Universal Theory of Residuated Distributive Lattice-Ordered Groupoids and Its Complexity

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Introduction

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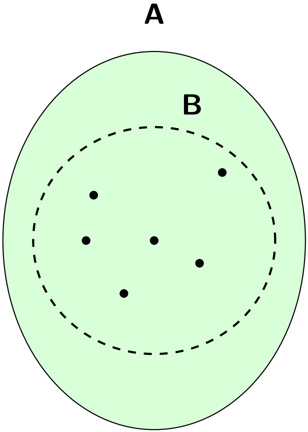
- $\mathrm{Th}_\forall(\mathbb{K})$ denotes the universal theory of $\mathbb{K}$.
- A usual way how to prove decidability of $\mathrm{Th}_\forall(\mathbb{K})$ is to establish the finite embeddability property for $\mathbb{K}$.

Definition

A class of algebras $\mathbb{K}$ has the finite embeddability property (FEP) if every finite partial subalgebra $\mathbf{B}$ of any algebra $\mathbf{A} \in \mathbb{K}$ is embeddable into a finite algebra $\mathbf{D} \in \mathbb{K}$.

FEP



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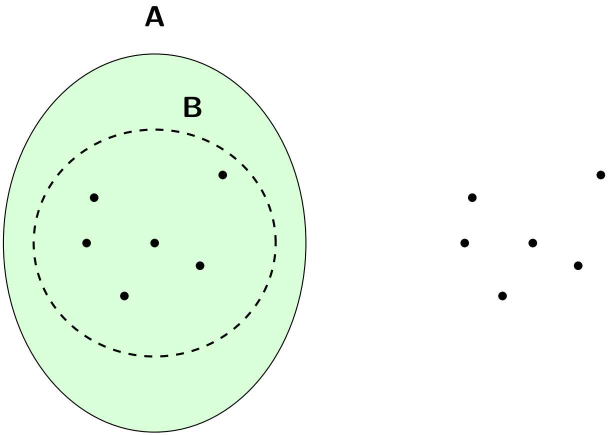

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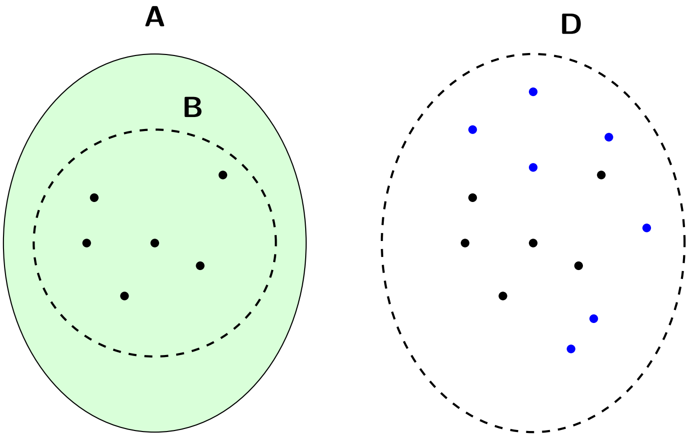

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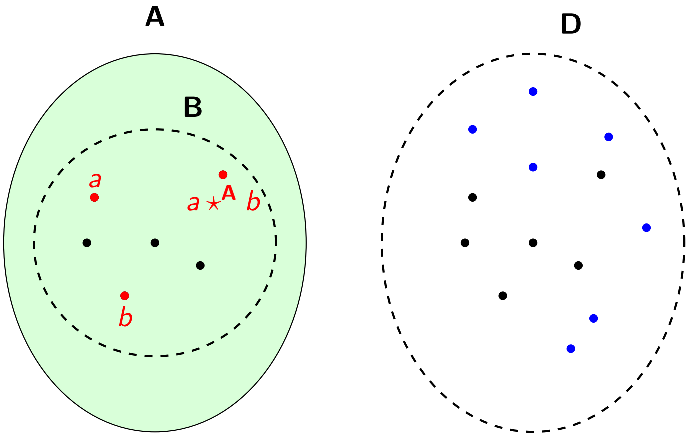

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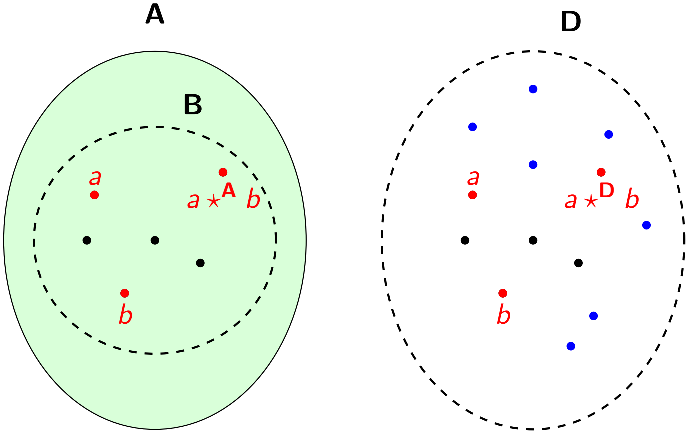

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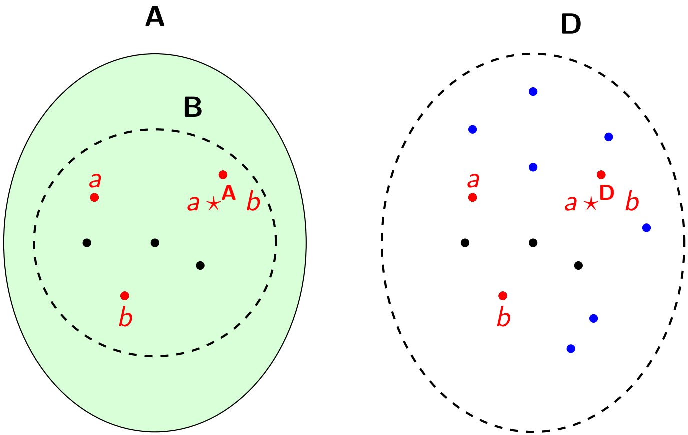

$$\mathbf{A} \not\models \Phi \implies \mathbf{B} = \text{eval. of subterms} \implies \mathbf{D} \not\models \Phi\,.$$

A bit of history

- McKinsey and Tarski 1946 – FEP for Heyting algebras

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Problem

Does $\mathbb{ROG}$ *have the FEP?*

Answer

- An affirmative answer was given by Farulewski 2008.

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- Farulewski's proof uses methods from proof-theory and also from algebra.
- Recall that $\mathbb{ROG}$ forms an algebraic semantics for nonassociative Lambek calculus NL.

Lemma (Buszkowski 2005)

Let $\mathcal{S} \cup \{X[Z] \Rightarrow C\}$ be a finite set of sequents and T the set of all subformulas occuring in $\mathcal{S} \cup \{X[Z] \Rightarrow C\}$. If $\mathcal{S} \vdash_{\mathsf{NL}} X[Z] \Rightarrow C$, then there exists an interpolant $D \in T$ such that $\mathcal{S} \vdash_{\mathsf{NL}} X[D] \Rightarrow C$ and $\mathcal{S} \vdash_{\mathsf{NL}} Z \Rightarrow D$.

Note that Z is a tree of formulas unlike D which is a single formula.

Residuated distributive lattice-ordered groupoids

Definition

A structure $\mathbf{A} = \langle A, \cdot, \backslash, / \leq \rangle$ is called residuated ordered groupoid (rog) if $\langle A, \cdot \rangle$ is a groupoid and for all $a, b, c \in A$:

$$ab \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c / b\,.$$

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A residuated distributive lattice-ordered groupoid (rdlog) $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, / \rangle$ is a rog such that $\langle A, \wedge, \vee \rangle$ is a distributive lattice.

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Every rog $\mathbf{A}$ embeds into a rdlog $\mathcal{O}(\mathbf{A})$ via $x \mapsto \downarrow\{x\}$.

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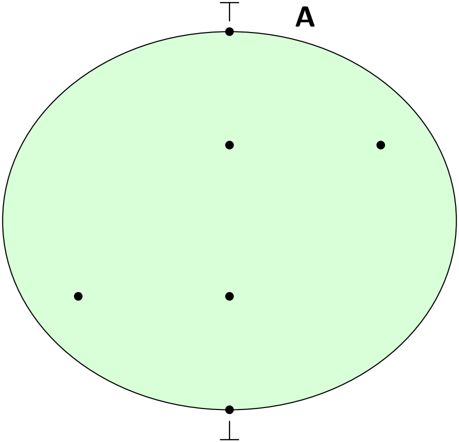
Every rog $\mathbf{A}$ *embeds into a rdlog* $\mathcal{O}(\mathbf{A})$ *via* $x \mapsto {\downarrow}\{x\}$.

Corollary

FEP for rdlogs $\implies$ *FEP for rogs.*

FEP for rdlogs



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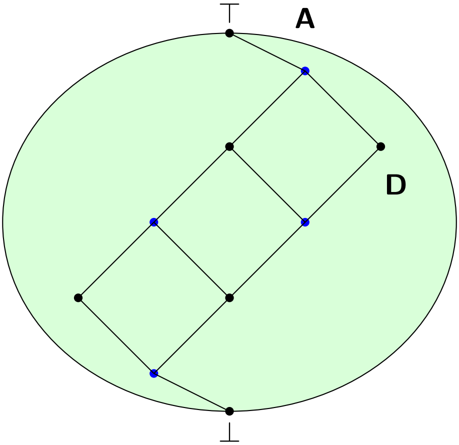

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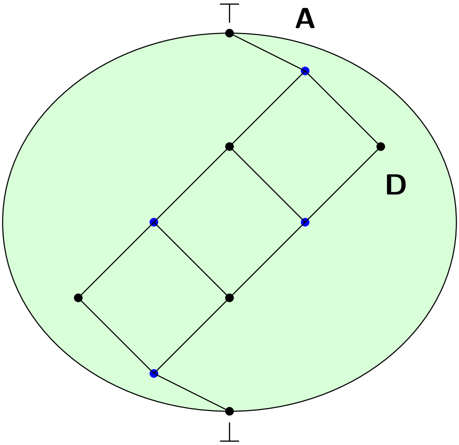

$$\begin{aligned}\gamma(x) &= \bigwedge\{y \in D \mid x \leq y\}\\ \sigma(x) &= \bigvee\{y \in D \mid y \leq x\}\\ \gamma[A] &= \sigma[A] = D\end{aligned}$$

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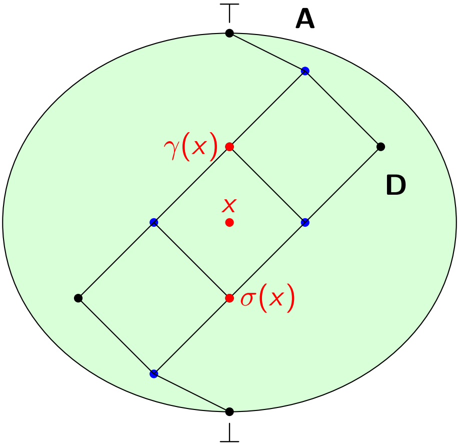

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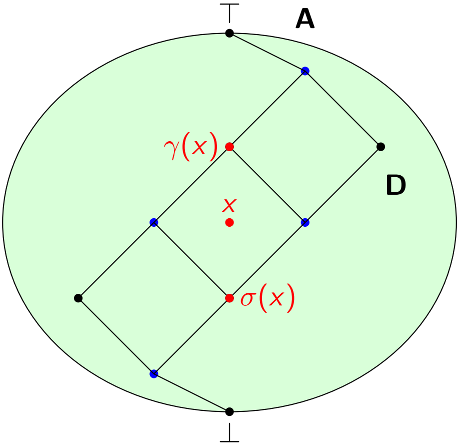

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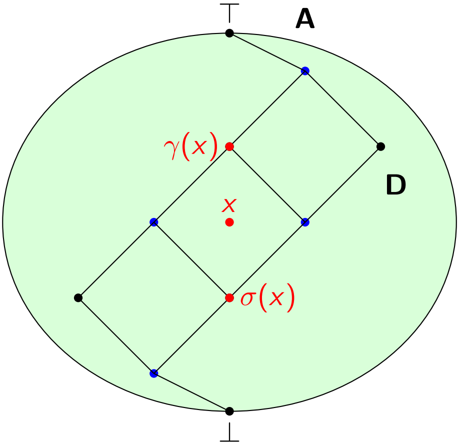

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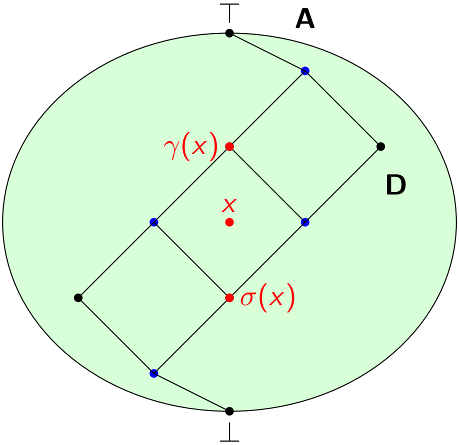

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$$x \circ y = \gamma(xy) \leq z \quad \text{iff} \quad xy \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad y \leq \sigma(x \backslash z) = x \mathbin{\backslash\!\backslash} z\,.$$

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Theorem

Let $\mathbb{RDLOG}$ *be the class of rdlogs. Then* $\mathbb{RDLOG}$ *has the FEP. The same holds for* $\mathbb{ROG}$*.*

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- What about computational complexity of $\mathrm{Th}_\forall(\mathbb{RDLOG})$?
- Buszkowski 2005 proved that the set of quasi-inequalities valid in $\mathbb{ROG}$ is in PTIME.
- Buszkowski, Farulewski 2008 claim that the quasi-equational theory of $\mathbb{RDLOG}$ is in 2-EXPTIME.

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- Thus $|\mathcal{J}(\mathbf{L})|$ is bounded by $2^n - 2$ (the number of join-irreducibles in the free n-generated distributive lattice).

Relational frames

Definition

A *frame* is a structure $\mathbf{W} = \langle W, \leq, R_\circ \rangle$ where $\langle W, \leq \rangle$ is a finite poset and $R_\circ \subseteq W^3$ such that for all $x, y, z, x', y', z' \in W$ we have

- $x \leq x'$ and $R_\circ xyz$ implies $R_\circ x'yz$,
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Having a finite rdlog $\mathbf{A}$, we define $\mathit{Stone}(\mathbf{A}) = \langle \mathcal{J}(\mathbf{A}), \leq, R_\circ \rangle$, where

$$R_\circ xyz \qquad \text{iff} \qquad z \leq xy\,.$$

Then $\mathit{Stone}(\mathbf{A})$ is a frame.

From frames to algebras

Having a frame $\mathbf{W}$, we define $Pred(\mathbf{W}) = \langle \mathcal{O}(\mathbf{W}), \cap, \cup, \cdot, \backslash, / \rangle$, where

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A finite rdlog $\mathbf{A}$ is isomorphic to $\mathit{PredStone}(\mathbf{A})$ via $\mu \colon \mathbf{A} \to \mathit{PredStone}(\mathbf{A})$ given by $\mu(x) = \mathcal{J}(\mathbf{A}) \cap {\downarrow}\{x\}$ for $x \in A$.

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To represent an n-generated rdlog $\mathbf{A}$, it suffices to store $\mathcal{J}(\mathbf{A})$ of cardinality $m \leq 2^n - 2$ and a relation $R_\circ$ of size m^3.

NEXPTIME

A problem P is in NEXPTIME if

$$P = \{x \mid \exists y \colon \langle x, y \rangle \in R\}$$

for some binary relation R such that

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Define R as a set of pairs $\langle \Phi, \mathcal{C} \rangle$, where the universal formula Φ is not valid in $\mathbb{RDLOG}$ and $\mathcal{C}$ is a frame $\mathbf{W}$ together with an evaluation e such that $\mathit{Pred}(\mathbf{W}) \not\models \Phi[e]$.

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Theorem

The universal theory $\mathrm{Th}_\forall(\mathbb{RDLOG})$ is in coNEXPTIME.

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  - exchange ($xy = yx$).

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Corollary

The quasi-equational theory of semilinear rdlogs is coNP-complete.

Open problems

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Thank you!