Algebraic Methods from Substructural Logics and Formal Languages

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A warm introduction

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1. *$\sim_L$ is the largest congruence such that $L = \bigcup_{w \in L} w/{\sim_L}$.*
2. *$\mathbf{M}(L)$ is finite iff L is regular (Myhill-Nerode Theorem).*

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\begin{aligned}
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- The syntactic congruence is known in AAL as Leibniz congruence which is used in the construction of Lindenbaum-Tarski algebra for a given theory.
- Can other constructions/ideas from (substructural) logics be used in the language theory?

Residuated lattices

Definition

Let $\mathbf{M} = \langle M, \cdot, 1\rangle$ be a monoid. A quasi-order $\leq$ on M is called compatible if for all $x, y, u, v \in M$:

$$x \leq y \implies uxv \leq uyv\,.$$

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A residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ is a monoid such that $\langle A, \wedge, \vee \rangle$ is a lattice and for all $a, b, c \in A$:

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b\,.$$

Powerset monoid

Example

Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a monoid. Then

$$\mathcal{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle$$

is a residuated lattice, where

$$\begin{aligned} X \cdot Y &= \{xy \in M \mid x \in X, y \in Y\}, \\ X \backslash Z &= \{y \in M \mid X \cdot \{y\} \subseteq Z\}, \\ Z / Y &= \{x \in M \mid \{x\} \cdot Y \subseteq Z\}. \end{aligned}$$

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Other examples can be obtained by introducing a suitable closure operator on $\mathcal{P}(M)$.

Closure operators

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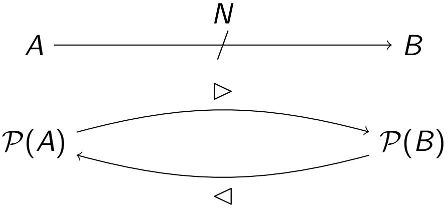

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X^{\triangleright} &= \{ b \in B \mid (\forall a \in X)(a \, N \, b) \} \, , \\
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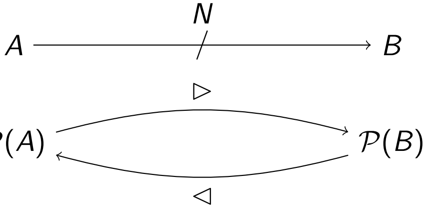

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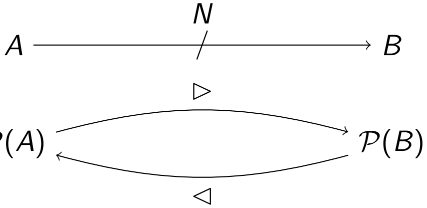

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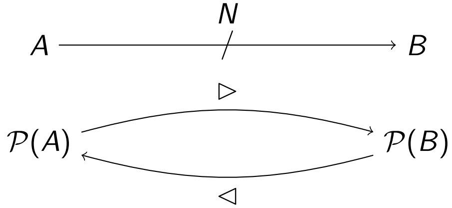

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- $\gamma(X) = X^{\triangleright\triangleleft}$ is a closure operator on $\mathcal{P}(A)$.
- $\{\{b\}^{\triangleleft} \mid b \in B\}$ is its basis.
- The collection of closed sets forms a complete lattice $\mathbf{W}^{+} = \langle \gamma[\mathcal{P}(A)], \cap, \cup_{\gamma} \rangle$, where

$$X \cup_{\gamma} Y = \gamma(X \cup Y)\,.$$

Residuated frames

- Given a monoid $\mathbf{A}$ and an frame $\mathbf{W} = \langle A, B, N \rangle$, define an extended frame $\widehat{\mathbf{W}} = \langle A, A^2 \times B, \widehat{N} \rangle$, where

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- Then $\widehat{\mathbf{W}}^+ = \langle \mathcal{P}(A)_\gamma, \cap, \cup_\gamma, \circ_\gamma, \backslash_\gamma, /_\gamma, \gamma\{1\} \rangle$ forms a complete residuated lattice, where $X \bullet_\gamma Y = \gamma(X \bullet Y)$ for $\bullet \in \{\circ, \backslash, /, \cup\}$.

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- The binary relation on A defined by

$$x \sqsubseteq y \quad \text{iff} \quad \gamma\{x\} \subseteq \gamma\{y\}$$

is a compatible quasi-order on **A**.

Syntactic residuated lattice

Definition

Let $L \subseteq \Sigma^*$ be a language. Define frame $\mathbf{W} = \langle \Sigma^*, \{\star\}, N \rangle$, where $N \subseteq \Sigma^* \times \{\star\}$ is defined by

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Then $\mathbf{R}(L) = \widehat{\mathbf{W}}^+$ is called the syntactic residuated lattice of L.

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3. $\mathbf{R}(L)$ *is finite iff L is regular.*

Generalized Myhill Theorem

The following theorem is the core of most decidability proofs we have for substructural logics.

Theorem

Let $\mathbf{A}$ *be a monoid and* $\mathbf{W} = \langle A, B, N \rangle$ *a frame where* B *is finite. Then* $\widehat{\mathbf{W}}^+$ *is finite iff there is a compatible dual well quasi-order* $\leq$ *on* $\mathbf{A}$ *such that*

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Corollary (Generalized Myhill Theorem – Ehrenfeucht, Rozenberg)

A language $L \subseteq \Sigma^*$ *is regular iff* L *is downward closed w.r.t. a compatible dual well quasi-order on* Σ^**.*

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Example

The language $a^+(b(a+b+c)^*b+b)c^+$ is closed under (r).

Application (cont.)

- Consider a closure operator $\gamma\colon \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$ s.t. its closed sets are closed under the rule:

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$$x \sqsubseteq y \quad \text{iff} \quad \gamma\{x\} \subseteq \gamma\{y\}\,.$$

- In order to show that L has to be regular, it suffices to show that $\sqsubseteq$ is a dual well quasi-order using the generalized Myhill theorem.

Higman's lemma

Definition

Let $\langle Q, \leq \rangle$ be a quasi-ordered set. Define a binary relation $\leq^*$ on Q^* by

> $a_1 \ldots a_n \leq^* b_1 \ldots b_m$ *iff there is a strictly increasing map* $f\colon [1, n] \to [1, m]$ *s.t.* $a_i \leq b_{f(i)}$ *for all* $i \in [1, n]$.

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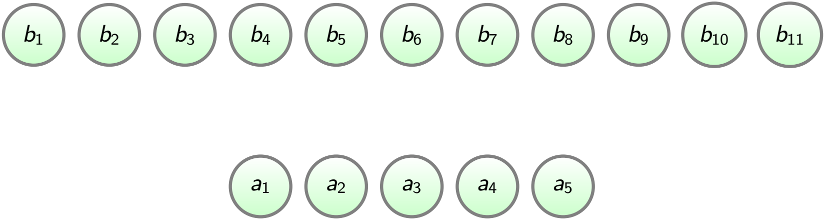

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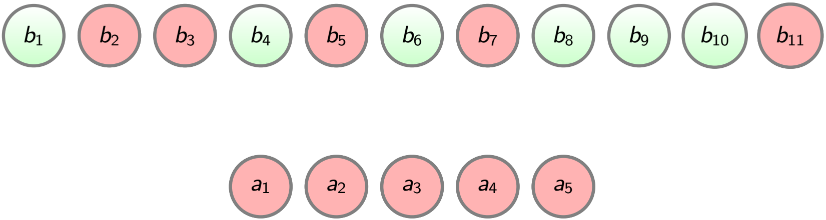

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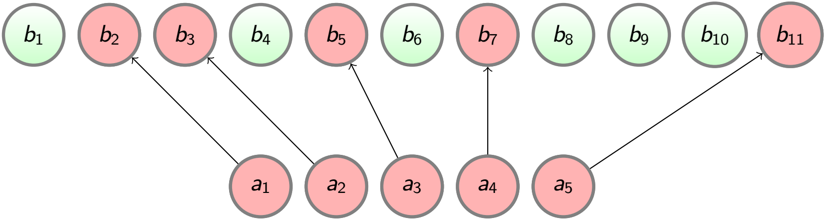

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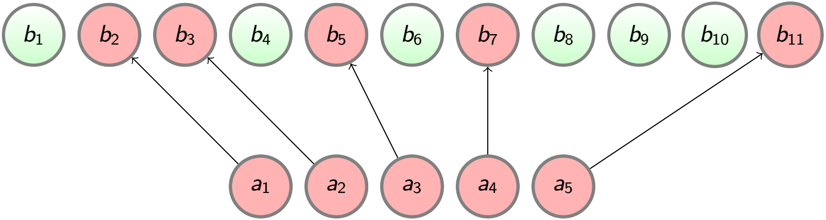

Lemma (Higman's lemma)

If $\langle Q, \leq \rangle$ is a well quasi-ordered set then so is $\langle Q^*, \leq^* \rangle$.

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Let $\langle Q, \leq \rangle$ be a quasi-ordered set. Define a binary relation $\leq^+$ on Q^+ by

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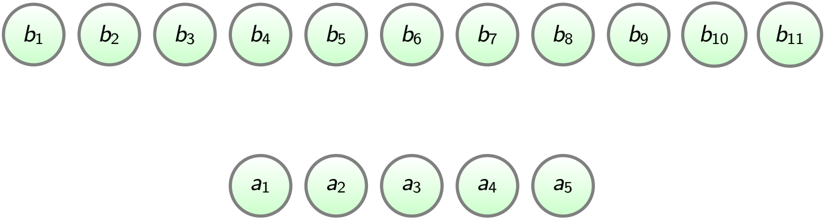

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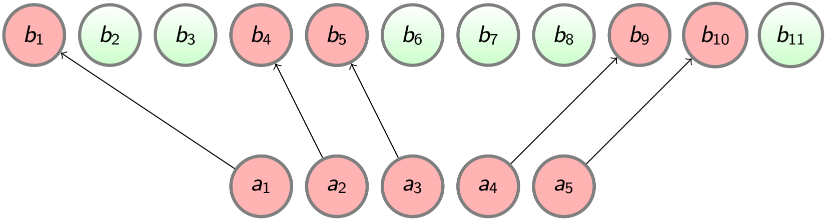

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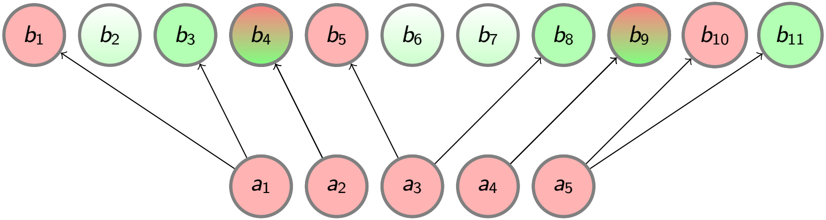

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Lemma

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Lemma

Let $w \in \Sigma^*$ *and* $\mathrm{Alph}(w) = \Gamma$. *Then* $wuw \sqsubseteq w$ *for every* $u \in \Gamma^*$.

Beyond regular languages?

Let $\Sigma = \{0, 1\}$.

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The rule (r) is equivalent to

$$1 \leq x \vee x^2 \vee x \backslash y.$$

Thus the languages L_1, L_2 can be separated by a variety of residuated lattices.

Thank you!