Quasiequational Theory of Square-increasing Residuated Lattices is Undecidable

Rostislav Horčík

Institute of Computer Science
Academy of Sciences of the Czech Republic

Logic, Algebra and Truth Degrees
Kanazawa, 10–14 September 2012

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- In the theory of residuated lattices are known as knotted rules/axioms (Hori, Ono, Schellinx).
- The aim of my talk: (un)decidability of the word problem for "Burnside" residuated lattices.

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Let $m, n \in \mathbb{N}$ and $m \neq n$. The variety of residuated lattices satisfying $x^m \leq x^n$ is denoted $\mathcal{RL}^n_m$.

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Theorem (van Alten)

Let $\mathcal{CRL}^n_m$ be the variety of commutative residuated lattices satisfying $x^m \leq x^n$. Then the universal theory (word problem) for $\mathcal{CRL}^n_m$ is decidable for all $m \neq n$.

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$$q_i \xrightarrow{c_1 := c_1 + 1} q_j$$

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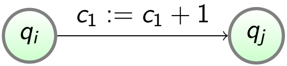

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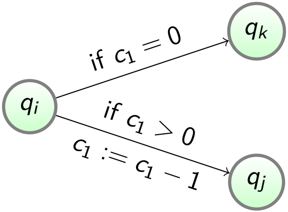

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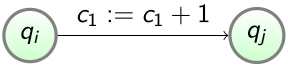

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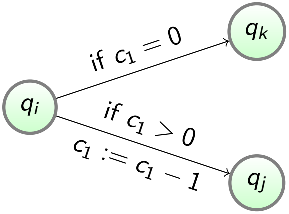
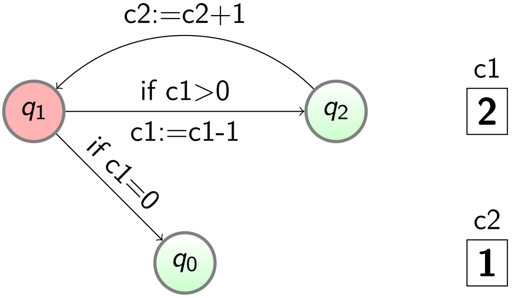


  - Analogously for the second counter.

A simple example



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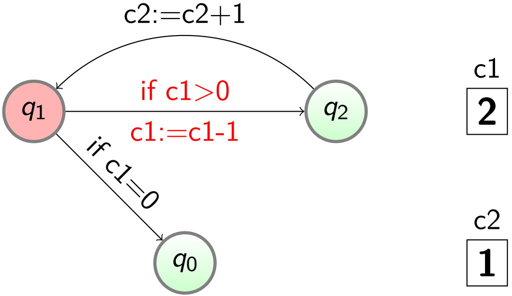

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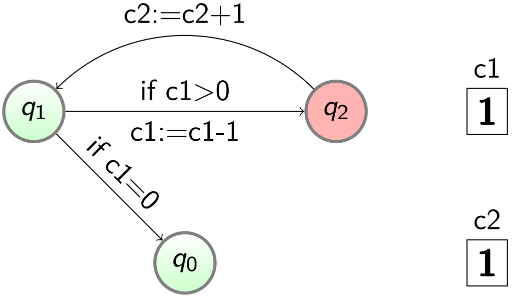

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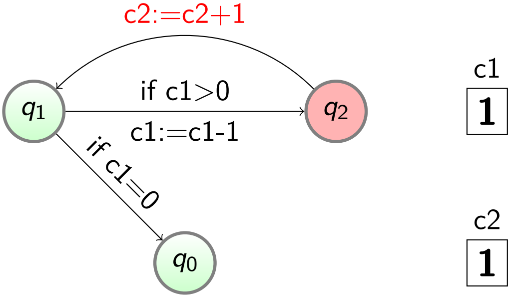

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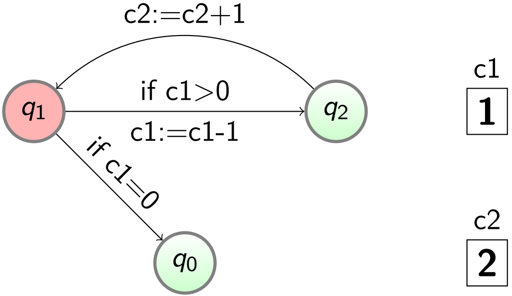

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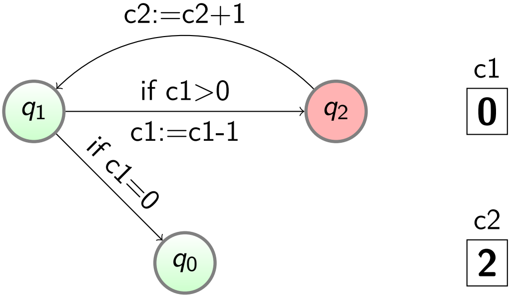

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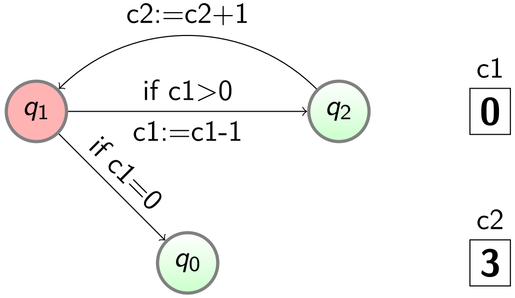

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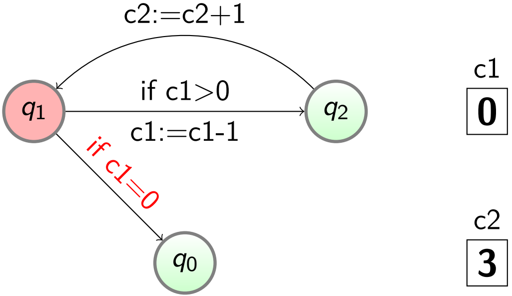

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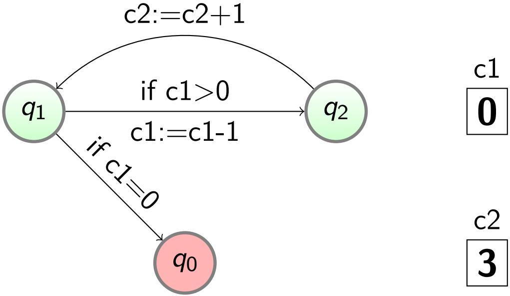

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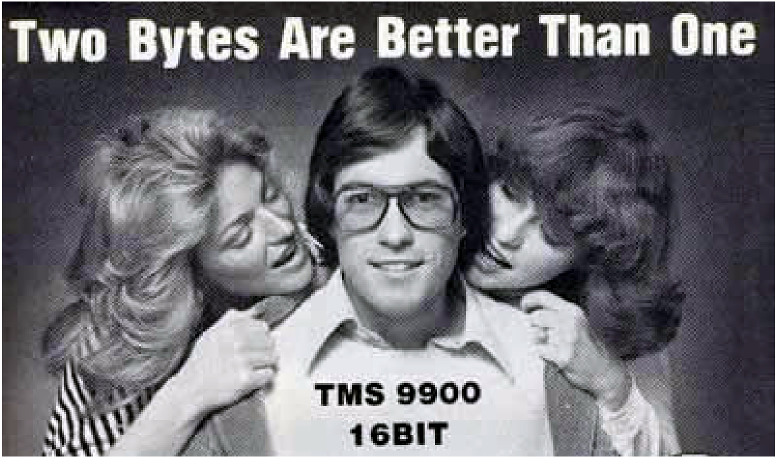
Theorem (Minsky, Lambek)

There is a Minsky machine (2CM) whose set of accepted configurations is undecidable.

2 counters are more than 1 counter



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2. *If $u \not\to_R^* v$ then $\Sigma^*/\sim_R$ does not satisfy (q).*

Simulating 2CM by Semi-Thue system

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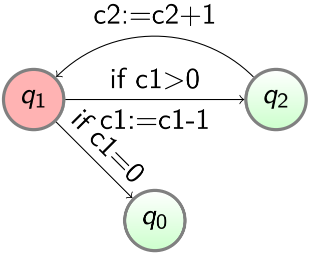

c1 $\boxed{2}$

c2 $\boxed{1}$

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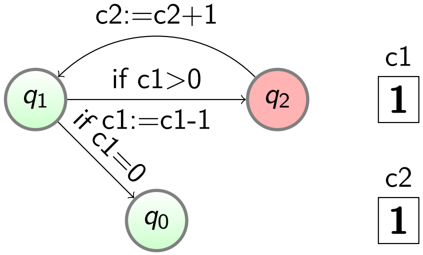
$$A a a q_1 a A \quad \to_R \quad A a q_2 a A$$



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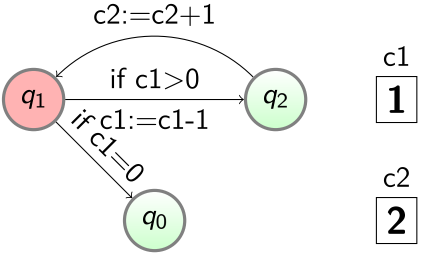

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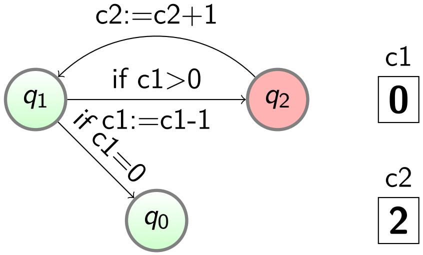

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- Configurations are encoded by words over a finite alphabet $\Sigma = \{q_0, \ldots, q_n, a, A\}$.

$$\langle q_i, c_1, c_2\rangle \quad \rightsquigarrow \quad Aa^{c_1} q_i a^{c_2} A$$

- Transition function is captured by the set R of rewriting rules:
  - $\tau(q_i) = \langle +, 1, q_j\rangle$: $q_i \to aq_j$
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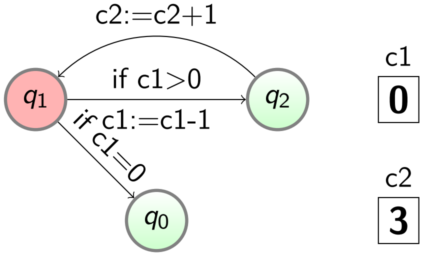

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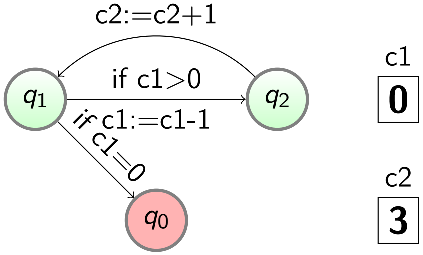

Burnside inequalities

Lemma

$\langle q_i, c_1, c_2 \rangle$ *is accepted iff* $Aa^{c_1} q_i a^{c_2} A \rightarrow_R^* Aq_0 A$.

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Definition

A word $w \in \Sigma^*$ contains square if it is of the form $w = u_1 x x u_2$ for some $u_1, u_2, x \in \Sigma^*$. Words containing no square are called square-free.

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Theorem (Thue 1906)

There is an infinite square-free word over Σ for $|\Sigma| \geq 3$.

Square-free morphisms – example

Let $\Sigma = \{a, b, c\}$. Define monoid endomorphism $h\colon \Sigma^* \to \Sigma^*$ as follows:

$$h(a) = abc\,, \quad h(b) = ac\,, \quad h(c) = b\,.$$

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Problem: $q_i h^{c_2}(a) \to q_j h^{c_2+1}(a)$, $q_i h^{c_2+1}(a) \to q_j h^{c_2}(a)$

Addition and substruction

$$h^{n+1}(a) = h(h^n(a)) = h(d_1 \dots d_k) = h(d_1) \dots h(d_k)$$

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Substruction can be treated similarly by

$$C^- h(d) \to d\, C^-$$

Resulting coding

- Alphabet: $\Sigma = \{q_0, \ldots, q_n, a, b, c, A, B, B^+, B^-, C, C^+, C^-\}$

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What have we achieved?

Lemma

$\mathcal{C} = \langle q_i, c_1, c_2 \rangle$ *is accepted iff* $A\overline{h^{c_1}(a)}Bq_iCh^{c_2}(a)A \to_R^* AaBq_0CaA$

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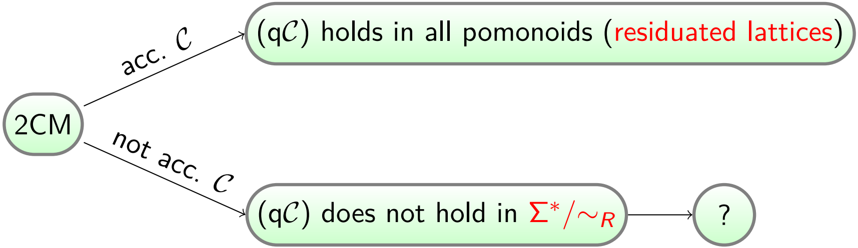
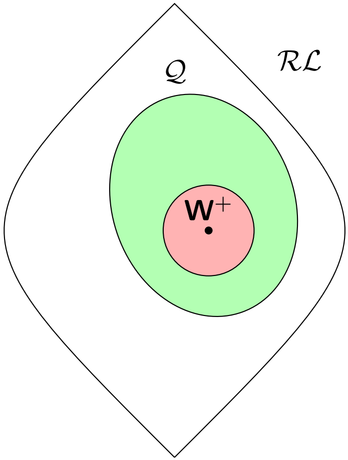


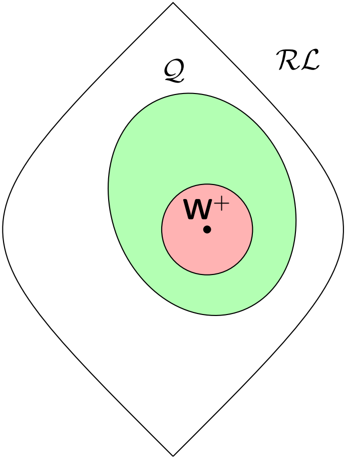
Construction of a residuated lattice

- We want a residuated lattice $\mathbf{W}^+$ satisfying as many (quasi-)identities as possible



Construction of a residuated lattice

- We want a residuated lattice $\mathbf{W}^+$ satisfying as many (quasi-)identities as possible
- but still being a countermodel for all (q$\mathcal{C}$)'s not valid in $\Sigma^*/{\sim_R}$.



Residuated frames (Galatos, Jipsen)

$$
\begin{tikzcd}
A \arrow[rr, "N", "/" marking] & & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{P}(A) \arrow[rr, bend left, "\triangleright"] & & \mathcal{P}(B) \arrow[ll, bend left, "\triangleleft"]
\end{tikzcd}
$$

Closure operator: $\gamma(X) = X^{\triangleright\triangleleft}$
The closed sets form a complete lattice $\mathbf{W}^+$.

Residuated frames (Galatos, Jipsen)

$$\begin{tikzcd}[column sep=huge]
\textcolor{red}{\text{monoid}} \quad A \arrow[r, "N", "/" marking] & B \\
\textcolor{red}{\text{res. lattice}} \quad \mathcal{P}(A) \arrow[r, bend left, "\rhd"] & \mathcal{P}(B) \arrow[l, bend left, "\lhd"]
\end{tikzcd}$$

Nucleus: $\gamma(X) = X^{\rhd\lhd}$
The closed sets form a residuated lattice $\mathbf{W}^+$.

$$xy \mathrel{N} z \quad \text{iff} \quad y \mathrel{N} x \mathbin{\backslash\!\!\backslash} z \quad \text{iff} \quad x \mathrel{N} z \mathbin{/\!\!/} y$$

Residuated frames (Galatos, Jipsen)

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The closed sets form a residuated lattice $\mathbf{W}^+$.

$$xy\,N\,z \quad \text{iff} \quad y\,N\,x \backslash\!\backslash z \quad \text{iff} \quad x\,N\,z /\!/ y$$

$f(x) = \gamma\{x\}$ is a monoid homomorphism from A to W^+.

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1. *N is nuclear and $\gamma\{t\} \subseteq \gamma\{s\}$ for all $t \to s \in R$.*

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1. N *is nuclear and* $\gamma\{t\} \subseteq \gamma\{s\}$ *for all* $t \to s \in R$.
2. *If* $w \not\to_R^* AaBq_0CaA$ *then*

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2. *If $w \not\to_R^* AaBq_0CaA$ then*

$$\gamma\{w\} \nsubseteq \gamma\{AaBq_0CaA\}\,.$$

3. *In particular, if a configuration $\mathcal{C} = \langle q_i, c_1, c_2\rangle$ is not accepted then*

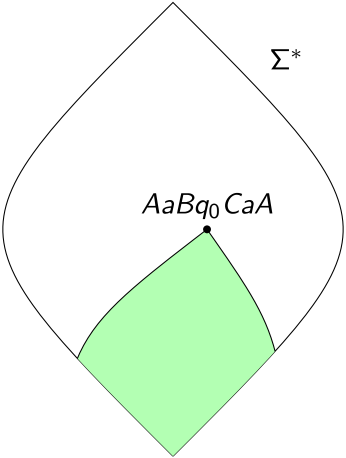
$$\&_{t\to s\in R}\ t \le s \implies A\overline{h^{c_1}(a)}Bq_iCh^{c_2}(a)A \le AaBq_0CaA \qquad (\mathrm{q}\mathcal{C})$$

does not hold in $\mathbf{W}^+$.

Properties

- γ is the pointwise greatest nucleus s.t.

$$\gamma\{AaBq_0CaA\} = \{u \in \Sigma^* \mid u \to_R^* AaBq_0CaA\}\,.$$



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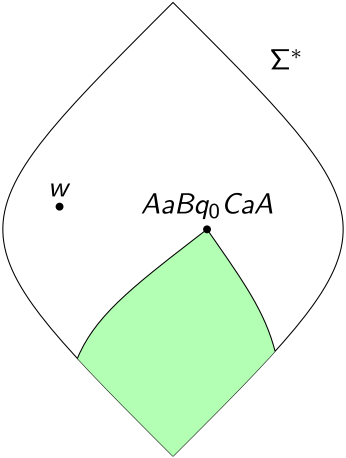

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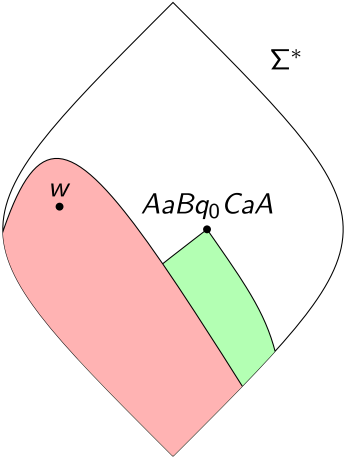

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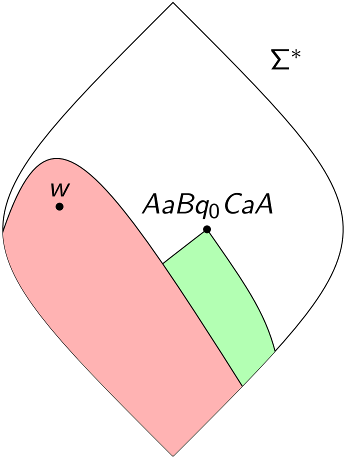
- Since $\gamma\{AaBq_0CaA\}$ contains only square-free words, the complex algebra $\mathbf{W}^+$ is a residuated lattice satisfying $x \leq x^2$ and $x^3 = x^2$.



Undecidability results

Let $\mathcal{C} = \langle q_i, c_1, c_2 \rangle$ be a configuration. Then $\mathcal{C}$ is accepted iff

$$\&_{t \to s \in R}\ t \leq s \implies A\overline{h^{c_1}(a)}Bq_i Ch^{c_2}(a)A \leq AaBq_0 CaA \qquad (\mathrm{q}\mathcal{C})$$

holds in $\mathcal{RL}^2_1 \cap \mathcal{RL}^2_3$.

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holds in $\mathcal{RL}_1^2 \cap \mathcal{RL}_3^2$.

Theorem

The word problem (quasi-equational theory) is undecidable in $\mathcal{RL}_m^n$ for $1 \leq n < m$ and $m < n \leq 2$.

Undecidability results

Let $\mathcal{C} = \langle q_i, c_1, c_2 \rangle$ be a configuration. Then $\mathcal{C}$ is accepted iff

$$\&_{t \to s \in R}\ t \leq s \implies A\overline{h^{c_1}(a)}Bq_iCh^{c_2}(a)A \leq AaBq_0CaA \qquad (\mathrm{q}\mathcal{C})$$

holds in $\mathcal{RL}_1^2 \cap \mathcal{RL}_3^2$.

Theorem

The word problem (quasi-equational theory) is undecidable in $\mathcal{RL}_m^n$ for $1 \leq n < m$ and $m < n \leq 2$.

Theorem

Let $\mathcal{DRL}_m^n$ be the variety of distributive residuated lattices satisfying $x^m \leq x^n$. Then the word problem (quasi-equational theory) is undecidable in $\mathcal{DRL}_m^n$ for $1 \leq n < m$.

What remains?

- $1 \leq x^n$ axiomatizes the trivial variety.

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The universal theory of integral residuated lattices (i.e., $\mathcal{RL}_m^0$) is decidable.

- The only remaining unknown cases are $x^m \leq x$ for $m \geq 2$.

Finite embeddability property

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Theorem (Blok, van Alten)

If for each $\mathbf{A} \in \mathcal{RL}^1_m$ every finitely generated submonoid B of $\mathbf{A}$ is dually well quasi-ordered then $\mathcal{RL}^1_m$ has the FEP.

Mingle $x^2 \leq x$

- Pomonoid subreducts of residuated lattices satisfying $x^2 \leq x$ are axiomatized by

$$uxv \leq z \;\&\; ux'v \leq z \implies uxx'v \leq z \tag{q}$$

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- Consider a closure operator $\delta \colon \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$ s.t. its closed sets are closed under the following rule:

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- Define a quasi-order on Σ^*:

$$x \sqsubseteq y \quad \text{iff} \quad \delta\{x\} \subseteq \delta\{y\} \quad \text{iff} \quad x \in \delta\{y\}\,.$$

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Lemma

Every finitely generated pomonoid $\mathbf{A}$ *satisfying* (q) *is a homomorphic image of* $\langle \Sigma^*, \sqsubseteq \rangle$ *for some finite* Σ.

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Lemma

The quasi-order $\sqsubseteq$ is a dual well quasi-order.

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Corollary

The universal theory of $\mathcal{RL}_2^1$ is decidable.

Conclusion

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Theorem (Green, Rees)

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- Is there a similar relation also for $m \geq 3$?

Thank you!