An Algebraic Proof of the Disjunction Property

Rostislav Horčík
joint work with Kazushige Terui

Institute of Computer Science
Academy of Sciences of the Czech Republic

Algebra & Coalgebra meet Proof Theory
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Complexity issues

- Proving a complexity of the decision problem of a (substructural) logic is often a tedious task usually based on a detailed inspection of the underlying (sequent) calculus.

A typical example is MALL which is known to be PSPACE-complete (Lincoln et al.). The proof is very long and technical.
Motivation

Complexity issues

- Proving a complexity of the decision problem of a (substructural) logic is often a tedious task usually based on a detailed inspection of the underlying (sequent) calculus.

- Such results often rely on proof theoretic methods and presuppose that the logic under consideration possesses a good sequent calculus for which cut-elimination holds.
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- A typical example is MALL which is known to be PSPACE-complete (Lincoln et al.). The proof is very long and technical.

- Can we have uniform methods which work for wider classes of substructural logics?
Our base logic is the full Lambek calculus **FL** (multiplicative-additive fragment of noncommutative intuitionistic linear logic).
Base logic

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Our base logic is the full Lambek calculus $FL$ (multiplicatively-additive fragment of noncommutative intuitionistic linear logic).

- Multiplicative connectives: $\cdot, \backslash, /, 1, 0,$
- Additive connectives: $\vee, \wedge, \bot, \top.$
Our base logic is the **full Lambek calculus FL** (multiplicative-additive fragment of noncommutative intuitionistic linear logic).

- Multiplicative connectives: \( \cdot, \\backslash, /, 1, 0, \)
- Additive connectives: \( \lor, \land, \bot, \top. \)

FL is given by a single-conclusion sequent calculus:

\[
\frac{\Gamma \Rightarrow \alpha}{\Pi, \Gamma, \Sigma \Rightarrow \varphi} \quad (\text{cut})
\]

\[
\frac{\Gamma \Rightarrow \alpha, \Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \lor \beta, \Sigma \Rightarrow \varphi} \quad (\lor \Rightarrow)
\]

\[
\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \lor \psi} \quad (\Rightarrow \lor)
\]

\[
\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \lor \psi} \quad (\Rightarrow \lor)
\]
A substructural logic is an extension of \( \mathbf{FL} \) by a set of rules (axioms) closed under substitutions having the form:

\[
\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n \\
\Gamma_0 \Rightarrow \varphi_0
\]
Substructural logics

Definition

A substructural logic is an extension of FL by a set of rules (axioms) closed under substitutions having the form:

\[
\frac{\Gamma_1 \Rightarrow \varphi_1 \ldots \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}
\]

Example

\[
\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi} \quad (c) \quad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi} \quad (e) \quad \frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi} \quad (i) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \varphi} \quad (o)
\]

- **MALL** = InFL\(_e\) = FL+(e)+(α \setminus 0) \setminus 0 \Rightarrow α,
- **Int** = FL+(e)+(c)+(i)+(o).
Definition

An FL-algebra is an algebra $A = \langle A, \land, \lor, \cdot, /, \backslash, 0, 1 \rangle$, where

- $\langle A, \land, \lor \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,
- 0 is an arbitrary element and
- the following condition holds:

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x \backslash z.$$
Algebraic semantics

Definition
An FL-algebra is an algebra $A = \langle A, \land, \lor, \cdot, /, \setminus, 0, 1 \rangle$, where
- $\langle A, \land, \lor \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,
- 0 is an arbitrary element and
- the following condition holds:

$$x \cdot y \leq z \iff x \leq z / y \iff y \leq x \setminus z.$$ 

Fact
The class of FL-algebras form a variety (i.e., an equational class).
Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of FL-algebras.
Algebraizability

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- Thus there is a dual isomorphism \( Q \) between the lattice of substructural logics and the sub-quasivariety lattice of FL-algebras.
Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of FL-algebras.

- Thus there is a dual isomorphism $Q$ between the lattice of substructural logics and the sub-quasivariety lattice of FL-algebras.

- Let $L$ be a substructural logic. Then we have the following equivalences:

  $\vdash_L \varphi \iff \models_{Q(L)} 1 = 1 \land \varphi \ [1 \leq \varphi]$. 

  $\models_{Q(L)} \varphi = \psi \iff \vdash_L (\varphi \setminus \psi) \land (\psi \setminus \varphi)$. 

By complexity of a substructural logic $L$ we mean the complexity of its set of theorems. Due to algebraizability it is the same as the complexity of the equational theory for $Q(L)$. 

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Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of **FL-algebras**.

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- Let **L** be a substructural logic. Then we have the following equivalences:

  \[ \vdash_L \varphi \iff \models_{Q(L)} 1 = 1 \land \varphi \quad [1 \leq \varphi]. \]

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- By **complexity** of a substructural logic **L** we mean the complexity of its set of theorems. Due to algebraizability it is the same as the complexity of the equational theory for **Q(L)**.
Correspondence between logic and algebra

<table>
<thead>
<tr>
<th>Logic</th>
<th>Algebra</th>
</tr>
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<tbody>
<tr>
<td>logic FL</td>
<td>variety FL</td>
</tr>
<tr>
<td>axiom ( \varphi )</td>
<td>identity ( 1 \leq \varphi )</td>
</tr>
<tr>
<td>inference rule ((r))</td>
<td>quasi-identity ((r^* ))</td>
</tr>
<tr>
<td>axiomatic extension ( L ) of FL</td>
<td>subvariety ( V(L) ) of FL</td>
</tr>
<tr>
<td>rule extension ( L ) of FL</td>
<td>subquasivariety ( Q(L) ) of FL</td>
</tr>
<tr>
<td>consistent</td>
<td>nontrivial</td>
</tr>
</tbody>
</table>

A logic \( L \) is **consistent** if there \( \varphi \) such that \( \not \vdash_L \varphi \).
Definition
Let $\mathbf{L}$ be a substructural logic. Then $\mathbf{L}$ satisfies the disjunction property (DP) if for all formulas $\varphi, \psi$

$$
\vdash_{\mathbf{L}} \varphi \lor \psi \quad \text{implies} \quad \vdash_{\mathbf{L}} \varphi \quad \text{or} \quad \vdash_{\mathbf{L}} \psi.
$$
Disjunction property (DP)

Disjunction Property

Definition

Let $L$ be a substructural logic. Then $L$ satisfies the disjunction property (DP) if for all formulas $\varphi, \psi$

\[ \vdash_L \varphi \lor \psi \text{ implies } \vdash_L \varphi \text{ or } \vdash_L \psi. \]

Analogously, we say the a quasivariety $K$ of FL-algebras has the DP if

\[ \models_K 1 \leq \varphi \lor \psi \text{ implies } \models_K 1 \leq \varphi \text{ or } \models_K 1 \leq \psi. \]
Theorem (Chagrov, Zakharyaschev)

*Every consistent superintuitionistic logic having the DP is PSPACE-hard.*
DP and complexity

**Theorem (Chagrov, Zakharyaschev)**

*Every consistent superintuitionistic logic having the DP is PSPACE-hard.*

**Theorem**

*Every consistent substructural logic having the DP is PSPACE-hard.*
Why is the DP interesting?

DP and complexity

Theorem (Chagrov, Zakharyaschev)

Every consistent superintuitionistic logic having the DP is PSPACE-hard.

Theorem

Every consistent substructural logic having the DP is PSPACE-hard.

Proof.

By reduction from the set of true quantified Boolean formulas.
Why is the DP interesting?

**DP and complexity**

**Theorem (Chagrov, Zakharyaschev)**

*Every consistent superintuitionistic logic having the DP is PSPACE-hard.*

**Theorem**

*Every consistent substructural logic having the DP is PSPACE-hard.*

**Proof.**

By reduction from the set of true quantified Boolean formulas. □

**Remark**

One cannot use the coding of quantifiers from MALL. It does not work for some logics having the DP, e.g. FL + αβ ∧ αγ ⇒ α(β ∧ γ).
How to prove the DP?

Proof-theoretic proof of DP

- How to prove the DP?

If our logic \( L \) enjoys the cut-elimination then one can use the following.

The very last rule in every cut-free proof of \( \Rightarrow \phi \lor \psi \) has to be \( (\Rightarrow \lor) \). Thus either \( \Rightarrow \phi \) or \( \Rightarrow \psi \) is provable.

What can we do if our logic does not have a cut-free presentation? E.g. if \( L \) is the extension of \( FL \) by \( \alpha \lor \beta \Rightarrow \beta \) and \( \beta \lor \alpha \Rightarrow \beta \).
Proof-theoretic proof of DP

- How to prove the DP?
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How to prove the DP?

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If our logic $L$ enjoys the cut-elimination then one can use the following.

The very last rule in every cut-free proof of $\Rightarrow \varphi \lor \psi$ has to be $(\Rightarrow \lor)$. Thus either $\Rightarrow \varphi$ or $\Rightarrow \psi$ is provable.

What can we do if our logic does not have a cut-free presentation? E.g. if $L$ is the extension of $FL$ by $\alpha \backslash \alpha \beta \Rightarrow \beta$ and $\beta \alpha / \alpha \Rightarrow \beta$. 
Definition

An FL-algebra $A$ is called well-connected if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$. 
Algebraic characterization of the DP

Definition
An FL-algebra $A$ is called well-connected if for all $x, y \in A$, $x \lor y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Theorem
Let $L$ be a substructural logic. Then $L$ has the DP iff the following condition holds:

(*) for every $A \in Q(L)$ there is a well-connected FL-algebra $C \in Q(L)$ such that $A$ is a homomorphic image of $C$. 
Algebraic characterization of the DP

Definition
An FL-algebra $\mathbf{A}$ is called well-connected if for all $x, y \in A$, $x \lor y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Theorem
Let $\mathbf{L}$ be a substructural logic. Then $\mathbf{L}$ has the DP iff the following condition holds:

(*) for every $\mathbf{A} \in \mathcal{Q}(\mathbf{L})$ there is a well-connected FL-algebra $\mathbf{C} \in \mathcal{Q}(\mathbf{L})$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{C}$.

Using this theorem we would like to find a large class of quasi-varieties of FL-algebras having the DP.
Definition

An \( \ell \)-monoidal quasi-identity is a quasi-identity

\[
t_1 \leq u_1 \quad \text{and} \quad \ldots \quad \text{and} \quad t_n \leq u_n \quad \implies \quad t_0 \leq u_0,
\]

where \( t_i \) is in the language \( \{ \cdot, \wedge, \vee, 1 \} \) and \( u_i \) is either 0 or in the language \( \{ \cdot, \wedge, \vee, 1 \} \).
**Definition**

An $\ell$-monoidal quasi-identity is a quasi-identity

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_n \leq u_n \implies t_0 \leq u_0,$$

where $t_i$ is in the language $\{\cdot, \wedge, \vee, 1\}$ and $u_i$ is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$.

Accordingly, an $\ell$-monoidal rule is a rule

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \ldots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

where $\Gamma_i$ is a sequence of formulas in the language $\{\cdot, \wedge, \vee, 1\}$ and $\varphi_i$ is either empty or a formula in the language $\{\cdot, \wedge, \vee, 1\}$. 
Useful algebra

Fix a quasivariety $K$ of FL-algebras defined by $\ell$-monoidal quasi-identities.
Useful algebra

- Fix a quasivariety $K$ of FL-algebras defined by $\ell$-monoidal quasi-identities.

**Lemma**

For any nontrivial algebra $A \in K$, there is an integral FL-algebra $B \in K$ which has a unique subcover of 1.
How to prove the DP?

**Useful algebra**

- Fix a quasivariety $K$ of FL-algebras defined by $\ell$-monoidal quasi-identities.

**Lemma**

For any nontrivial algebra $A \in K$, there is an integral FL-algebra $B \in K$ which has a unique subcover of $1$.

**Proof.**

Let $a \in A$ such that $a < 1$ and $B = \{a^n \mid n \geq 0\}$. The submonoid $B$ gives rise to an FL-algebra $B$ by setting

$$x \to y = \bigvee \{z \in B \mid xz \leq y\}$$

$$0_B = 1 \text{ or } a \text{ (depending whether } \models_A 1 \leq 0 \text{ or not).}$$
Construction of a well-connected algebra
Construction of a well-connected algebra

\[ \langle 1_A, 1_B \rangle \]

\[ \langle 1_A, s \rangle \]

\[ A \times B \]
Construction of a well-connected algebra
Construction of a well-connected algebra

- $A \times B$ belongs to $K$. 

\[ \sigma[A \times B] \text{ is a subalgebra of } A \times B \text{ with respect to the language } \{·, ∧, ∨, 1, 0\}. \]

\[ \sigma[A \times B] \text{ is an image of an interior operator } \sigma. \]

Moreover, we have
\[ \sigma(x) \sigma(y) \leq \sigma(xy), \] i.e., $\sigma$ is a conucleus.

Thus $\sigma[A \times B]$ is an FL-algebra ($x \;\sigma\; y = \sigma(x \;\sigma\; y)$).

$\sigma[A \times B]$ is well-connected.

$A$ is a homomorphic image of $\sigma[A \times B]$. 
Construction of a well-connected algebra

- $A \times B$ belongs to $K$.
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Construction of a well-connected algebra

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- Thus $\sigma[A \times B]$ is an FL-algebra ($x \setminus_\sigma y = \sigma(x \setminus y)$).
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- Moreover, we have $\sigma(x)\sigma(y) \leq \sigma(xy)$, i.e., $\sigma$ is a conucleus.
- Thus $\sigma[A \times B]$ is an FL-algebra ($x \setminus_\sigma y = \sigma(x \setminus y)$).
- $\sigma[A \times B]$ is well-connected.
- $A$ is a homomorphic image of $\sigma[A \times B]$. 
DP for \( \ell \)-monoidal extensions

**Theorem**

*Every quasivariety of FL-algebras defined by \( \ell \)-monoidal quasi-identities has the DP. Every extension of FL by \( \ell \)-monoidal rules has the DP.*
DP for $\ell$-monoidal extensions

Theorem

*Every quasivariety of FL-algebras defined by $\ell$-monoidal quasi-identities has the DP.*

*Every extension of FL by $\ell$-monoidal rules has the DP.*

Example

- Every extension of FL by structural rules (e), (c), (i), (o) enjoys the DP.
DP for $\ell$-monoidal extensions

Theorem

Every quasivariety of FL-algebras defined by $\ell$-monoidal quasi-identities has the DP.

Every extension of FL by $\ell$-monoidal rules has the DP.

Example

- Every extension of FL by structural rules (e), (c), (i), (o) enjoys the DP.
- The extension of FL by the rule

$$\Rightarrow \varphi \cdot \psi \Rightarrow \varphi$$

has the DP. It defines a proper subquasivariety of FL.
Note that $xy/y = x = y \setminus yx$ are equivalent to $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$. 
How to prove the DP?

\( M_2 \)-axioms

- Note that \( xy/y = x = y \setminus yx \) are equivalent to \( xz = yz \Rightarrow x = y \) and \( zx = zy \Rightarrow x = y \).

**Definition (Class \( M_2 \))**

Let \( \mathcal{V} \) be a set of variables. Given a set \( T \) of terms, let \( T^\circ \) be its closure under the operations \( \{ \cdot, \wedge, \vee, 1 \} \).
\textbf{\(M_2\)-axioms}

- Note that \(xy/y = x = y \setminus yx\) are equivalent to \(xz = yz \Rightarrow x = y\) and \(zx = zy \Rightarrow x = y\).

\textbf{Definition (Class \(M_2\))}

Let \(\mathcal{V}\) be a set of variables. Given a set \(T\) of terms, let \(T^\circ\) be its closure under the operations \(\{\cdot, \land, \lor, 1\}\). Likewise, let \(T^\bullet\) be its closure under the following rules:

- \(0 \in T^\bullet\), \(\mathcal{V}^\circ \subseteq T^\bullet\);
- if \(t, u \in T^\bullet\) then \(t \land u \in T^\bullet\);
- if \(t \in T^\circ\) and \(u \in T^\bullet\), then \(t \setminus u, u/t \in T^\bullet\).
How to prove the DP?

$\mathcal{M}_2$-axioms

- Note that $xy/y = x = y \setminus yx$ are equivalent to $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$.

Definition (Class $\mathcal{M}_2$)

Let $\mathcal{V}$ be a set of variables. Given a set $T$ of terms, let $T^\circ$ be its closure under the operations $\{\cdot, \land, \lor, 1\}$. Likewise, let $T^\bullet$ be its closure under the following rules:

- $0 \in T^\bullet$, $\mathcal{V}^\circ \subseteq T^\bullet$;
- if $t, u \in T^\bullet$ then $t \land u \in T^\bullet$;
- if $t \in T^\circ$ and $u \in T^\bullet$, then $t \setminus u, u/t \in T^\bullet$.

We define $\mathcal{M}_1 = \mathcal{V}^\bullet$ and $\mathcal{M}_2 = \mathcal{M}_1^\bullet$. An identity $t \leq u$ belongs to $\mathcal{M}_2$ if $t \in \mathcal{M}_1^\circ$ and $u \in \mathcal{M}_2$. Analogously, $\alpha \Rightarrow \beta \in \mathcal{M}_2$ if $\alpha \in \mathcal{M}_1^\circ$ and $\beta \in \mathcal{M}_2$. 

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### Examples of $\mathcal{M}_2$-axioms

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Name</th>
</tr>
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<tbody>
<tr>
<td>$\alpha \beta \Rightarrow \beta \alpha$</td>
<td>exchange (e)</td>
</tr>
<tr>
<td>$\alpha \Rightarrow 1$</td>
<td>integrality, left weakening (i)</td>
</tr>
<tr>
<td>$0 \Rightarrow \alpha$</td>
<td>right weakening (o)</td>
</tr>
<tr>
<td>$\alpha \Rightarrow \alpha \alpha$</td>
<td>contraction (c)</td>
</tr>
<tr>
<td>$\alpha^n \Rightarrow \alpha^m$</td>
<td>knotted axioms ($n, m \geq 0$)</td>
</tr>
<tr>
<td>$\alpha \land (\alpha \setminus 0) \Rightarrow$</td>
<td>no-contradiction</td>
</tr>
<tr>
<td>$\alpha \beta/\beta \Rightarrow \alpha$, $\alpha \setminus \alpha \beta \Rightarrow \beta$</td>
<td>cancellativity</td>
</tr>
<tr>
<td>$\alpha \land (\beta \lor \gamma) \Rightarrow (\alpha \land \beta) \lor (\alpha \land \gamma)$</td>
<td>distributivity</td>
</tr>
<tr>
<td>$((\alpha \land \beta) \lor \gamma) \land \beta \Rightarrow (\alpha \land \beta) \lor (\gamma \land \beta)$</td>
<td>modularity</td>
</tr>
<tr>
<td>$\alpha \beta \land \alpha \gamma \Rightarrow \alpha(\beta \land \gamma)$</td>
<td>$(\cdot, \land)$-distributivity</td>
</tr>
<tr>
<td>$\alpha \land (\beta \gamma) \Rightarrow (\alpha \land \beta)(\alpha \land \gamma)$</td>
<td>$(\land, \cdot)$-distributivity</td>
</tr>
</tbody>
</table>
How to prove the DP?

**DP for extensions by $\mathcal{M}_2$-axioms**

**Theorem**

*Every identity in $\mathcal{M}_2$ is equivalent in FL to a set of $\ell$-monoidal quasi-identities.*

**Corollary**

*Every extension of FL by $\mathcal{M}_2$-axioms has the DP.*
Negations: $\sim \varphi = \varphi \setminus 0$, $-\varphi = 0/\varphi$. 

Let $L$ be a substructural logic. Then $In_{L}$ is $L + (DN)$. 

Theorem: Every extension of $In_{FL}$ and $In_{FL} e (MALL)$ by inference rules in the language $\{\&, \lor, 1\}$ has the DP.

Example: The distributive extension of $In_{FL} e$ has the DP. Thus the relevance logic $RW$ has the DP.
Involution substructural logics

- Negations: \( \sim \varphi = \varphi \setminus 0, \quad \neg \varphi = 0/\varphi \).
- Double negation elimination laws (DN): \( \sim \neg \varphi \Rightarrow \varphi, \quad \neg \sim \varphi \Rightarrow \varphi \).
How to prove the DP?

Involution substructural logics

- Negations: $\sim \varphi = \varphi \setminus 0$, $-\varphi = 0/\varphi$.
- Double negation elimination laws (DN): $\sim -\varphi \Rightarrow \varphi$, $-\sim \varphi \Rightarrow \varphi$.
- Let $L$ be a substructural logic. Then $\text{InL}$ is $L+(DN)$.
Involution substructural logics

- Negations: $\neg \varphi = \varphi \setminus 0$, $\neg \varphi = 0/\varphi$.
- Double negation elimination laws (DN): $\neg \neg \varphi \Rightarrow \varphi$, $\neg \varphi \Rightarrow \varphi$.
- Let $L$ be a substructural logic. Then $\text{In}_L$ is $L+(\text{DN})$.

**Theorem**

*Every extension of InFL and InFL$_e$ (MALL) by inference rules in the language $\{\land, \lor, 1\}$ has the DP.*
Involutive substructural logics

- Negations: \( \sim \varphi = \varphi \setminus 0, \neg \varphi = 0/\varphi \).
- Double negation elimination laws (DN): \( \sim \neg \varphi \Rightarrow \varphi, \neg \sim \varphi \Rightarrow \varphi \).
- Let \( L \) be a substructural logic. Then \( \text{InL} \) is \( L+(DN) \).

**Theorem**

Every extension of \( \text{InFL} \) and \( \text{InFL}_e \) (MALL) by inference rules in the language \( \{ \wedge, \vee, 1 \} \) has the DP.

**Example**

The distributive extension of \( \text{InFL}_e \) has the DP. Thus the relevance logic \( \text{RW} \) has the DP.
How to prove the DP?

Construction for involutive logics

\[ \langle 1_A, 1 \rangle \]
\[ \langle 1_A, 1/2 \rangle \]
\[ \langle 0_A, 1/2 \rangle \]
\[ \langle 0_A, 0 \rangle \]
Theorem

Let $L$ be a consistent substructural logic. The decision problem for $L$ is coNP-hard. If $L$ further satisfies the DP, then it is PSPACE-hard.
Conclusions

Theorem

Let $L$ be a consistent substructural logic. The decision problem for $L$ is coNP-hard. If $L$ further satisfies the DP, then it is PSPACE-hard.

Corollary

Let $L$ be a consistent extension of $FL$ by $\ell$-monoidal inference rules and/or $M_2$-axioms. Then the decision problem for $L$ is PSPACE-hard. The same is true also for every consistent extension of $InFL$ or $InFL_e$ by inference rules in the language $\{\land, \lor, 1\}$. 
How to prove the DP?

Conclusions

Theorem

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The DP is a sufficient condition for PSPACE-hardness but not a necessary one. A counterexample is $LQ$ obtained by extending intuitionistic logic with the law $\neg\alpha \lor \neg\neg\alpha$. 