An Algebraic Proof of the Disjunction Property

Rostislav Horčík
joint work with Kazushige Terui

Institute of Computer Science
Academy of Sciences of the Czech Republic

Algebra & Coalgebra meet Proof Theory
Bern 2011

Outline

Outline

Outline

Outline

Outline

1. Motivation
2. Substructural logics
3. Disjunction property (DP)
4. Why is the DP interesting?
5. How to prove the DP?

Complexity issues

- Proving a complexity of the decision problem of a (substructural) logic is often a tedious task usually based on a detailed inspection of the underlying (sequent) calculus.

Complexity issues

- Proving a complexity of the decision problem of a (substructural) logic is often a tedious task usually based on a detailed inspection of the underlying (sequent) calculus.

- Such results often rely on proof theoretic methods and presuppose that the logic under consideration possesses a good sequent calculus for which cut-elimination holds.

Complexity issues

- Proving a complexity of the decision problem of a (substructural) logic is often a tedious task usually based on a detailed inspection of the underlying (sequent) calculus.
- Such results often rely on proof theoretic methods and presuppose that the logic under consideration possesses a good sequent calculus for which cut-elimination holds.
- A typical example is MALL which is known to be PSPACE-complete (Lincoln et al.). The proof is very long and technical.

Complexity issues

- Proving a complexity of the decision problem of a (substructural) logic is often a tedious task usually based on a detailed inspection of the underlying (sequent) calculus.
- Such results often rely on proof theoretic methods and presuppose that the logic under consideration possesses a good sequent calculus for which cut-elimination holds.
- A typical example is MALL which is known to be PSPACE-complete (Lincoln et al.). The proof is very long and technical.
- Can we have uniform methods which work for wider classes of substructural logics?

Base logic

- Our base logic is the full Lambek calculus **FL** (multiplicative-additive fragment of noncommutative intuitionistic linear logic).

Base logic

- Our base logic is the full Lambek calculus **FL** (multiplicative-additive fragment of noncommutative intuitionistic linear logic).
- Multiplicative connectives: $\cdot, \backslash, /, 1, 0,$

Base logic

- Our base logic is the full Lambek calculus **FL** (multiplicative-additive fragment of noncommutative intuitionistic linear logic).
- Multiplicative connectives: $\cdot, \backslash, /, 1, 0$,
- Additive connectives: $\vee, \wedge, \bot, \top$.

Base logic

- Our base logic is the full Lambek calculus **FL** (multiplicative-additive fragment of noncommutative intuitionistic linear logic).
- Multiplicative connectives: $\cdot, \backslash, /, 1, 0$,
- Additive connectives: $\vee, \wedge, \bot, \top$.
- **FL** is given by a single-conclusion sequent calculus:

$$\alpha \Rightarrow \alpha \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \alpha, \Sigma \Rightarrow \varphi}{\Pi, \Gamma, \Sigma \Rightarrow \varphi}\ (\text{cut})$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \varphi \qquad \Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Sigma \Rightarrow \varphi}\ (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow\vee) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow\vee)$$

$$\vdots$$

Substructural logics

Definition

A substructural logic is an extension of **FL** by a set of rules (axioms) closed under substitutions having the form:

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

Substructural logics

Definition

A substructural logic is an extension of **FL** by a set of rules (axioms) closed under substitutions having the form:

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

Example

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}\text{(c)} \quad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}\text{(e)} \quad \frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}\text{(i)} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\text{(o)}$$

- $\mathbf{MALL} = \mathbf{InFL_e} = \mathbf{FL}$+(e)+$(\alpha \setminus 0) \setminus 0 \Rightarrow \alpha$,
- $\mathbf{Int} = \mathbf{FL}$+(e)+(c)+(i)+(o).

Algebraic semantics

Definition

An FL-algebra is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1\rangle$, where

- $\langle A, \wedge, \vee\rangle$ is a lattice,
- $\langle A, \cdot, 1\rangle$ is a monoid,
- 0 is an arbitrary element and
- the following condition holds:

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z\,.$$

Algebraic semantics

Definition

An FL-algebra is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle$, where

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,
- 0 is an arbitrary element and
- the following condition holds:

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z \,.$$

Fact

The class of FL-algebras form a variety (i.e., an equational class).

Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of FL-algebras.

Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of FL-algebras.
- Thus there is a dual isomorphism Q between the lattice of substructural logics and the sub-quasivariety lattice of FL-algebras.

Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of FL-algebras.
- Thus there is a dual isomorphism Q between the lattice of substructural logics and the sub-quasivariety lattice of FL-algebras.
- Let **L** be a substructural logic. Then we have the following equivalences:

$$\vdash_{\mathbf{L}} \varphi \quad \text{iff} \quad \models_{\mathsf{Q}(\mathbf{L})} 1 = 1 \wedge \varphi \quad [1 \leq \varphi]\,.$$

$$\models_{\mathsf{Q}(\mathbf{L})} \varphi = \psi \quad \text{iff} \quad \vdash_{\mathbf{L}} (\varphi \backslash \psi) \wedge (\psi \backslash \varphi)\,.$$

Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of FL-algebras.
- Thus there is a dual isomorphism Q between the lattice of substructural logics and the sub-quasivariety lattice of FL-algebras.
- Let **L** be a substructural logic. Then we have the following equivalences:

$$\vdash_{\mathbf{L}} \varphi \quad \text{iff} \quad \models_{\mathrm{Q}(\mathbf{L})} 1 = 1 \wedge \varphi \quad [1 \leq \varphi]\,.$$

$$\models_{\mathrm{Q}(\mathbf{L})} \varphi = \psi \quad \text{iff} \quad \vdash_{\mathbf{L}} (\varphi \backslash \psi) \wedge (\psi \backslash \varphi)\,.$$

- By complexity of a substructural logic **L** we mean the complexity of its set of theorems. Due to algebraizability it is the same as the complexity of the equational theory for Q(**L**).

Correspondence between logic and algebra

Logic	Algebra
logic **FL**	variety FL
axiom φ	identity $1 \leq \varphi$
inference rule (r)	quasi-identity $(r^{\cdot})$
axiomatic extension **L** of **FL**	subvariety V(**L**) of FL
rule extension **L** of **FL**	subquasivariety Q(**L**) of FL
consistent	nontrivial

- A logic **L** is consistent if there φ such that $\nvdash_{\mathbf{L}} \varphi$.

Disjunction Property

Definition

Let **L** be a substructural logic. Then **L** satisfies the disjunction property (DP) if for all formulas φ, ψ

$$\vdash_{\mathbf{L}} \varphi \vee \psi \quad \text{implies} \quad \vdash_{\mathbf{L}} \varphi \quad \text{or} \quad \vdash_{\mathbf{L}} \psi.$$

Disjunction Property

Definition

Let **L** be a substructural logic. Then **L** satisfies the disjunction property (DP) if for all formulas φ, ψ

$$\vdash_{\mathbf{L}} \varphi \vee \psi \quad \text{implies} \quad \vdash_{\mathbf{L}} \varphi \quad \text{or} \quad \vdash_{\mathbf{L}} \psi.$$

Analogously, we say the a quasivariety K of FL-algebras has the DP if

$$\models_{\mathsf{K}} 1 \leq \varphi \vee \psi \quad \text{implies} \quad \models_{\mathsf{K}} 1 \leq \varphi \quad \text{or} \quad \models_{\mathsf{K}} 1 \leq \psi.$$

DP and complexity

Theorem (Chagrov, Zakharyaschev)

Every consistent superintuitionistic logic having the DP is PSPACE-hard.

DP and complexity

Theorem (Chagrov, Zakharyaschev)

Every consistent superintuitionistic logic having the DP is PSPACE-hard.

Theorem

Every consistent substructural logic having the DP is PSPACE-hard.

DP and complexity

Theorem (Chagrov, Zakharyaschev)

Every consistent superintuitionistic logic having the DP is PSPACE-hard.

Theorem

Every consistent substructural logic having the DP is PSPACE-hard.

Proof.

By reduction from the set of true quantified Boolean formulas. □

DP and complexity

Theorem (Chagrov, Zakharyaschev)

Every consistent superintuitionistic logic having the DP is PSPACE-hard.

Theorem

Every consistent substructural logic having the DP is PSPACE-hard.

Proof.

By reduction from the set of true quantified Boolean formulas. □

Remark

One cannot use the coding of quantifiers from MALL. It does not work for some logics having the DP, e.g. **FL** $+ \alpha\beta \wedge \alpha\gamma \Rightarrow \alpha(\beta \wedge \gamma)$.

Proof-theoretic proof of DP

- How to prove the DP?

Proof-theoretic proof of DP

- How to prove the DP?
- If our logic **L** enjoys the cut-elimination then one can use the following.

Proof-theoretic proof of DP

- How to prove the DP?
- If our logic **L** enjoys the cut-elimination then one can use the following.
- The very last rule in every cut-free proof of $\Rightarrow \varphi \vee \psi$ has to be $(\Rightarrow\vee)$. Thus either $\Rightarrow \varphi$ or $\Rightarrow \psi$ is provable.

Proof-theoretic proof of DP

- How to prove the DP?
- If our logic **L** enjoys the cut-elimination then one can use the following.
- The very last rule in every cut-free proof of $\Rightarrow \varphi \vee \psi$ has to be $(\Rightarrow\vee)$. Thus either $\Rightarrow \varphi$ or $\Rightarrow \psi$ is provable.
- What can we do if our logic does not have a cut-free presentation? E.g. if **L** is the extension of **FL** by $\alpha \backslash \alpha\beta \Rightarrow \beta$ and $\beta\alpha/\alpha \Rightarrow \beta$.

Algebraic characterization of the DP

Definition

An FL-algebra $\boldsymbol{A}$ is called well-connected if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Algebraic characterization of the DP

Definition

An FL-algebra $\boldsymbol{A}$ is called well-connected if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Theorem

Let $\mathbf{L}$ *be a substructural logic. Then* $\mathbf{L}$ *has the DP iff the following condition holds:*

(*) *for every* $\boldsymbol{A} \in \mathrm{Q}(\mathbf{L})$ *there is a well-connected FL-algebra* $\boldsymbol{C} \in \mathrm{Q}(\mathbf{L})$ *such that* $\boldsymbol{A}$ *is a homomorphic image of* $\boldsymbol{C}$*.*

Algebraic characterization of the DP

Definition

An FL-algebra $\boldsymbol{A}$ is called well-connected if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Theorem

Let $\mathbf{L}$ *be a substructural logic. Then* $\mathbf{L}$ *has the DP iff the following condition holds:*

(*) *for every* $\boldsymbol{A} \in \mathrm{Q}(\mathbf{L})$ *there is a well-connected FL-algebra* $\boldsymbol{C} \in \mathrm{Q}(\mathbf{L})$ *such that* $\boldsymbol{A}$ *is a homomorphic image of* $\boldsymbol{C}$.

Using this theorem we would like to find a large class of quasi-varieties of FL-algebras having the DP.

ℓ-monoidal quasi-identities (rules)

Definition

An ℓ-monoidal quasi-identity is a quasi-identity

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_n \leq u_n \implies t_0 \leq u_0 \,,$$

where t_i is in the language $\{\cdot, \wedge, \vee, 1\}$ and u_i is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$.

ℓ-monoidal quasi-identities (rules)

Definition

An ℓ-monoidal quasi-identity is a quasi-identity

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_n \leq u_n \implies t_0 \leq u_0\,,$$

where t_i is in the language $\{\cdot, \wedge, \vee, 1\}$ and u_i is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$.

Accordingly, an ℓ-monoidal rule is a rule

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

where Γ_i is a sequence of formulas in the language $\{\cdot, \wedge, \vee, 1\}$ and φ_i is either empty or a formula in the language $\{\cdot, \wedge, \vee, 1\}$.

Useful algebra

- Fix a quasivariety K of FL-algebras defined by ℓ-monoidal quasi-identities.

Useful algebra

- Fix a quasivariety K of FL-algebras defined by ℓ-monoidal quasi-identities.

Lemma

For any nontrivial algebra $\boldsymbol{A} \in \mathsf{K}$*, there is an integral FL-algebra* $\boldsymbol{B} \in \mathsf{K}$ *which has a unique subcover of* 1*.*

Useful algebra

- Fix a quasivariety K of FL-algebras defined by ℓ-monoidal quasi-identities.

Lemma

For any nontrivial algebra $\boldsymbol{A} \in \mathsf{K}$, there is an integral FL-algebra $\boldsymbol{B} \in \mathsf{K}$ which has a unique subcover of 1.

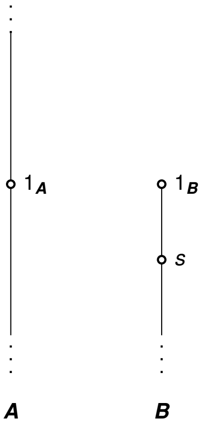
Proof.

Let $a \in A$ such that $a < 1$ and $B = \{a^n \mid n \geq 0\}$. The submonoid B gives rise to an FL-algebra $\boldsymbol{B}$ by setting

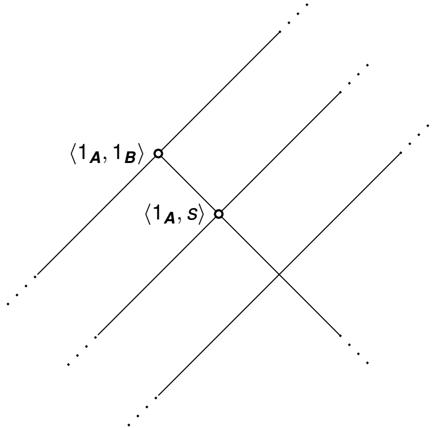
$$
\begin{aligned}
x \to y &= \bigvee\{z \in B \mid xz \leq y\} \\
0_{\boldsymbol{B}} &= 1 \text{ or } a \text{ (depending whether } \models_{\boldsymbol{A}} 1 \leq 0 \text{ or not).}
\end{aligned}
$$

□

Construction of a well-connected algebra

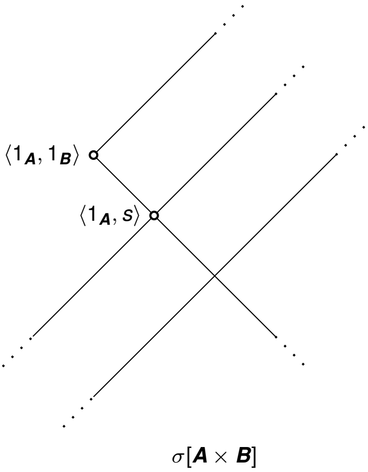


Construction of a well-connected algebra



$\boldsymbol{A} \times \boldsymbol{B}$

Construction of a well-connected algebra



Construction of a well-connected algebra

- $\boldsymbol{A} \times \boldsymbol{B}$ belongs to K.

Construction of a well-connected algebra

- $\boldsymbol{A} \times \boldsymbol{B}$ belongs to K.
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is forms a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$

Construction of a well-connected algebra

- $\boldsymbol{A} \times \boldsymbol{B}$ belongs to K.
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is forms a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an image of an interior operator σ.

Construction of a well-connected algebra

- $\boldsymbol{A} \times \boldsymbol{B}$ belongs to K.
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is forms a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an image of an interior operator σ.
- Moreover, we have $\sigma(x)\sigma(y) \leq \sigma(xy)$, i.e., σ is a conucleus.

Construction of a well-connected algebra

- $\boldsymbol{A} \times \boldsymbol{B}$ belongs to K.
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is forms a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an image of an interior operator σ.
- Moreover, we have $\sigma(x)\sigma(y) \leq \sigma(xy)$, i.e., σ is a conucleus.
- Thus $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an FL-algebra ($x \setminus_\sigma y = \sigma(x \setminus y)$).

Construction of a well-connected algebra

- $\boldsymbol{A} \times \boldsymbol{B}$ belongs to K.
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is forms a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an image of an interior operator σ.
- Moreover, we have $\sigma(x)\sigma(y) \leq \sigma(xy)$, i.e., σ is a conucleus.
- Thus $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an FL-algebra ($x \backslash_\sigma y = \sigma(x \backslash y)$).
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is well-connected.

Construction of a well-connected algebra

- $\boldsymbol{A} \times \boldsymbol{B}$ belongs to K.
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is forms a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an image of an interior operator σ.
- Moreover, we have $\sigma(x)\sigma(y) \leq \sigma(xy)$, i.e., σ is a conucleus.
- Thus $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is an FL-algebra ($x \setminus_\sigma y = \sigma(x \setminus y)$).
- $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is well-connected.
- $\boldsymbol{A}$ is a homomorphic image of $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$.

DP for ℓ-monoidal extensions

Theorem

Every quasivariety of FL-algebras defined by ℓ-monoidal quasi-identities has the DP.
Every extension of **FL** *by ℓ-monoidal rules has the DP.*

DP for ℓ-monoidal extensions

Theorem

Every quasivariety of FL-algebras defined by ℓ-monoidal quasi-identities has the DP.
Every extension of **FL** *by ℓ-monoidal rules has the DP.*

Example

- Every extension of **FL** by structural rules (e), (c), (i), (o) enjoys the DP.

DP for ℓ-monoidal extensions

Theorem

Every quasivariety of FL-algebras defined by ℓ-monoidal quasi-identities has the DP.
Every extension of **FL** *by ℓ-monoidal rules has the DP.*

Example

- Every extension of **FL** by structural rules (e), (c), (i), (o) enjoys the DP.
- The extension of **FL** by the rule

$$\frac{\Rightarrow \varphi \cdot \psi}{\Rightarrow \varphi}$$

has the DP. It defines a proper subquasivariety of FL

$\mathcal{M}_2$-axioms

- Note that $xy/y = x = y \backslash yx$ are equivalent to $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$.

$\mathcal{M}_2$-axioms

- Note that $xy/y = x = y \backslash yx$ are equivalent to $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$.

Definition (Class $\mathcal{M}_2$)

Let $\mathcal{V}$ be a set of variables. Given a set T of terms, let T° be its closure under the operations $\{\cdot, \wedge, \vee, 1\}$.

$\mathcal{M}_2$-axioms

- Note that $xy/y = x = y \backslash yx$ are equivalent to $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$.

Definition (Class $\mathcal{M}_2$)

Let $\mathcal{V}$ be a set of variables. Given a set T of terms, let T° be its closure under the operations $\{\cdot, \wedge, \vee, 1\}$. Likewise, let $T^\bullet$ be its closure under the following rules:

- $0 \in T^\bullet$, $\mathcal{V}^\circ \subseteq T^\bullet$;
- if $t, u \in T^\bullet$ then $t \wedge u \in T^\bullet$;
- if $t \in T^\circ$ and $u \in T^\bullet$, then $t \backslash u, u/t \in T^\bullet$.

$\mathcal{M}_2$-axioms

- Note that $xy/y = x = y \backslash yx$ are equivalent to $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$.

Definition (Class $\mathcal{M}_2$)

Let $\mathcal{V}$ be a set of variables. Given a set T of terms, let T° be its closure under the operations $\{\cdot, \wedge, \vee, 1\}$. Likewise, let $T^\bullet$ be its closure under the following rules:

- $0 \in T^\bullet$, $\mathcal{V}^\circ \subseteq T^\bullet$;
- if $t, u \in T^\bullet$ then $t \wedge u \in T^\bullet$;
- if $t \in T^\circ$ and $u \in T^\bullet$, then $t \backslash u, u/t \in T^\bullet$.

We define $\mathcal{M}_1 = \mathcal{V}^\bullet$ and $\mathcal{M}_2 = \mathcal{M}_1^\bullet$. An identity $t \leq u$ belongs to $\mathcal{M}_2$ if $t \in \mathcal{M}_1^\circ$ and $u \in \mathcal{M}_2$. Analogously, $\alpha \Rightarrow \beta \in \mathcal{M}_2$ if $\alpha \in \mathcal{M}_1^\circ$ and $\beta \in \mathcal{M}_2$.

Examples of $\mathcal{M}_2$-axioms

Axiom	Name
$\alpha\beta \Rightarrow \beta\alpha$	exchange (e)
$\alpha \Rightarrow 1$	integrality, left weakening (i)
$0 \Rightarrow \alpha$	right weakening (o)
$\alpha \Rightarrow \alpha\alpha$	contraction (c)
$\alpha^n \Rightarrow \alpha^m$	knotted axioms ($n, m \geq 0$)
$\alpha \wedge (\alpha \backslash 0) \Rightarrow$	no-contradiction
$\alpha\beta/\beta \Rightarrow \alpha,\ \alpha \backslash \alpha\beta \Rightarrow \beta$	cancellativity
$\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$	distributivity
$((\alpha \wedge \beta) \vee \gamma) \wedge \beta \Rightarrow (\alpha \wedge \beta) \vee (\gamma \wedge \beta)$	modularity
$\alpha\beta \wedge \alpha\gamma \Rightarrow \alpha(\beta \wedge \gamma)$	$(\cdot, \wedge)$-distributivity
$\alpha \wedge (\beta\gamma) \Rightarrow (\alpha \wedge \beta)(\alpha \wedge \gamma)$	$(\wedge, \cdot)$-distributivity

DP for extensions by $\mathcal{M}_2$-axioms

Theorem

Every identity in $\mathcal{M}_2$ is equivalent in FL *to a set of ℓ-monoidal quasi-identities.*

Corollary

Every extension of **FL** *by $\mathcal{M}_2$-axioms has the DP.*

Involutive substructural logics

- Negations: ${\sim}\varphi = \varphi \setminus 0$, $-\varphi = 0/\varphi$.

Involutive substructural logics

- Negations: ${\sim}\varphi = \varphi \setminus 0$, $-\varphi = 0/\varphi$.
- Double negation elimination laws (DN): ${\sim}{-}\varphi \Rightarrow \varphi\,,\; -{\sim}\varphi \Rightarrow \varphi\,.$

Involutive substructural logics

- Negations: $\sim\varphi = \varphi \setminus 0$, $-\varphi = 0 / \varphi$.
- Double negation elimination laws (DN): $\sim -\varphi \Rightarrow \varphi\,,\ -\sim\varphi \Rightarrow \varphi\,.$
- Let **L** be a substructural logic. Then **InL** is **L**+(DN).

Involutive substructural logics

- Negations: $\sim\varphi = \varphi \backslash 0$, $-\varphi = 0/\varphi$.
- Double negation elimination laws (DN): $\sim-\varphi \Rightarrow \varphi\,,\ -\sim\varphi \Rightarrow \varphi\,.$
- Let **L** be a substructural logic. Then **InL** is **L**+(DN).

Theorem

Every extension of **InFL** *and* $\mathbf{InFL_e}$ *(***MALL***) by inference rules in the language* $\{\wedge, \vee, 1\}$ *has the DP.*

Involutive substructural logics

- Negations: $\sim\varphi = \varphi \backslash 0$, $-\varphi = 0/\varphi$.
- Double negation elimination laws (DN): $\sim-\varphi \Rightarrow \varphi\,,\; -\sim\varphi \Rightarrow \varphi\,.$
- Let **L** be a substructural logic. Then **InL** is **L**+(DN).

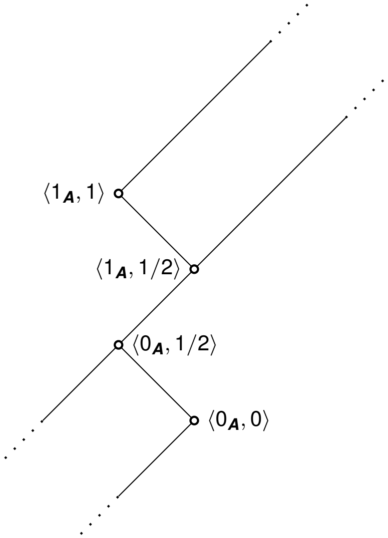
Theorem

Every extension of **InFL** *and* **InFL**$_e$ *(***MALL***) by inference rules in the language* $\{\wedge, \vee, 1\}$ *has the DP.*

Example

The distributive extension of **InFL**$_e$ has the DP. Thus the relevance logic **RW** has the DP.

Construction for involutive logics



Conclusions

Theorem

Let **L** *be a consistent substructural logic. The decision problem for* **L** *is coNP-hard. If* **L** *further satisfies the DP, then it is PSPACE-hard.*

Conclusions

Theorem

Let **L** *be a consistent substructural logic. The decision problem for* **L** *is coNP-hard. If* **L** *further satisfies the DP, then it is PSPACE-hard.*

Corollary

Let **L** *be a consistent extension of* **FL** *by* ℓ*-monoidal inference rules and/or* $\mathcal{M}_2$*-axioms. Then the decision problem for* **L** *is PSPACE-hard. The same is true also for every consistent extension of* **InFL** *or* $\mathbf{InFL_e}$ *by inference rules in the language* $\{\wedge, \vee, 1\}$.

Conclusions

Theorem

Let **L** *be a consistent substructural logic. The decision problem for* **L** *is coNP-hard. If* **L** *further satisfies the DP, then it is PSPACE-hard.*

Corollary

Let **L** *be a consistent extension of* **FL** *by ℓ-monoidal inference rules and/or $\mathcal{M}_2$-axioms. Then the decision problem for* **L** *is PSPACE-hard. The same is true also for every consistent extension of* **InFL** *or* $\mathbf{InFL_e}$ *by inference rules in the language $\{\wedge, \vee, 1\}$.*

The DP is a sufficient condition for PSPACE-hardness but not a necessary one. A counterexample is **LQ** obtained by extending intuitionistic logic with the law $\neg\alpha \vee \neg\neg\alpha$.