Sketch of the proof



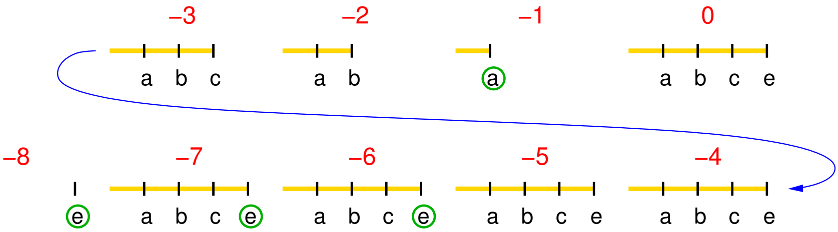
First, we have $g^8 = \gamma(\langle -1, a\rangle^8) = \gamma(\langle -8, a^8\rangle) = \langle -8, e\rangle$.

Sketch of the proof



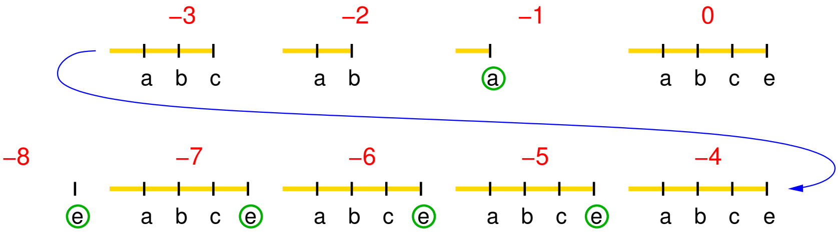
Then $g \to_\sigma g^8 = \sigma(\langle -1, a\rangle \to \langle -8, e\rangle) = \sigma(\langle -7, e\rangle) = \langle -7, e\rangle$.

Sketch of the proof



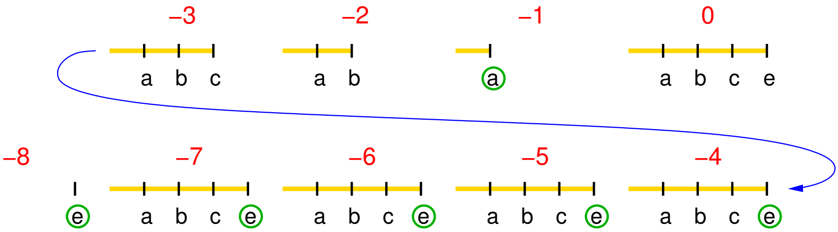
Then $g^2 \to_\sigma g^8 = \sigma(\langle -2, a^2 \rangle \to \langle -8, e \rangle) = \sigma(\langle -6, e \rangle) = \langle -6, e \rangle$.

Sketch of the proof



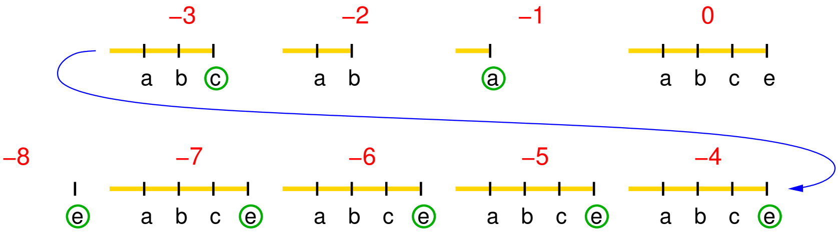
Then $g^3 \to_\sigma g^8 = \sigma(\langle -3, a^3\rangle \to \langle -8, e\rangle) = \sigma(\langle -5, e\rangle) = \langle -5, e\rangle$.

Sketch of the proof



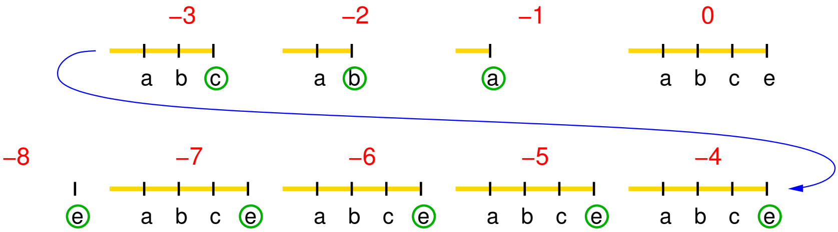
Then $g^4 \to_\sigma g^8 = \sigma(\langle -4, a^4 \rangle \to \langle -8, e \rangle) = \sigma(\langle -4, e \rangle) = \langle -4, e \rangle$.

Sketch of the proof



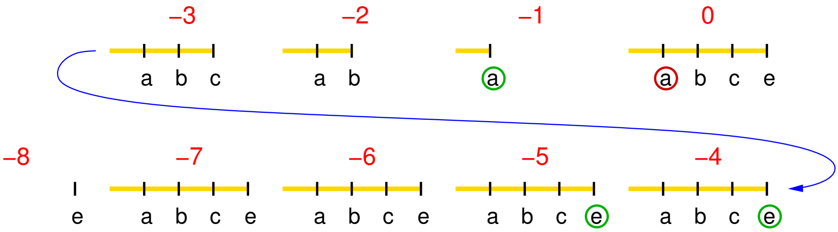
Then $g^5 \to_\sigma g^8 = \sigma(\langle -5, a^5\rangle \to \langle -8, e\rangle) = \sigma(\langle -3, e\rangle) = \langle -3, c\rangle$.

Sketch of the proof



Then $g^6 \to_\sigma g^8 = \sigma(\langle -6, a^6\rangle \to \langle -8, e\rangle) = \sigma(\langle -2, e\rangle) = \langle -2, b\rangle$.

Sketch of the proof



We have
$\langle -5, e\rangle \rightarrow_\sigma \langle -1, a\rangle\langle -4, e\rangle = \sigma(\langle -5, e\rangle \rightarrow \langle -5, a\rangle) = \sigma(\langle 0, a\rangle) = \langle 0, a\rangle$.

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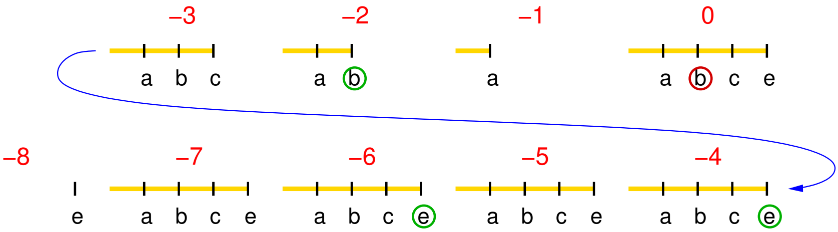

We have

$\langle -6, e\rangle \rightarrow_\sigma \langle -2, b\rangle\langle -4, e\rangle = \sigma(\langle -6, e\rangle \rightarrow \langle -6, b\rangle) = \sigma(\langle 0, b\rangle) = \langle 0, b\rangle$.

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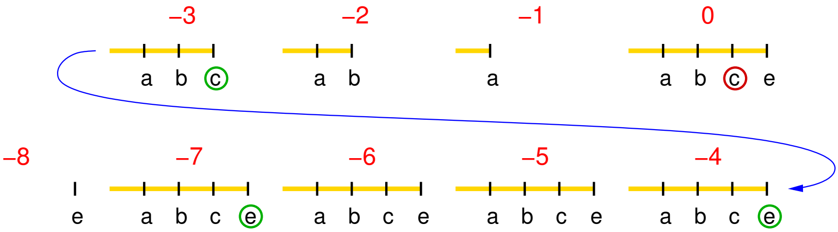

We have
$\langle -7, e\rangle \to_\sigma \langle -3, c\rangle\langle -4, e\rangle = \sigma(\langle -7, e\rangle \to \langle -7, c\rangle) = \sigma(\langle 0, c\rangle) = \langle 0, c\rangle$.

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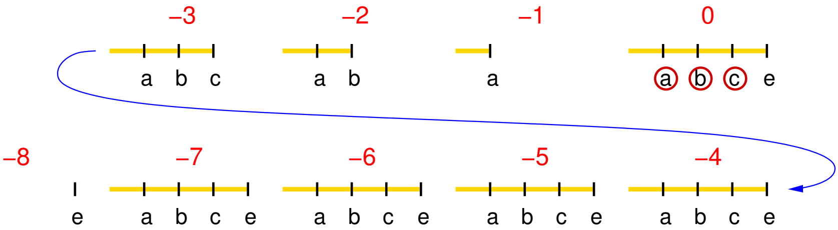

Thus $\langle 0, a\rangle, \langle 0, b\rangle, \langle 0, c\rangle \in$ **C**, i.e. **C** contains an isomorphic copy of **A**.

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Let T be a finite theory and φ be a formula such that $T \not\vdash \varphi$. Then there is a substitution σ such that $\sigma(T) \not\vdash \sigma(\varphi)$ and $\sigma(T), \sigma(\varphi)$ contain only a single propositional variable.

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Moreover, if $|\psi| = n$ and $Var(\psi) = \{v_1, \ldots, v_m\}$ then

$$\sigma(v_i) = (p^{n+1-k} \to p^{2(n+1)}) \to ((p^{n+1} \to p^{2(n+1)}) \cdot (p^{2(n+1)-k} \to p^{2(n+1)}))$$

for some $1 \leq k \leq n$.

There is a function $f \in \mathcal{O}(n^2)$ such that for any $\psi \in T \cup \{\varphi\}$ we have $|\sigma(\psi)| \leq f(|\psi|)$.

Time complexity

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Theorem

There exists a polynomial-time reduction from $\mathrm{TAUT}(\mathrm{MTL}_m^+)$ *to* $\mathrm{TAUT}(\mathrm{MTL}_1^+)$*.*

The translation is of this form:

$$\varphi'(p) = \bigvee_{(k_1,\ldots,k_m)\in\{1,\ldots,n\}^m} \sigma_{(k_1,\ldots,k_m)}(\varphi)\,.$$

Space complexity

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Remarks

We have to fix the number of variables m otherwise the length of φ' is bounded only by n^n.

Space complexity

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Remarks

We have to fix the number of variables m otherwise the length of φ' is bounded only by n^n.

However, to check that a formula φ is in TAUT(MTL$^+$) it suffices to go through all the disjuncts in φ' and check if they belong to TAUT(MTL$_1^+$) or not. In order to do this, we need "essentially" the same space.

For instance, if we would know that TAUT(MTL$_1^+$) is in PSPACE then we can infer that TAUT(MTL$^+$) is in PSPACE as well.