Sizes of Countable Sets

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Two great questions

1. The existence of actual infinity? (B. Bolzano, G. Cantor)
2. How to measure infinities? What are their "sizes"?

Galileo's paradox

$$\begin{array}{cccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots \\ 1, & 4, & 9, & 16, & 25, & 36, & 49, & 64, & 81, & \ldots \end{array}$$

- Part-whole principle. The whole is greater than its part.
- Cantor's principle. One-to-one correspondence.

Bolzano and one-to-one correspondence

- „Let us turn now to the consideration of a highly remarkable pecularity . . . Merely from this circumstances we can in no way conclude that these two multitudes are equal to one another.“ (Bolzano, B. *Paradoxes of the Infinite* §20, 1848.)
- „An equality of these multiplicities may only be concluded if some other reason is added, such as that both sets have one and the same determining ground e.g. they have exactly the same way of being formed “ (§21).

Galileo's paradox: How do we get the squares?

- We select them from numbers.
- We create them from numbers.

In Zermelo-Fraenkel Set Theory:

- Schema of Specification $A = \{n;\ n \in \mathbb{N} \wedge (\exists m)(m \in \mathbb{N} \wedge n = m^2)\}$.
- Schema of Replacement $B = \{n^2, n \in \mathbb{N}\}$.

Bolzano's infinite series

- $P \ = 1 + 2 + 3 + \ 4 + \cdots + \text{in inf.}$
- $S \ = 1 + 4 + 9 + 16 + \cdots + \text{in inf.}$
- $\overset{0}{N} = 1 + 1 + 1 + 1 + \cdots + \text{in inf.}$
- $\overset{m}{N} = \underbrace{\ldots}_{m} 1 + 1 + 1 + \cdots + \text{in inf.}$, the first m terms are omitted.

1. Infinite series have one and the same multitude of terms.
2. Unless explicitly stated otherwise, as $\overset{m}{N}$. Then $\overset{0}{N} - \overset{m}{N} = m$.
3. $S > P$ for every corresponding term of S is greater than in P.
4. $S >> P$, i.e. S is greater than every finite multiple of P.
5. If we change the order of finitely many terms of the series the quantity doesn't change.

Interpretation

Bolzano's series $\sim$ non-decreasing sequences of partial sums.

- $P = 1 + 2 + 3 + 4 + \ldots$ in inf. $\sim (1, 3, 6, 10, \ldots) = (\frac{n\cdot(n+1)}{2})_n$
- $S = 1 + 4 + 9 + 16 + \ldots$ in inf. $\sim (1, 5, 16, 32, \ldots) = (\frac{n(n+1)(2n+1)}{6})_n$
- $\overset{0}{N} = 1 + 1 + 1 + 1 + \ldots$ in inf. $\sim (1, 2, 3, \ldots) = (n)_n$
- $\overset{m}{N} = \underbrace{\ldots}_{m} 1 + 1 + 1 + \ldots$ in inf. $\sim (\underbrace{0 \ldots 0}_{m}, 1, 2, 3, \ldots) = (n - m)_n$
- $m = 1 + \cdots + 1 \sim (1, 2 \ldots m, m, m, \ldots)$.

Let S be the set of non-decreasing sequences of natural numbers.

1. Sum $+$ and product $\cdot$ are defined componentwise.
2. $(a_n) =_{\mathcal{F}} (b_n)$ if and only if $(\exists m)(\forall n)(n > m \Rightarrow a_n = b_n)$.
3. $(a_n) <_{\mathcal{F}} (b_n)$ if and only if $(\exists m)(\forall n)(n > m \Rightarrow a_n < b_n)$.

Then $(S, +, \cdot, =_{\mathcal{F}}, <_{\mathcal{F}})$ is a partial ordered non-Archimedean semiring.
Bolzano's assertions are valid.

Set-sizes

A set A is *canonically countable* if there is an arrangement of A into finite disjoint subsets of A (components) according its determining ground

$$A = \bigcup \{A_n, n \in \mathbb{N}\}.$$

- A *set-size* of A is a Bolzano's series

$$|A_1| + |A_2| + |A_3| + \dots \text{in inf.}$$

- A *characteristic sequence* $\chi(A)$

$$\chi(A) = (|A_1|, |A_2|, |A_3| \dots)$$

- A *size sequence* $\sigma(A)$

$$\sigma(A) = (|A_1|, |A_1| + |A_2|, |A_1| + |A_2| + |A_3|, \dots) = (\sum_{i=1}^{n} |A_i|)_n$$

Canonical arrangement

Let $A = \bigcup\{A_n, n \in \mathbb{N}\}$, $B = \bigcup\{B_n, n \in \mathbb{N}\}$ be canonically countable

1. Then
$$A \subseteq B \Rightarrow (\forall n)(n \in \mathbb{N} \Rightarrow A_n \subseteq B_n).$$

2. A canonical arrangement of a Cartesian product $A \times B$

	A_1	A_2	A_3	A_4	$\dots$
B_1	$A_1 \times B_1$	$A_2 \times B_1$	$A_3 \times B_1$	$A_4 \times B_1$	$\dots$
B_2	$A_1 \times B_2$	$A_2 \times B_2$	$A_3 \times B_2$	$A_4 \times B_2$	$\dots$
B_3	$A_1 \times B_3$	$A_2 \times B_3$	$A_3 \times B_3$	$A_3 \times B_3$	$\dots$
B_4	$A_1 \times B_4$	$A_2 \times B_4$	$A_3 \times B_4$	$A_4 \times B_4$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

$$(A \times B)_n = \bigcup\{A_i \times B_j, n = \max\{i, j\}\}.$$

Natural numbers and their subsets

A canonical arrangement of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ is $A_n = \{n\}$

$$\mathbb{N} = \{1\} \cup \{2\} \cup \{3\} \cup \dots$$

1. $\chi(\mathbb{N}) = (1, 1, 1, 1, \dots)$, $\sigma(\mathbb{N}) = (1, 2, 3, 4, \dots) = \alpha$.
2. $\chi(\{3, 4\}) = (0, 0, 1, 1, 0, 0 \dots)$, $\sigma(\{3, 4\}) = (0, 0, 1, 2, 2, 2, \dots) =_{\mathcal{F}} 2$.
3. $\chi(\mathbb{N} \setminus \{3, 4\}) = (1, 1, 0, 0, 1, \dots)$, $\sigma(\mathbb{N} \setminus \{3, 4\}) = (1, 2, 2, 2, 3, \dots)$
   $\sigma(\mathbb{N} \setminus \{3, 4\}) + \sigma(\{3, 4\}) = \alpha$.
4. Even numbers E, $\chi(E) = (0, 1, 0, 1, \dots)$, $\sigma(E) = (0, 1, 1, 2, \dots)$
5. Odd numbers O, $\chi(O) = (1, 0, 1, 0, \dots)$, $\sigma(O) = (1, 1, 2, 2, \dots)$

$$\sigma(E) + \sigma(O) = \alpha, \quad \sigma(E) \leq \sigma(O), \quad \sigma(O) - \sigma(E) \leq 1.$$

Primes, squares, k-multiples

1. Squares $S = \{1, 4, 9, 16, \dots\}$, $\chi(S) = (1, 0, 0, 1, 0, 0, 0, 0, 1, 0 \dots)$, $\sigma(S) = (1, 1, 1, 2, 2, 2, 2, 2, 3, 3 \dots) <_{\mathcal{F}} \alpha$, $\sigma_n(S) = \lfloor \sqrt{n} \rfloor$.
2. k-multiples $M_k = \{k, 2k, 3k, \dots\}$. $\sigma(M_k) = (\underbrace{0, \dots 0}_{k-1}, \underbrace{1, \dots 1}_{k}, \underbrace{2 \dots 2}_{k}, \dots \dots)$, $\sigma_n(M_k) = \lfloor \frac{n}{k} \rfloor$.
3. Let f be an injective function defined on natural numbers $\mathbb{N}$. Let $A \subseteq \mathbb{N}$ such that $A = \{n \in \mathbb{N}; (\exists m)(m \in \mathbb{N} \wedge n = f(m)\}$. Then
$$\sigma_n(A) = \lfloor f^{-1}(n) \rfloor.$$
4. Primes $P = \{1, 2, 3, 5, \dots\}$, $\sigma_n(P) = (\pi(n))$, prime counting function $\frac{n}{\log n} \leq \pi(n) \leq \frac{3n}{\log n}$.

$$\sigma(S) <<_{\mathcal{F}} \sigma(P) <<_{\mathcal{F}} \sigma(M_k).$$

Consequences

If A, B are canonically countable sets then

1. $|A| = n$ is finite if and only if $\sigma(A) =_{\mathcal{F}} n$.
2. $\sigma(A \cup B) = \sigma(A) + \sigma(B) - \sigma(A \cap B)$
3. $\sigma(A \times B) = \sigma(A) \cdot \sigma(B)$
4. If $A \subset B$ then $\sigma(A) <_{\mathcal{F}} \sigma(B)$. Part-whole principle.

Integers

- Negative numbers $\mathbb{N}^{-} = \mathbb{N} \times \{0\}$ have the same canonical arrangement as $\mathbb{N}$.

  The set-size

$$\sigma(\mathbb{N}^{-}) = \sigma(\mathbb{N}) = (1, 2, 3, 4, \dots) = \alpha$$

- Integers $\mathbb{Z} = \mathbb{N} \cup \mathbb{N}^{-} \cup \{0\}$.

  The set-size

$$\sigma(\mathbb{Z}) = \sigma(\mathbb{N}) + \sigma(\mathbb{N}^{-}) + \sigma(\{0\}) = (3, 5, 7, 9, 11, \dots) = 2\alpha + 1.$$

Rational numbers - the interval $\mathbb{I} = (0,1]_{\mathbb{Q}} \subseteq \mathbb{Q}$

$\mathbb{I} = \{\frac{m}{n} \sim [m,n];\ m,n \text{ are coprime and } m < n \text{ or } m = n = 1\} \subseteq \mathbb{N} \times \mathbb{N}$

$$\mathbb{I}_n = \{\frac{m}{n};\ m,n \text{ are coprime and } m < n\}, \quad \mathbb{I}_1 = \{1\}$$

- The canonical arrangement $\mathbb{I} = \{1\} \cup \{\frac{1}{2}\} \cup \{\frac{1}{3}, \frac{2}{3}\} \dots$
- The characteristic sequence $\chi_n(\mathbb{I}) = \varphi(n)$ Euler function

$$\chi(\mathbb{I}) = (1,1,2,2,4,2,6,4,6,\dots).$$

- The set-size $\sigma_n(\mathbb{I}) = \Phi(n)$ totient summatory function

$$\sigma(\mathbb{I}) = (1,2,4,6,10,12,18,22,28,\dots) = \varphi.$$

$$\frac{3}{10} \cdot \alpha^2 < \varphi < \frac{\alpha^2 - \alpha}{2}$$

Rational numbers $\mathbb{Q}$

Positive rational numbers $\mathbb{Q}^+$

$$\mathbb{Q}^+ \sim \mathbb{N}_0 \times \mathbb{I}$$

The set-size $\sigma(\mathbb{Q}^+) = \sigma(\mathbb{N}_0) \cdot \sigma(\mathbb{I}) = (\alpha + 1) \cdot \varphi$

$$\frac{3}{10} \cdot \alpha^3 < \frac{3}{10}(\alpha^3 + \alpha) < \sigma(\mathbb{Q}^+) < \frac{1}{2}(\alpha^3 - \alpha) < \frac{1}{2} \cdot \alpha^3.$$

Rational numbers $\mathbb{Q}$

$$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$$

The set-size $\sigma(\mathbb{Q}) = 2 \cdot \sigma(\mathbb{Q}^+) + 1$

$$\frac{3}{5} \cdot \alpha^3 < \frac{3}{5}(\alpha^3 + \alpha) < \sigma(\mathbb{Q}) < \alpha^3 - \alpha < \alpha^3.$$

All intervals of rational numbers of the same size have the same size.

Theory of numerosity

Benci, Vieri & Di Nasso, Mauro (2003). Numerosities of Labelled Sets: a New Way of Counting, *Advances in Mathematics* 173, p. 50-67.

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- Labelled sets = canonically countable sets.
- An ultrafilter instead of Fréchet filter for $=_{\mathcal{F}}$ and $<_{\mathcal{F}}$.
- Results depend on the choice of an ultrafilter.
- Set-sizes are linearly ordered. *Quis pro quo.*
- A basis of α-calculus, non-standard analysis.
- Basic notions are defined and not justified.
- Some assumption are arbitrary, $\sigma((0,1]_{\mathbb{Q}}) = \alpha$.

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