First-Order Logic of Questions

Vít Punčochář

Institute of Philosophy,
Czech Academy of Sciences,
Czech Republic

Predecessors of inquisitive semantics

Alternative semantics:

- Hamblin, C. L. (1973). Questions in Montague English.
- Karttunen, L. (1977). Syntax and Semantics of Questions.

Partition semantics:

- Groenendijk, J., Stokhof, M. (1984). Studies in the Semantics of Questions and the Pragmatics of Answers.
- Groenendijk, J. (1999). The Logic of Interrogation.

Inquisitive indifference semantics:

- Groenendijk, J. (2009). Inquisitive Semantics: Two Possibilities for Disjunction.
- Mascarenhas, S. (2009). Inquisitive Semantics and Logic. (Master thesis)

The current framework of inquisitive semantics

- Ciardelli, I. (2009). Inquisitive Semantics and Intermediate Logics. (Master thesis)
- Ciardelli, I. (2016) Questions in Logic. (Ph.D. thesis)
- Ciardelli, I., Groenendijk, J., Roelofsen, F. (2019). Inquisitive Semantics.
- Grilletti, G. (2020). Questions and Quantification. (Ph.D. thesis)
- Ciardelli, I. (to appear). Questions in Logic.

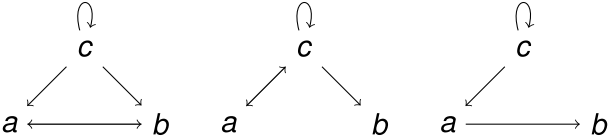
Three aspects of inquisitive logic

1. Questions are types of types (information types)
2. One can define a consequence relation among information types
3. Information types can be combined by logical connectives

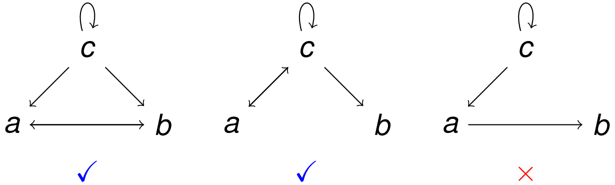
Questions are types of types

- Statements classify structures.
- Questions classify statements.

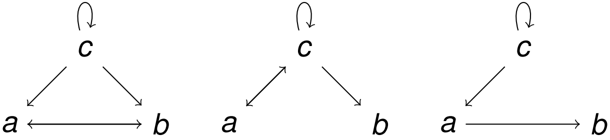
$\forall x Rcx$



$\forall x Rcx$

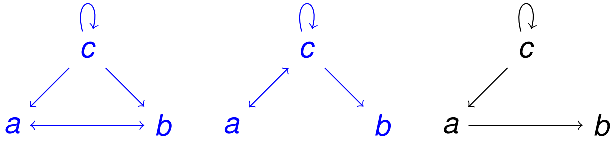


$?\forall x Rcx$



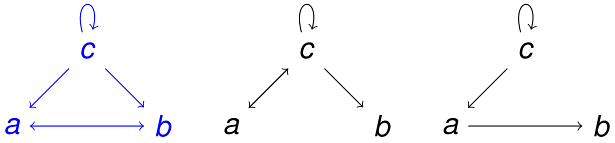
$?\forall x Rcx$

✓ (this state provides the answer YES)



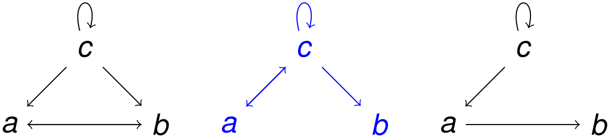
$?\forall x Rcx$

✓ (this state provides the answer YES)



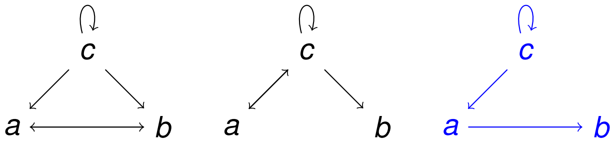
$?\forall x Rcx$

✓ (this state provides the answer YES)



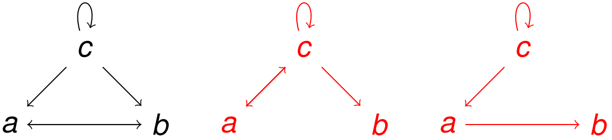
$?\forall x Rcx$

$\checkmark$ (this state provides the answer NO)



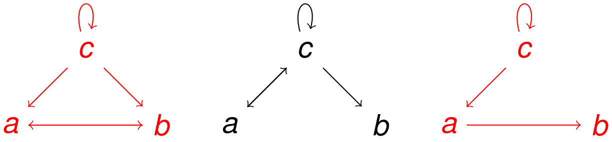
$?\forall xRcx$

× (this state provides no answer)



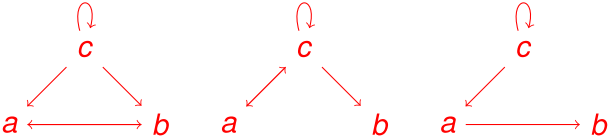
$?\forall x Rcx$

× (this state provides no answer)



?$\forall x Rcx$

× (this state provides no answer)



Algebras of information tokens and of their types

$$A \qquad B$$

structures

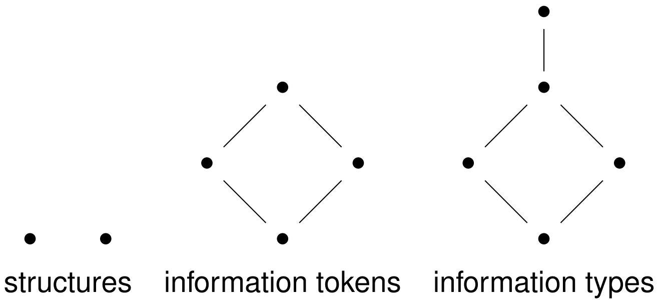
$$\begin{tikzcd}[column sep=small]
& \{A, B\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{A\} \arrow[dr, dash] & & \{B\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

information tokens

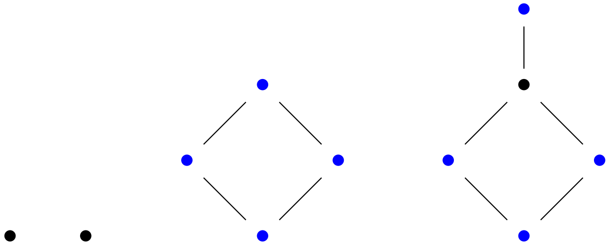
$$\begin{tikzcd}[column sep=small]
& \{\{A, B\}, \{A\}, \{B\}, \emptyset\} \arrow[d, dash] & \\
& \{\{A\}, \{B\}, \emptyset\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{\{A\}, \emptyset\} \arrow[dr, dash] & & \{\{B\}, \emptyset\} \arrow[dl, dash] \\
& \{\emptyset\} &
\end{tikzcd}$$

information types

Algebras of information tokens and of their types

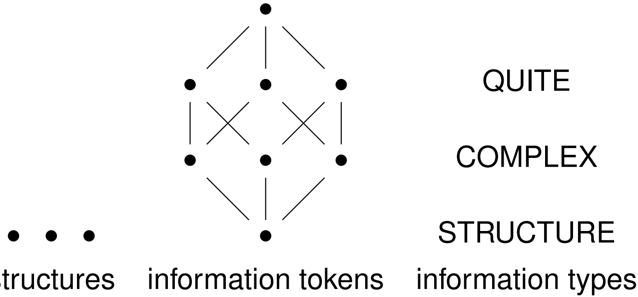


Algebras of information tokens and of their types



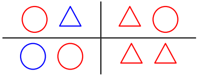
structures   information tokens   information types

Algebras of information states and their types



Entailment among types of information

The space of possibilities S:



Information tokens:

a is a circle, b is a triangle, a is red, . . .

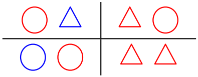
Information types:

shape of a, shape of b, colour of a, colour of b

- *a is a triangle* $\models_S$ *b is red*
- *a is a circle* $\not\models_S$ *b is red*
- *colour of b, shape of a* $\models_S$ *colour of a*
- *colour of b, shape of a* $\not\models_S$ *shape of b*

Entailment among types of information

The space of possibilities S:



Information tokens:

a is a circle, b is a triangle, a is red, . . .

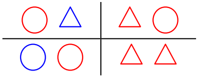
Information types:

shape of a, shape of b, colour of a, colour of b

- *a is a triangle* $\models_S$ *b is red*
- *a is a circle* $\not\models_S$ *b is red*
- *colour of b, shape of a* $\models_S$ *colour of a*
- *colour of b, shape of a* $\not\models_S$ *shape of b*

Entailment among types of information

The space of possibilities S:



Information tokens:

a is a circle, b is a triangle, a is red, ...

Information types:

shape of a, shape of b, colour of a, colour of b

- *a is a triangle* $\models_S$ *b is red*
- *a is a circle* $\not\models_S$ *b is red*
- *colour of b, shape of a* $\models_S$ *colour of a*
- *colour of b, shape of a* $\not\models_S$ *shape of b*

Combining information types

- the shape of *a* and the colour of *b* (an instance: *a* is a circle and *b* is blue)
- the colour of all objects (an instance: *a* is red and *b* is blue)
- dependence of the shape of *b* on the colour of *a* (an instance: if *a* is red then *b* a triangle and if *a* is blue then *b* is a circle)

First-order language

Terms are defined in the usual way. Complex formulas are defined as follows:

$$\varphi ::= \bot \mid t_1 = t_2 \mid Pt_1 \ldots t_n \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x\varphi \mid \varphi \mathbin{\mathbb{V}} \varphi \mid \exists\!\!\exists x\varphi$$

- $\neg\varphi =_{def} \varphi \to \bot$
- $\varphi \vee \psi =_{def} \neg(\neg\varphi \wedge \neg\psi)$
- $\exists x\varphi =_{def} \neg\forall x\neg\varphi$
- $?\varphi =_{def} \varphi \mathbin{\mathbb{V}} \neg\varphi$

- $Pa \mathbin{\mathbb{V}} Qa$ represents the question *whether a has the property P or the property Q*
- $\exists\!\!\exists xPx$ represents the question that asks *what is an object that has the property P*

Some examples

- Is Alice married to Bob? $?Mab$
- Is Alice married to Bob$^{\uparrow}$ or to Charlie$^{\downarrow}$? $Mab \mathbin{\vee\!\!\!\vee} Mac$
- Is Allice married to Bob or to Charlie$^{\uparrow}$ $?(Mab \vee Mac)$
- Who did Alice invite to her wedding? $\forall x ?Iax$
- What is Bob's favorite dish? $\exists xFbx$

Some examples

$$\exists! x \varphi(x) =_{def} \exists x(\varphi(x) \land \forall y(\varphi(y) \rightarrow y = x))$$

- What is the largest city in the world?
- Who is the current president of France?
- Who was the best man at your wedding?

Inquisitive model

An inquisitive model (for a given signature) is a pair $\mathcal{M} = \langle D, W \rangle$, where

- D is a nonempty set,
- W is a set of first-order structures on the domain D.

We can assume that the interpretations of names and function symbols are rigid. Given an evaluation of variables e every term t has a fixed value $t^{\mathcal{M},e}$.

An information state in $\mathcal{M}$ is a subset of W.

Inquisitive semantics

Given an inquisitive model $\mathcal{M} = \langle D, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \ldots t_n$ iff $M \vDash_e Pt_1 \ldots t_n$, for every $M \in W$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \to \psi$ iff for every $t \subseteq s$, if $t \Vdash_e \varphi$, then $t \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \mathbin{\rotatebox[origin=c]{-90}{\(\geqslant\)}} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Inquisitive semantics

Given an inquisitive model $\mathcal{M} = \langle D, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \dots t_n$ iff $M \vDash_e Pt_1 \dots t_n$, for every $M \in W$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \rightarrow \psi$ iff for every $t \subseteq s$, if $t \Vdash_e \varphi$, then $t \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Key properties

Proposition

The following two properties hold generally for every formula φ:

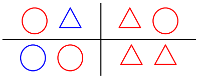
1. *Empty-set property: $\emptyset \Vdash_e \varphi$,*
2. *Persistence: $s \Vdash_e \varphi$ and $t \subseteq s$ implies $t \Vdash_e \varphi$.*

The following property holds for every $\{\exists, \mathbb{V}\}$-free formula α:

3. *Truth-support bridge: $s \Vdash_e \alpha$ iff for all $M \in s$, $M \vDash_e \alpha$.*

Inquisitive vs. declarative existential quantifier

- $s \Vdash_e \exists x P x$ means: in every structure from s there is some object that has the property P.
- $s \Vdash_e \exists x P x$ means: there is some object that in every structure from s has the property P.

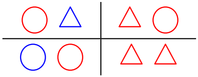


In this state s we have

- $s \Vdash_e \exists x R x$,
- but $s \nVdash_e \exists x P x$.

Inquisitive vs. declarative disjunction

- $s \Vdash_e Pa \vee Qa$ means: in every structure from s, the object a either has the property P or the property Q.
- $s \Vdash_e Pa \mathbin{\vee\!\!\!\vee} Qa$ means: either the object a has the property P in all structures from s, or the object a has the property Q in all structures from s.



In this state s we have

- $s \Vdash_e Ca \vee Ra$,
- but $s \nVdash_e Ca \mathbin{\vee\!\!\!\vee} Ra$.

Inquisitive consequence relation

We define the consequence relation $\models$ as preservation of support.

Proposition

For the $\{\exists, \mathbb{V}\}$-free fragment of the language, the logic corresponds to classical first-order logic.

Disjunction and existence property

Theorem (Grilletti 2018)

Let Γ *be a set of* $\{\exists, \mathbb{V}\}$*-free formulas and* φ, ψ *arbitrary formulas. Then*

(a) *if* $\Gamma \vDash \varphi \mathbb{V} \psi$ *then* $\Gamma \vDash \varphi$ *or* $\Gamma \vDash \psi$,

(b) *if* $\Gamma \vDash \exists x \varphi$ *then for some term* t, $\Gamma \vDash \varphi[t/x]$.

Compactness

Theorem

If every finite subset of Δ is satisfiable then Δ is satisfiable.

Compactness for entailment is an open problem:

- if $\Delta \models \varphi$ then for some finite $\Delta' \subseteq \Delta$, $\Delta' \models \varphi$.

More open problems

- Is the set of valid formulas recursively enumerable? (axiomatization)
- If φ is not valid, is there a counterexample $\langle D, W \rangle$ with countable D and W? (Löwenheim-Skolem)

A fragment of the language $\mathcal{L}^{-}_{inq}$

Only declarative antecedents are allowed:

$$\varphi ::= \bot \mid t_1 = t_2 \mid Pt_1 \dots t_n \mid \varphi \land \varphi \mid \alpha \to \varphi \mid \forall x \varphi \mid \varphi \mathbin{\mathbb{V}} \varphi \mid \exists x \varphi$$

where α is $\{\exists, \mathbb{V}\}$-free

Inquisitive logic in the language $\mathcal{L}^{-}_{inq}$

Intuitionistic logic plus (where α is declarative)

DN $\neg\neg\alpha \to \alpha$,

CD $\forall x(\varphi \mathbin{\mathbb{V}} \psi) \to (\varphi \mathbin{\mathbb{V}} \forall x\psi)$, if x is not free in φ,

$\mathbb{V}$-split $(\alpha \to (\varphi \mathbin{\mathbb{V}} \psi)) \to ((\alpha \to \varphi) \mathbin{\mathbb{V}} (\alpha \to \psi))$,

$\exists$-split $(\alpha \to \exists x\varphi) \to \exists x(\alpha \to \varphi)$, if x is not free in φ.

The derivability relation is denoted by $\vdash$.

Theorem (Grilletti 2020)

Let $\Phi \cup \{\varphi\}$ be a set of $\mathcal{L}^{-}_{inq}$-sentences. Then,

$$\Phi \models \varphi \text{ iff } \Phi \vdash \varphi.$$

Mention-some fragment

$$\chi ::= \alpha \mid \chi \mathbin{\rotatebox[origin=c]{180}{$\mathbb{\wedge}$}} \chi \mid \exists x \chi \mid \chi \wedge \chi \mid \alpha \to \chi$$

where α is $\{\exists, \mathbb{V}\}$-free

Theorem (Ciardelli 2016)

For every χ from the mention-some fragment there are declarative $\alpha_1, \ldots, \alpha_n$ and tuples of variables $\overline{x}_1, \ldots, \overline{x}_n$ such that:

$$\vdash \chi \leftrightarrow \exists \overline{x}_1 \alpha_1 \mathbb{V} \ldots \mathbb{V} \exists \overline{x}_n \alpha_n.$$

Antecedents from the mention-some fragment

$$\chi ::= \alpha \mid \chi \vee\!\!\!\vee \chi \mid \exists x \chi \mid \chi \wedge \chi \mid \alpha \to \chi$$

$$\varphi ::= \bot \mid t_1 = t_2 \mid P t_1 \dots t_n \mid \varphi \wedge \varphi \mid \chi \to \varphi \mid \forall x \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \exists x \varphi$$

where α is $\{\exists, \vee\!\!\!\vee\}$-free

What creates the problem

Formulas like this:

- $\forall x ? Px \rightarrow \exists x Sx$

Two ways of fuzzyfication

(a) Fuzzyfication of the information states
(b) Fuzzyfication of the support relation

Information states

An inquisitive model (for a given signature) is a pair $\mathcal{M} = \langle D, W \rangle$, where

- D is a nonempty set,
- W is a set of first-order structures on the domain D.

We can assume that the interpretations of names and function symbols are rigid. Given an evaluation of variables e every term t has a fixed value $t^{\mathcal{M},e}$.

An information state in $\mathcal{M}$ is a subset of W.

Fuzzy information states

A fuzzy inquisitive model (for a given signature) is a tuple $\mathcal{M} = \langle D, A, W \rangle$, where

- D is a nonempty set,
- A is an algebra of values,
- W is a set of first-order fuzzy structures on the domain D.

We can assume that the interpretations of names and function symbols are rigid. Given an evaluation of variables e every term t has a fixed value $t^{\mathcal{M},e}$.

A fuzzy information state is a fuzzy subset of W.

Crisp and fuzzy information states

- A crisp information state s in an inquisitive model $\mathcal{M}$ is a set of structures from $\mathcal{M}$ that are compatible with the information available in s.
- A fuzzy information state s in a fuzzy inquisitive model $\mathcal{M}$ assigns to M from $\mathcal{M}$ the degree to which M is compatible with the information available in s.

Inquisitive semantics

Given an inquisitive model $\mathcal{M} = \langle D, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \ldots t_n$ iff for all $M \in s$, $M \vDash_e Pt_1 \ldots t_n$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \to \psi$ iff for every $t \subseteq s$, if $t \Vdash_e \varphi$, then $t \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \mathbin{\mathbb{V}} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Inquisitive semantics

Given a fuzzy inquisitive model $\mathcal{M} = \langle D, A, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \ldots t_n$ iff for all $M \in s$, $M \vDash_e Pt_1 \ldots t_n$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \to \psi$ iff for every $t \subseteq s$, if $t \Vdash_e \varphi$, then $t \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \bigvee \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Inquisitive semantics

Given a fuzzy inquisitive model $\mathcal{M} = \langle D, A, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \ldots t_n$ iff for all $M \in W$, $s(M) \leq M^e(Pt_1 \ldots t_1)$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \to \psi$ iff for every $t \sqsubseteq s$, if $t \Vdash_e \varphi$, then $t \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Inquisitive semantics

Given a fuzzy inquisitive model $\mathcal{M} = \langle D, A, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \ldots t_n$ iff for all $M \in W$, $s(M) \leq M^e(Pt_1 \ldots t_1)$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \to \psi$ iff for every t, if $t \Vdash_e \varphi$, then $t \sqcap s \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Inquisitive semantics

Given a fuzzy inquisitive model $\mathcal{M} = \langle D, A, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \ldots t_n$ iff for all $M \in W$, $s(M) \leq M^e(Pt_1 \ldots t_1)$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \to \psi$ iff for every t, if $t \Vdash_e \varphi$, then $t * s \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \mathbin{\rotatebox[origin=c]{180}{$\wedge$}\!\!\!\vee} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Key properties

Proposition

The following two properties hold generally for every formula φ:

1. *Empty-set property: $\emptyset \Vdash_e \varphi$,*
2. *Persistence: $s \Vdash_e \varphi$ and $t \sqsubseteq s$ implies $t \Vdash_e \varphi$.*

The following property holds for every $\{\exists, \mathbb{V}\}$-free formula α:

3. *Truth-support bridge: $s \Vdash_e \alpha$ iff $s(M) \leq M^e(\alpha)$, for all $M \in W$.*

Proposition

For the $\{\exists, \mathbb{V}\}$-free fragment of the language, the logic of fuzzy information states is the corresponding fuzzy first-order logic.

Fuzzy support relation

Consider the question:

- What is the color of *a*?

The statement *a* is blue resolves the question to a greater degree than the statement *a* is blue or green.

We can define the degree to which an information state supports a formula.

Fuzzy support relation

$s \Vdash_e \alpha$ iff $s(M) \leq M^e(\alpha)$, for all $M \in W$

$s[\alpha] = 1$ iff $\bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha)) = 1$

$s[\alpha] = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$

Fuzzy support relation

$s \Vdash_e \alpha$ iff $s(M) \leq M^e(\alpha)$, for all $M \in W$

$s[\alpha] = 1$ iff $\bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha)) = 1$

$s[\alpha] = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$

Fuzzy support relation

$s \Vdash_e \alpha$ iff $s(M) \leq M^e(\alpha)$, for all $M \in W$

$s[\alpha] = 1$ iff $\bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha)) = 1$

$s[\alpha] = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$

Inquisitive semantics

Given a fuzzy inquisitive model $\mathcal{M} = \langle D, A, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define the degree to which an information state supports a formula.

- $s[\bot]_e = \bigwedge_{M \in W}(s(M) \Rightarrow 0)$,
- $s[t_1 = t_2]$ is 0 or 1 according to whether $t_1^{\mathcal{M},e} = t_2^{\mathcal{M},e}$,
- $s[Pt_1 \ldots t_n]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(Pt_1 \ldots t_1))$,
- $s[\varphi \wedge \psi]_e = min\{s[\varphi]_e, s[\psi]_e\}$,
- $s[\varphi \to \psi]_e = \bigwedge_{t \in States} t[\varphi]_e \Rightarrow t{*}s[\psi]_e$,
- $s[\forall x \varphi]_e = \bigwedge_{o \in D} s[\varphi]_{e(o/x)}$,
- $s[\varphi \mathbin{\rotatebox[origin=c]{180}{$\mathbin{\bar{\wedge}}$}} \psi]_e = max\{s[\varphi]_e, s[\psi]_e\}$,
- $s[\exists x \varphi]_e = \bigvee_{o \in D} s[\varphi]_{e(o/x)}$.

Key properties

Proposition

The following two properties hold generally for every formula φ:

1. *Empty-set property*: $\emptyset[\varphi]_e = 1$,
2. *Persistence*: $t \sqsubseteq s$ *implies* $s[\varphi]_e \leq t[\psi]_e$.

The following property holds for every $\{\exists, \mathbb{V}\}$-free formula α:

3. *Truth-support bridge*: $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$.

Truth-support bridge: the inductive step for implication

If it holds for every state s

- $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$,
- $s[\beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\beta))$

then it holds for every state s

- $s[\alpha \to \beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha \to \beta))$.

left side of the equation: $\bigwedge_{t \in State}(t[\alpha]_e \Rightarrow s{*}t[\beta]_e)$, i.e.

$$\bigwedge_{t \in State}(\bigwedge_{M \in W}(t(M) \Rightarrow M^e(\alpha)) \Rightarrow \bigwedge_{M \in W}(t{*}s(M) \Rightarrow M^e(\beta))).$$

Truth-support bridge: the inductive step for implication

If it holds for every state s

- $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$,
- $s[\beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\beta))$

then it holds for every state s

- $s[\alpha \to \beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha \to \beta))$.

left side of the equation: $\bigwedge_{t \in State}(t[\alpha]_e \Rightarrow s{*}t[\beta]_e)$, i.e.

$$\bigwedge_{t \in State}(\bigwedge_{M \in W}(t(M) \Rightarrow M^e(\alpha)) \Rightarrow \bigwedge_{M \in W}(t{*}s(M) \Rightarrow M^e(\beta))).$$

Truth-support bridge: the inductive step for implication

If it holds for every state s

- $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$,
- $s[\beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\beta))$

then it holds for every state s

- $s[\alpha \to \beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha \to \beta))$.

left side of the equation: $\bigwedge_{t \in State}(t[\alpha]_e \Rightarrow s{*}t[\beta]_e)$, i.e.

$$\bigwedge_{t \in State}(\bigwedge_{M \in W}(t(M) \Rightarrow M^e(\alpha)) \Rightarrow \bigwedge_{M \in W}(t{*}s(M) \Rightarrow M^e(\beta))).$$

Truth-support bridge: the inductive step for implication

If it holds for every state s

- $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$,
- $s[\beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\beta))$

then it holds for every state s

- $s[\alpha \to \beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha \to \beta))$.

left side of the equation: $\bigwedge_{t \in State}(t[\alpha]_e \Rightarrow s{*}t[\beta]_e)$, i.e.

$\bigwedge_{t \in State}(\bigwedge_{M \in W}(t(M) \Rightarrow M^e(\alpha)) \Rightarrow \bigwedge_{M \in W}(t{*}s(M) \Rightarrow M^e(\beta)))$.

Truth-support bridge: the inductive step for implication

If it holds for every state s

- $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$,
- $s[\beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\beta))$

then it holds for every state s

- $s[\alpha \to \beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha \to \beta))$.

left side of the equation: $\bigwedge_{t \in State}(t[\alpha]_e \Rightarrow s{*}t[\beta]_e)$, i.e.

$\bigwedge_{t \in State}(\bigwedge_{M \in W}(t(M) \Rightarrow M^e(\alpha)) \Rightarrow \bigwedge_{M \in W}(t{*}s(M) \Rightarrow M^e(\beta)))$.

Truth-support bridge: the inductive step for implication

If it holds for every state s

- $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$,
- $s[\beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\beta))$

then it holds for every state s

- $s[\alpha \to \beta]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha \to \beta))$.

left side of the equation: $\bigwedge_{t \in State}(t[\alpha]_e \Rightarrow s{*}t[\beta]_e)$, i.e.

$$\bigwedge_{t \in State}(\bigwedge_{M \in W}(t(M) \Rightarrow M^e(\alpha)) \Rightarrow \bigwedge_{M \in W}(t{*}s(M) \Rightarrow M^e(\beta))).$$

(the algebra of value can be any bounded commutative residuated lattice)

Truth-support bridge: the inductive step for implication

- $s \Vdash_e \alpha$ iff for all $M \in s$, $M \vDash_e \alpha$.
- $s \Vdash_e \alpha$ iff for all $M \in W$, $s(M) \leq M^e(\alpha)$.
- $s[\alpha]_e = \bigwedge_{M \in W}(s(M) \Rightarrow M^e(\alpha))$.

Proposition

For the $\{\exists, \mathbb{V}\}$-free fragment of the language, the logic of fuzzy information states is the corresponding fuzzy first-order logic.

Algebraic semantics

- Punčochář, V. (2021) Inquisitive Heyting algebras. *Studia Logica*, 109(5), 995-1017.
- Quadrellaro, D.E. On intermediate inquisitive and dependence logics. To appear in Annals of Pure and Applied Logic.

Complete infinitely distributive Heyting algebras

A *complete infinitely distributive Heyting algebra* (H-algebra) is a structure $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0 \rangle$ where

- $\langle H, \bigsqcup, \bigsqcap \rangle$ is a complete lattice in which:

  $s \sqcup \bigsqcap_{i \in I} t_i = \bigsqcap_{i \in I} (s \sqcup t_i)$,
  $s \sqcap \bigsqcup_{i \in I} t_i = \bigsqcup_{i \in I} (s \sqcap t_i)$.

- $\Rightarrow$ is the relative pseudocomplement:

  $s \sqcap t \leq u$ iff $s \leq t \Rightarrow u$.

- 0 is the least element.

Note that $\Rightarrow$ and 0 can be defined in terms of the join and meet:

- $x \Rightarrow y = \bigsqcup \{ z \in H \mid z \sqcap x \leq y \}$,
- $0 = \bigsqcap H$.

Inquisitive and declarative propositions

- information tokens (statements) correspond to principal ideals in the Boolean algebra of information states
- information types (questions) correspond to nonempty downward closed sets in the Boolean algebra of information states

Cored algebras

A *cored algebra* is a structure $\mathcal{A} = \langle A, C(A), \bigsqcup, \bigsqcap, \Rightarrow, 0\rangle$ satisfying the following conditions:

(a) $\mathcal{A}^* = \langle A, \bigsqcup, \bigsqcap, \Rightarrow, 0\rangle$ forms an H-algebra,

(b) $C(A)$ is a subset of A that contains 0 and is closed under $\bigsqcap$ and $\Rightarrow$ ($C(A)$ is called the *core* of $\mathcal{A}$),

(c) $\mathcal{A}_* = \langle C(A), \bigsqcap, \Rightarrow, 0\rangle$ (with $\bigsqcap$ and $\Rightarrow$ restricted to $C(A)$) forms a power-set algebra (i.e. complete atomic Boolean algebra).

Downsets cored algebras

For any power-set algebra $\mathcal{B}$ we define the structure $\mathit{Dw}\mathcal{B} = \langle \mathit{DwB}, \mathit{PDwB}, \bigcup, \bigcap, \Rrightarrow, \{0\}\rangle$, where

- DwB is the set of all non-empty downsets of $\mathcal{B}$,
- PDwB is the set of principal downsets,
- the operations $\bigcup$ and $\bigcap$ are (infinitary) union and intersection,
- and $\Rrightarrow$ is defined as follows:
  - $X \Rrightarrow Y = \bigcup\{Z \in \mathit{DwB} \mid Z \cap X \subseteq Y\}$.

Cored models

A *cored model* is a tuple $\mathcal{N} = \langle \mathcal{A}, D, V \rangle$, where

- $\mathcal{A} = \langle A, C(A), \bigsqcup, \bigsqcap, \Rightarrow, 0 \rangle$ is a cored algebra,
- D is a non-empty set
- V is a valuation, i.e. a function that assigns
  - to every name an element of D,
  - to every n-ary predicate a function that assigns to every n-tuple of elements from D an element of the core.

Algebraic semantics for inquisitive logic

Given $\mathcal{N} = \langle \mathcal{A}, D, V \rangle$ where $\mathcal{A} = \langle A, C(A), \bigsqcup, \bigsqcap, \Rightarrow, 0 \rangle$

- $|\bot|^{\mathcal{N}}_e = 0$,
- $|Pt_1 \dots t_n|^{\mathcal{N}}_e = V(P)(V^e(t_1), \dots, V^e(t_n))$,
- $|\varphi \wedge \psi|^{\mathcal{N}}_e = |\varphi|^{\mathcal{N}}_e \sqcap |\psi|^{\mathcal{N}}_e$,
- $|\varphi \to \psi|^{\mathcal{N}}_e = |\varphi|^{\mathcal{N}}_e \Rightarrow |\psi|^{\mathcal{N}}_e$,
- $|\varphi \vee\!\!\!\vee \psi|^{\mathcal{N}}_e = |\varphi|^{\mathcal{N}}_e \sqcup |\psi|^{\mathcal{N}}_e$,
- $|\forall x \varphi|^{\mathcal{N}}_e = \bigsqcap_{o \in D} |\varphi|^{\mathcal{N}}_{e(o/x)}$,
- $|\exists x \varphi|^{\mathcal{N}}_e = \bigsqcup_{o \in D} |\varphi|^{\mathcal{N}}_{e(o/x)}$.

$\vDash^{alg}_{\mathcal{C}}$ is preservation of validity in algebraic models.

Inquisitive algebras

An *inquisitive algebra* is a cored algebra $\mathcal{A}$ which satisfies the following conditions:

(a) for every $x \in A$, $x = \bigsqcup Y$ for some $Y \subseteq C(A)$,

(b) $C(A)$ is the set of $\bigsqcup$-irreducible elements of $\mathcal{A}$.

A characterization of inquisitive algebras

Theorem

For every power-set algebra $\mathcal{B}$, the structure Dw$\mathcal{B}$ is an inquisitive algebra. Moreover, every inquisitive algebra is c-isomorphic to Dw$\mathcal{B}$ for some power-set algebra $\mathcal{B}$.

Theorem

Let $\mathcal{I}$ be the class of inquisitive algebras. Let $\Phi \cup \{\varphi\}$ be a set of $\mathcal{L}_{inq}$-sentences. Then,

$$\Phi \vDash_{\mathcal{I}}^{alg} \varphi \text{ iff } \Phi \vDash \varphi.$$

Complete c-homomorphism

Consider two cored algebras:

- $\mathcal{A} = \langle A, C(A), \bigsqcup^{\mathcal{A}}, \bigsqcap^{\mathcal{A}}, \Rightarrow^{\mathcal{A}}, 0^{\mathcal{A}} \rangle$,
- $\mathcal{B} = \langle B, C(B), \bigsqcup^{\mathcal{B}}, \bigsqcap^{\mathcal{B}}, \Rightarrow^{\mathcal{B}}, 0^{\mathcal{B}} \rangle$.

and a function h from A to B. Then, h is called a *(complete) c-homomorphism from $\mathcal{A}$ to $\mathcal{B}$* if it satisfies the following conditions for every $x, y, x_i \in A$ (for all $i \in I$ of some index set I):

- $h(C(A)) = C(B)$,
- $h(\bigsqcup^{\mathcal{A}}_{i \in I} x_i) = \bigsqcup^{\mathcal{B}}_{i \in I} h(x_i)$,
- $h(\bigsqcap^{\mathcal{A}}_{i \in I} x_i) = \bigsqcap^{\mathcal{B}}_{i \in I} h(x_i)$,
- $h(x \Rightarrow^{\mathcal{A}} y) = h(x) \Rightarrow^{\mathcal{B}} h(y)$,
- $h(0^{\mathcal{A}}) = 0^{\mathcal{B}}$.

If h is moreover a bijection, it is called a *c-isomorphism*.

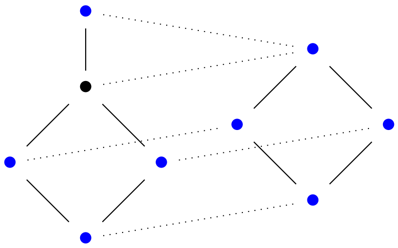
c-homomorphisms preserve and reflect validity

Lemma

Let $\mathcal{A}$, $\mathcal{B}$ be cored algebras, h a c-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, U a non-empty set, and φ an $\mathcal{L}_{inq}$-formula. Then, the following hold:

$$\varphi \text{ is valid in } \langle \mathcal{A}, U \rangle \text{ iff } \varphi \text{ is valid in } \langle \mathcal{B}, U \rangle.$$

Inquisitive algebras are not closed under complete c-homomorphic images



A characterization of c-homomorphic images of inquisitive algebras

Theorem

Let $\mathcal{A} = \langle A, C(A), \bigsqcup, \bigsqcap, \Rightarrow, 0 \rangle$ be a cored algebra. $\mathcal{A}$ is a c-homomorphic image of some inquisitive algebra if and only if it satisfies the following two conditions for every index sets I, J and all $a_{ij}, a, b_i \in C(A)$, where $i \in I$ and $j \in J$:

(1) $\bigsqcap_{i \in I} \bigsqcup_{j \in J} a_{ij} = \bigsqcup_{f\colon I \to J} \bigsqcap_{i \in I} a_{if(i)}$,

(2) $a \Rightarrow \bigsqcup_{i \in I} b_i = \bigsqcup_{i \in I} (a \Rightarrow b_i)$.

Theorem

Let $\mathcal{I}^+$ be the class of all cored models based on cored algebras that satisfy the conditions (1) and (2) above. Let $\Phi \cup \{\varphi\}$ be a set of $\mathcal{L}_{inq}$-sentences. Then,

$$\Phi \vDash^{alg}_{\mathcal{I}^+} \varphi \text{ iff } \Phi \vDash \varphi.$$

Resolutions in propositional logic

- $\mathcal{R}(p) = \{p\}$, $\mathcal{R}(\bot) = \{\bot\}$,
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \alpha \to f(\alpha) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

Theorem

$\varphi \equiv_{InqL} \vee\!\!\!\vee \mathcal{R}(\varphi)$.

Resolutions in predicate logic

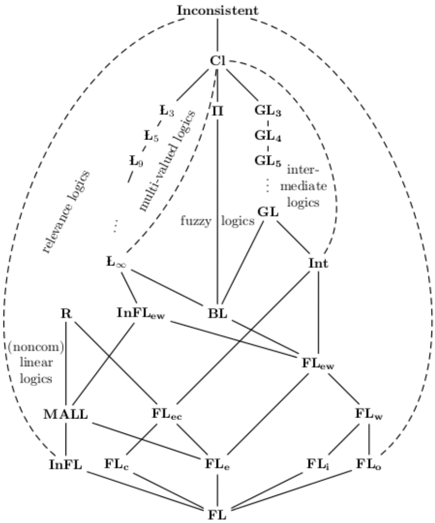
- $\mathcal{R}^e(Pt_1 \ldots t_n) = \{|Pt_1 \ldots t_n|_e\}$, $\mathcal{R}^e(\bot) = \{0\}$,
- $\mathcal{R}^e(\varphi \wedge \psi) = \{a \sqcap b \mid a \in \mathcal{R}^e(\varphi), b \in \mathcal{R}^e(\psi)\}$,
- $\mathcal{R}^e(\varphi \to \psi) = \{\bigsqcap_{a \in \mathcal{R}^e(\varphi)} (a \Rightarrow f(a)) \mid f : \mathcal{R}^e(\varphi) \to \mathcal{R}^e(\psi)\}$,
- $\mathcal{R}^e(\forall x \varphi) = \{\bigsqcap_{m \in U} f(m) \mid f \in x \rightsquigarrow \mathcal{R}^e(\varphi)\}$,
- $\mathcal{R}^e(\varphi \vee \psi) = \mathcal{R}^e(\varphi) \cup \mathcal{R}^e(\psi)$,
- $\mathcal{R}^e(\exists x \varphi) = \bigcup_{m \in U} \mathcal{R}^{e(m/x)}(\varphi)$,

where

$$x \rightsquigarrow \mathcal{R}^e(\varphi) = \{f \mid \text{for all } m \in U : f(m) \in \mathcal{R}^{e(m/x)}(\varphi)\}.$$

Theorem

In cored algebras satisfying (1) and (2), $|\varphi|_e = \bigsqcup \mathcal{R}^e(\varphi)$.



Picture taken from Galatos, N. Jipsen, P. Kowalski, T., Ono, H. (2007) *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier Science.