Varieties of Cancellative Prelinear Semihoops Covering the Variety Generated by Negative Integers

Rostislav Horčík

Institute of Computer Science
Academy of Sciences of the Czech Republic

Logic, Algebra and Truth Degrees
University of Siena
September 8–11, 2008

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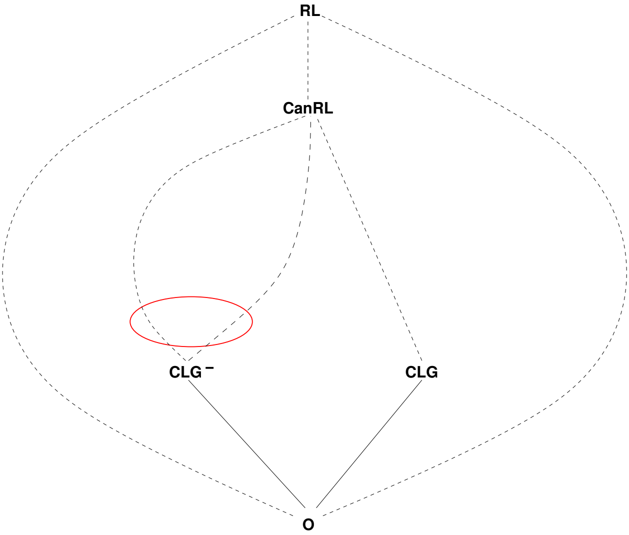
- The second one is the variety CLG$^-$ of negative cones of Abelian ℓ-groups. CLG$^-$ is generated by $\mathbb{Z}^-$.
- We are going to discuss what is above CLG$^-$.
- The obtained results can be applied also to the lattice of fuzzy logics extending MTL.

Lattice of subvarieties $\Lambda(\mathsf{RL})$



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- There are uncountably many covers of CLG^-. It follows from the fact that there are uncountably many covers of CLG and the result by BCGJT. This results shows that the mapping assigning to a class of ℓ-groups their negative cones is a lattice embedding.

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- What about integral commutative representable covers?
- We will show that there are infinitely many of such covers.
- However, it remains still open whether there are only countably many of them or uncountably many.

Algebras of interest

- A residuated lattice (RL) is an algebra $\mathbf{A} = (A, \cdot, /, \backslash, \wedge, \vee, 1)$ where the following conditions are satisfied:
  - $(A, \cdot, 1)$ is a monoid,
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- A CanICRC belongs to CLG^- iff it satisfies $x \wedge y = x(x \rightarrow y)$.

2-generated submonoids of $\mathbb{Z}^-$

- The fact that there are only two cancellative atoms in $\Lambda(\mathsf{RL})$ is proved by showing that for any CanRL there is a 1-generated subalgebra isomorphic either to $\mathbf{Z}$ or $\mathbf{Z}^-$ where
  - $\mathbf{Z} = (\mathbb{Z}, +, -, \min, \max, 0)$ and
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- The varieties generated by such CanICRCs will serve as good candidates for covers of CLG$^-$.

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- Then $\langle a, b\rangle$ contains the following elements:

$$\begin{aligned} 0 &< 5 < 10 < 12 < 15 < 17 < 20 < 22 < 24 < 25 \\ &< 27 < 29 < 30 < 32 < 34 < 35 < 36 < 37 < 39 \\ &< 40 < 41 < 42 < 44 < \color{red}{45 < 46 < 47 < 48 < \cdots} \end{aligned}$$

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- 45 is the first multiple of a above which there are no gaps.

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- The monoidal operation is defined as follows:

$$(x, y) + (u, v) = \begin{cases} (x + u + 1, y +_a v) & \text{if } y + v \geq a, \\ (x + u, y +_a v) & \text{otherwise.} \end{cases}$$

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- Let $a, b \in \mathbb{N}$. The submonoid $\langle a, b \rangle$ can be embedded into $\mathbb{N} \times \mathbb{Z}_a$.

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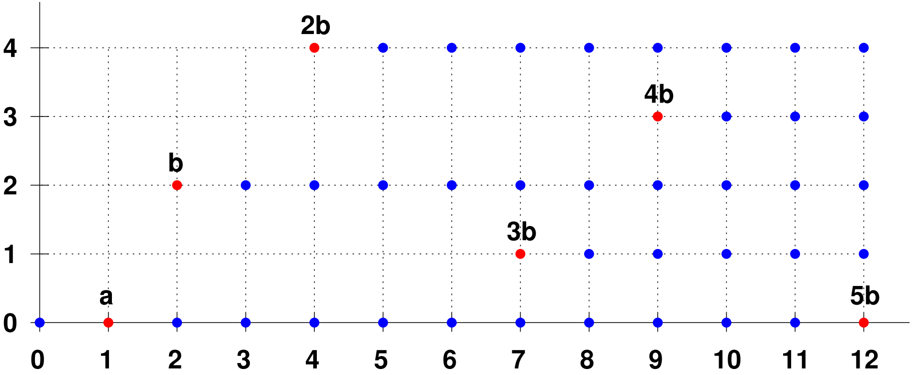

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2. *For each $x < n$ there is $y \in \mathbb{Z}_a \setminus \{0\}$ such that $(x, y) \notin \langle a, b\rangle$.*

CanICRCs arising from $\langle a, b \rangle$

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- Then $\mathbf{M}(a,b) = (M(a,b), +, \to, \min, \max, 0)$ is a simple CanICRC where
$$M(a,b) = \{-ka - lb \mid k, l \in \mathbb{N}\},$$
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- We will consider varieties $\mathsf{V}(\mathbf{M}(a, b))$ for $0 < a < b$, a, b coprime, and a prime.
$$G = \{(a, b) \in \mathbb{N}^2 \mid 0 < a < b,\ a, b \text{ coprime},\ a \text{ prime}\}.$$

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Lemma

Let $(a, b) \in G$. *Then* $\mathbf{M}(a, b)$ *satisfies the identity*

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3. $\mathsf{V}(\mathbf{M}(a, b)) \neq \mathsf{V}(\mathbf{M}(c, d))$.

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- We will use Jónsson's lemma telling that each SI-algebra in $\mathsf{V}(\mathbf{M}(a, b))$ belongs to $\mathsf{HSP}_{\mathsf{U}}(\mathbf{M}(a, b))$.

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It remains to discuss what happens when $\mathbf{A} \in \mathsf{ISP_U}(\mathbf{M}(a,b))$.

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Corollary

There are infinitely many varieties of Π*MTL-algebras covering the variety of product algebras.*