Going nuclear in Prague

Czech Gathering of Logicians

Jun 17, 2022

Tadeusz Litak

Describable Nuclei

Negative Translations and

Extension Stability

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Tadeusz Litak with the help of

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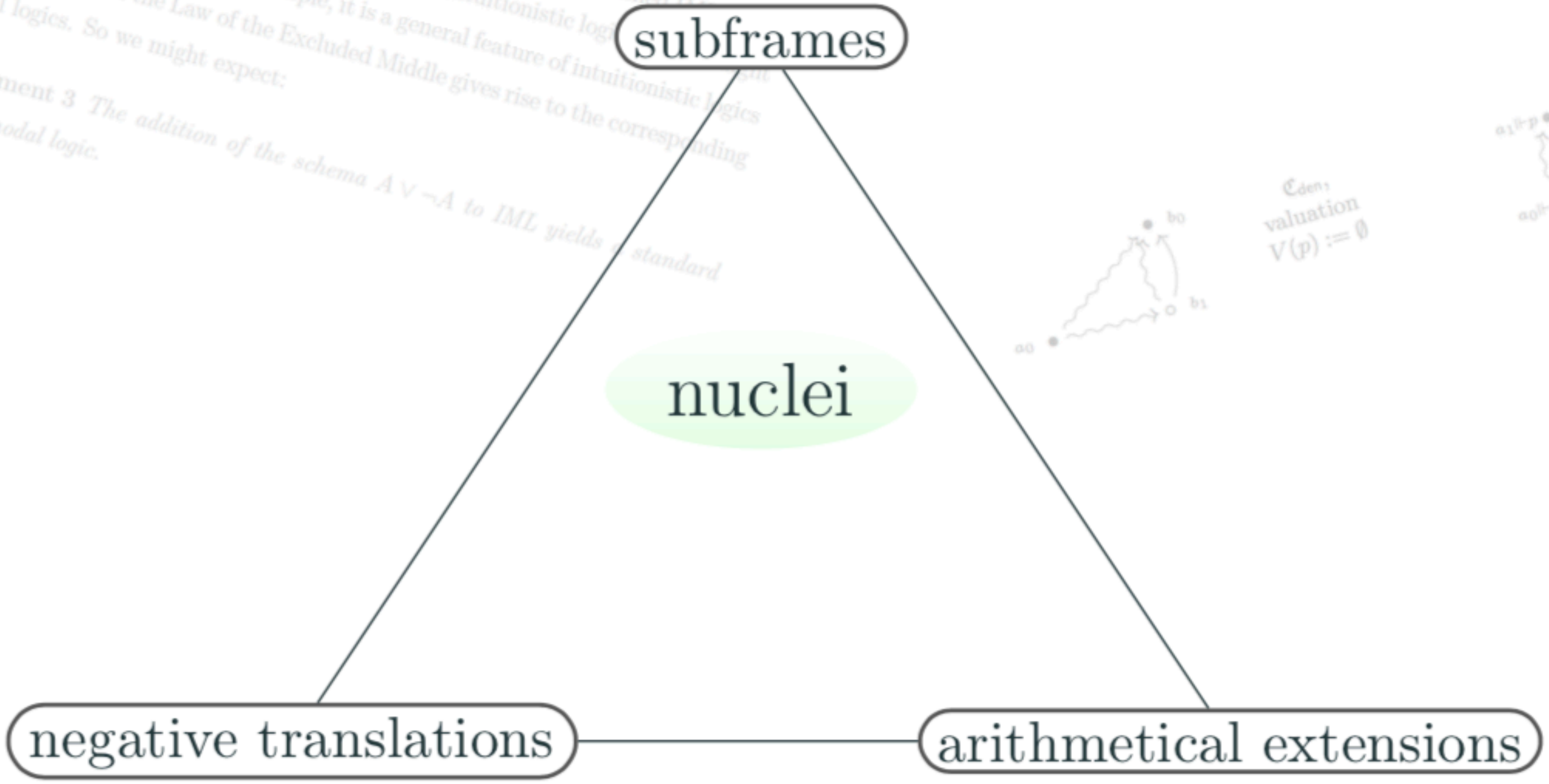
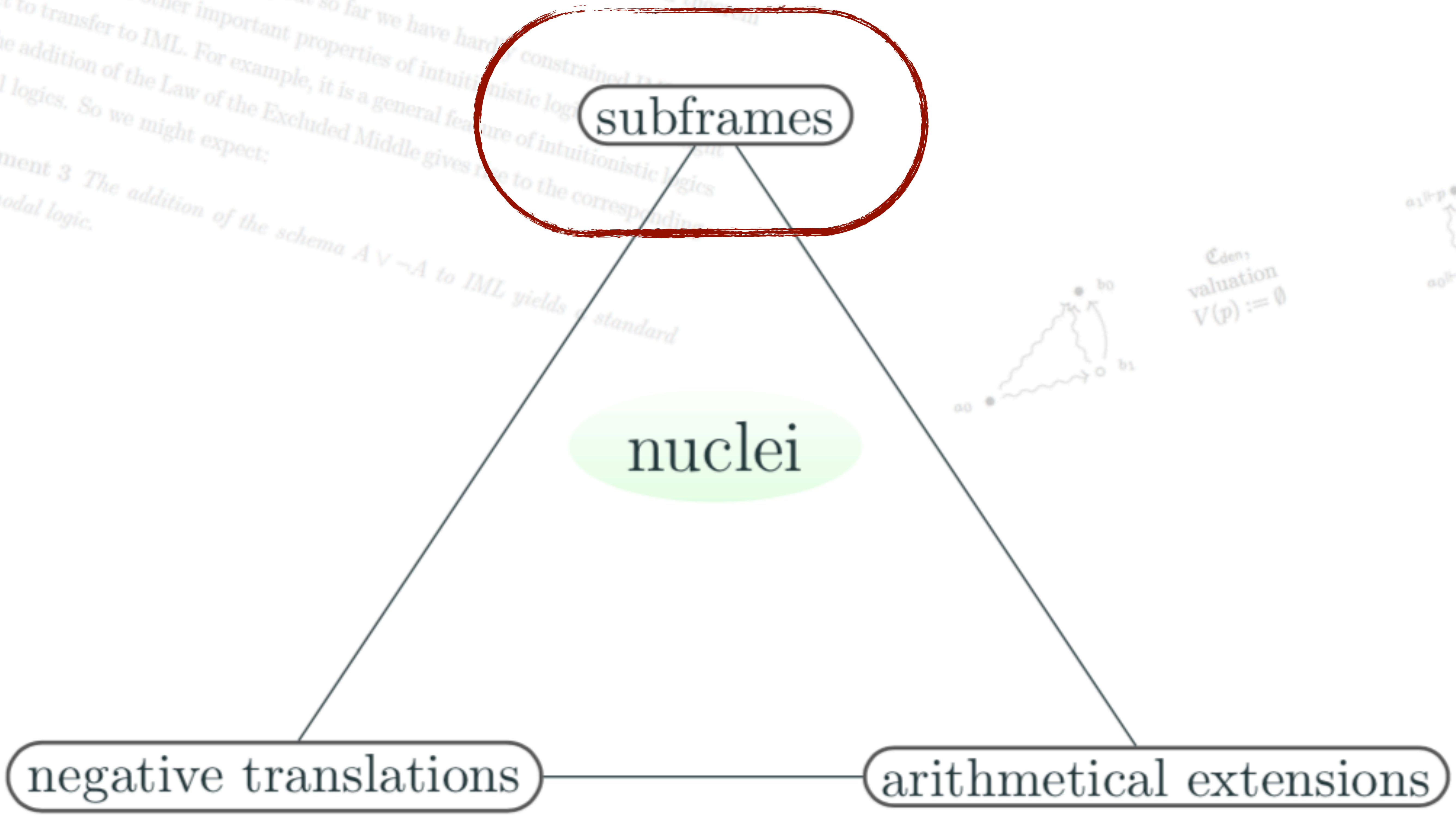
stability of a logic under …



stability of a logic under . . .



Nuclei on BAOs & Heyting Algebras

Nuclei

- Given any Boolean (or Heyting) algebra $\mathfrak{A}$ and $a \in A$,
  $\mathsf{J}_a : A \to A$ defined as $\mathsf{J}_a(x) = x \vee a$ is a nucleus
  which we can also call a strong monad on a poset category
  which we can also call a multiplicative closure operator
  which we can also call a lax modality

- Axioms of nuclei: $x \leq j(x)$, $j(x) = j(j(x))$ and $j(x \wedge y) = j(x) \wedge j(y)$

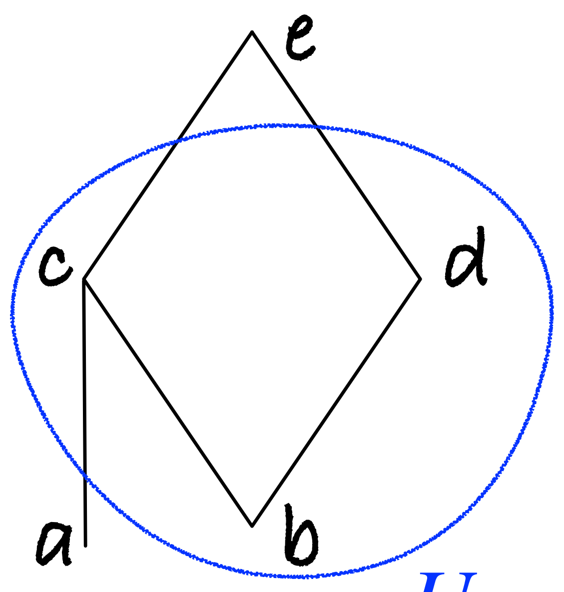
- Boolean algebras are a Kindergarten setting for nuclei:
  any nucleus on a Boolean algebra $\mathfrak{A}$ is of the form J_a for some $a \in A$
  In Fourman-Scott terminology, any Boolean nucleic quotient is closed
  Note that we could use also the open quotient $\mathsf{J}^a : A \to A$ defined as $\mathsf{J}^a(x) = a \to x$

Subframe construction, dually ...

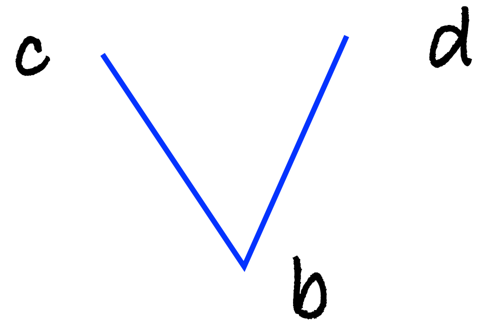
Modal frames

$W = \{a, b, c, d, e\}$

$\mathfrak{F} = \langle W, R \rangle$



$U = \{b, c, d\}$



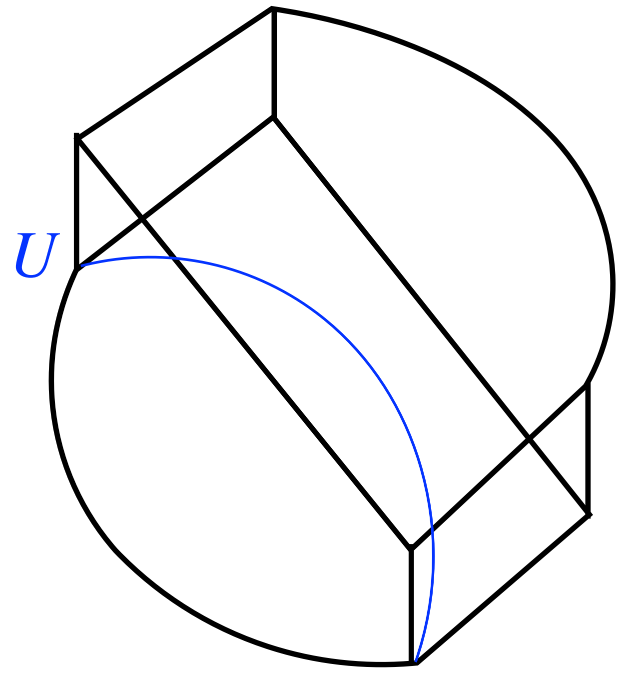
$\mathfrak{F}_U = \langle U, R\restriction_S \rangle$

Dual

modal algebras

$(\mathfrak{F}^+)_\Diamond = \langle \mathscr{P}(W), \Diamond_R \rangle$

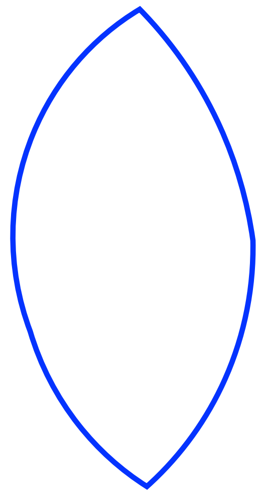
$\top = W$



$\bot = \varnothing$

$h^U_\Diamond(X) = X \cap U$

$\top = U$



$\bot = \varnothing$

$(\mathfrak{F}^+_U)_\Diamond = \langle \mathscr{P}(U), \Diamond^U_R \rangle$

$\Diamond^U_R Y = U \cap \Diamond_R(U \cap Y)$

Subframe construction, dually ...

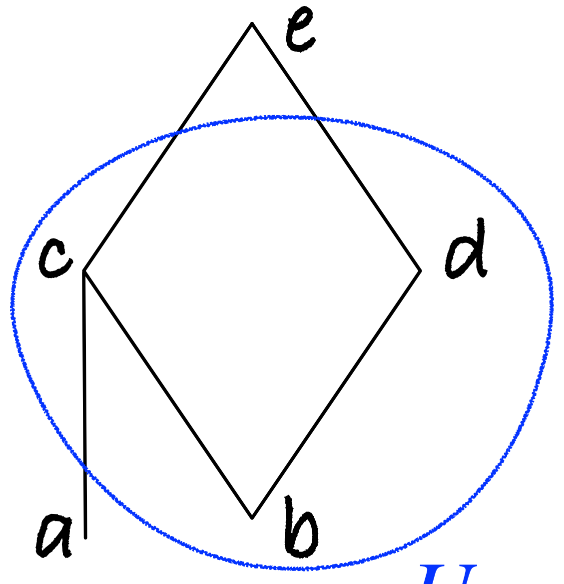
Modal frames

Dual modal algebras

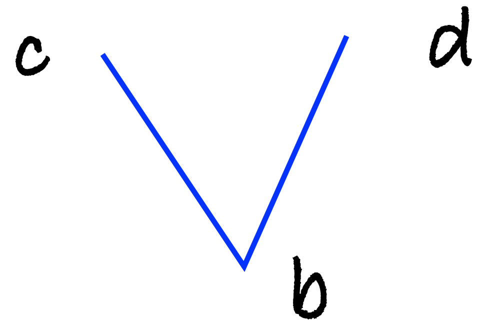
$W = \{a, b, c, d, e\}$

$\mathfrak{F} = \langle W, R \rangle$

$(\mathfrak{F}^+)_\Diamond = \langle \mathscr{P}(W), \Diamond_R \rangle$

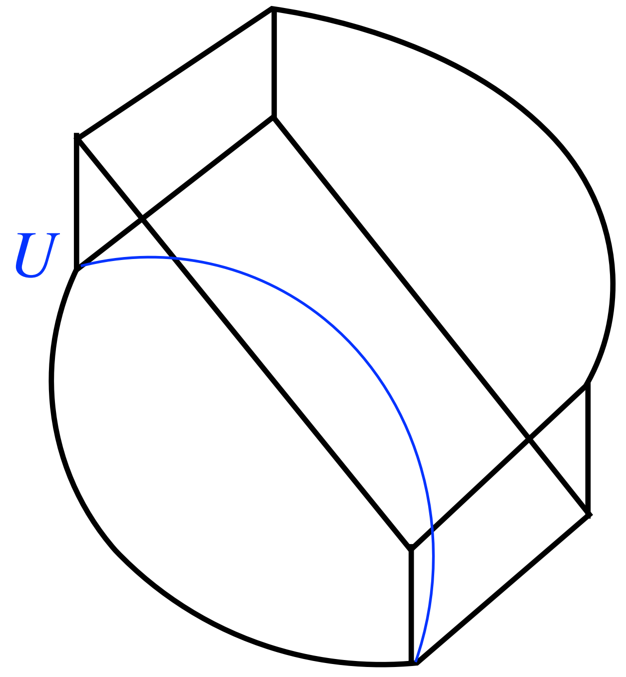


$U = \{b, c, d\}$



$\mathfrak{F}_U = \langle U, R \restriction_S \rangle$

$\top = W$

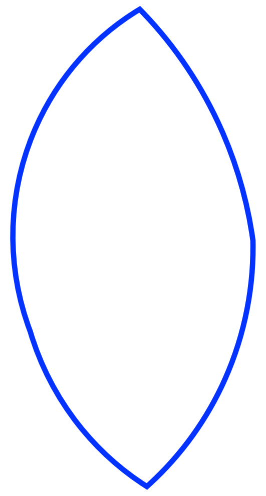


$\bot = \varnothing$

$\iota^U_\Diamond(Y) = Y$

$h^U_\Diamond(X) = X \cap U$

$\top = U$



$\bot = \varnothing$

$(\mathfrak{F}^+_U)_\Diamond = \langle \mathscr{P}(U), \Diamond^U_R \rangle$

$\Diamond^U_R Y = U \cap \Diamond_R(U \cap Y)$

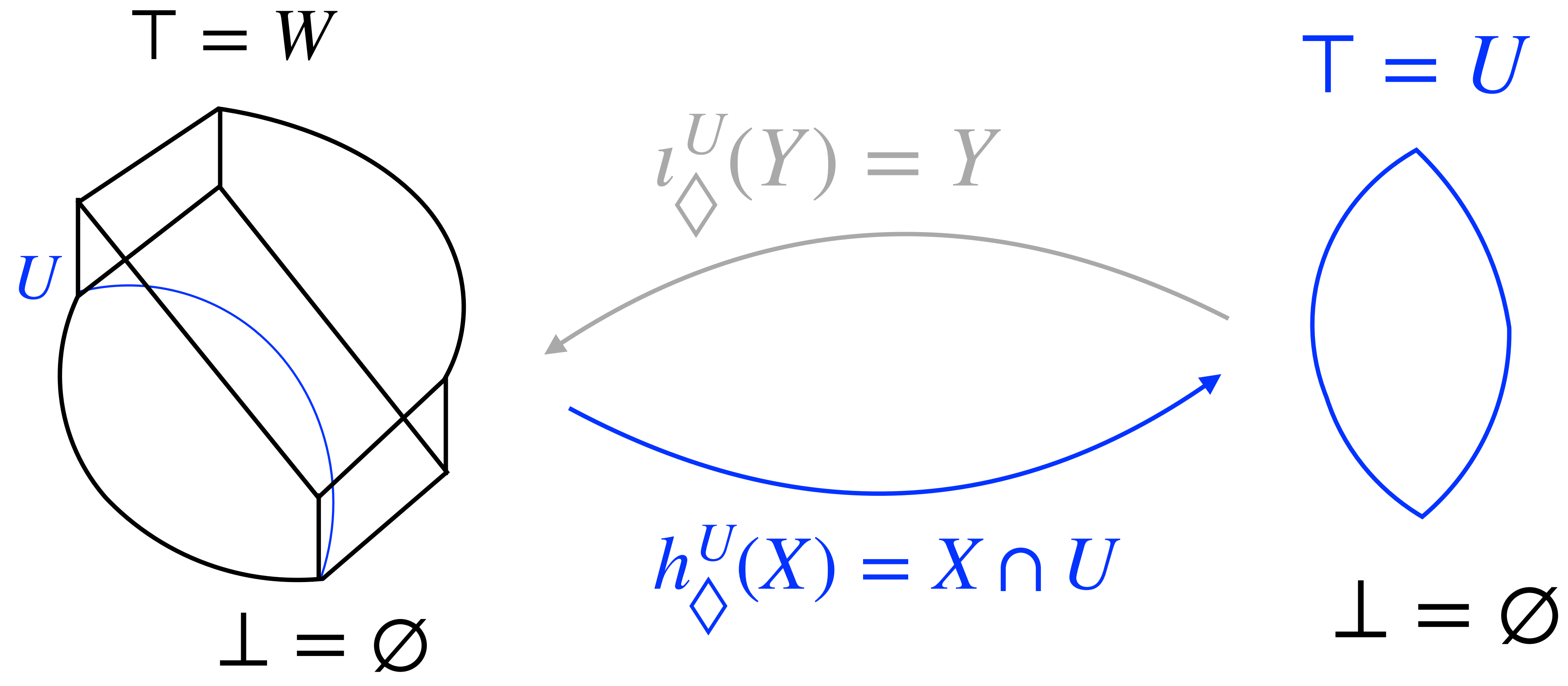
$= h^U_\Diamond \Diamond_R(\iota^U_\Diamond Y)$

$\iota^U_\Diamond$ not a Boolean morphism and $h^U_\Diamond$ in general not a $\Diamond$-morphism: pick $Y = \{e\}$ to get $h^U_\Diamond(\Diamond_R Y) \neq \Diamond^U_R(h^U_\Diamond Y)$

… or maybe ?

Dual modal algebras

$(\mathfrak{F}^+)_\Diamond = \langle \mathscr{P}(W), \Diamond_R \rangle$



$(\mathfrak{F}^+_U)_\Diamond = \langle \mathscr{P}(U), \Diamond^U_R \rangle$

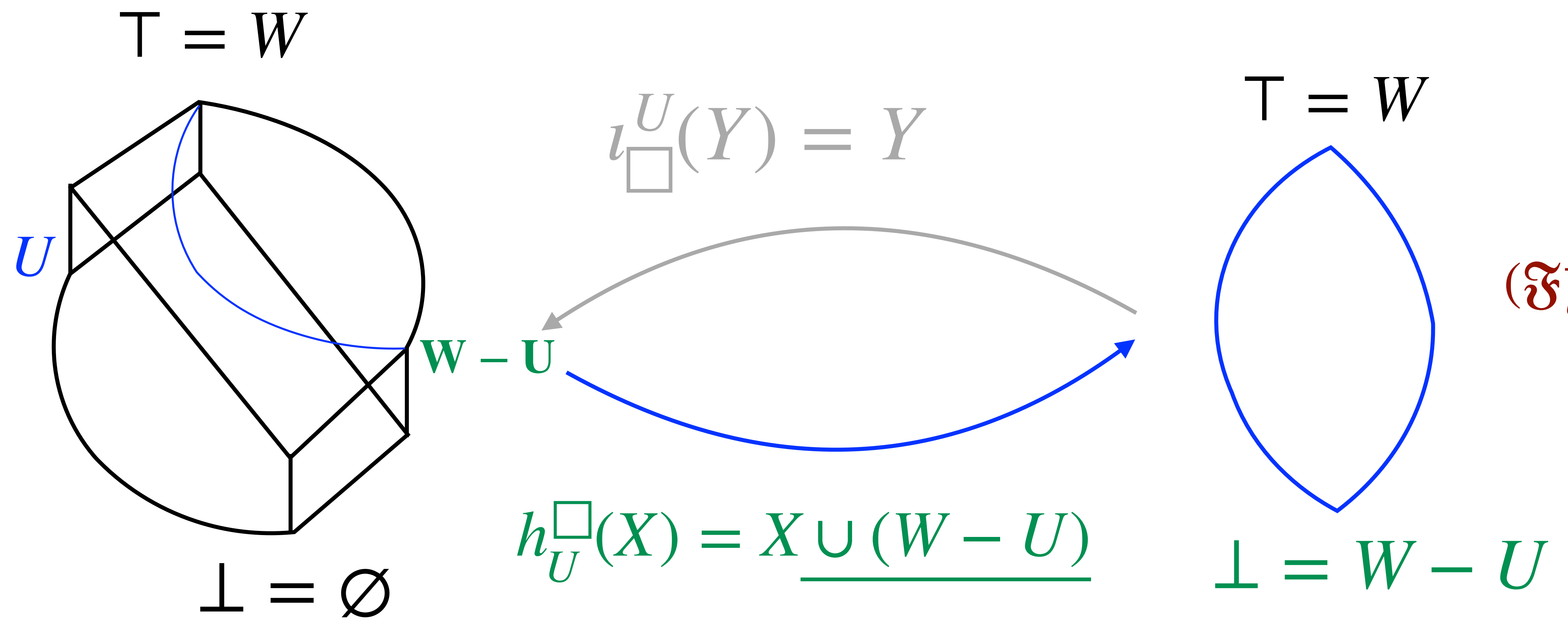
$\Diamond^U_R Y = U \cap \Diamond_R(U \cap Y)$

$= h^U_\Diamond \Diamond_R(\iota^U_\Diamond Y)$

term equivalent

Also dual modal algebras

$(\mathfrak{F}^+)_\Box = \langle \mathscr{P}(W), \Box_R \rangle$



$(\mathfrak{F}^+_U)_\Box = \langle \uparrow_{\mathscr{P}(W)} (W - U), \Box^U_R \rangle$

$\Box^U_R Y = W - \Diamond_R(U - Y)$

$= h^U_\Box \Box_R(\iota^U_\Box Y)$

$h^U_\Diamond$ and $h^U_\Box$ restrict to mutually inverse Boolean isomorphisms between $(\mathfrak{F}^+_U)_\Diamond$ and $(\mathfrak{F}^+_U)_\Box$

- For any $\mathfrak{A}$ and any nucleus $j : A \to A$, we can define $A_j = \{a \in A \mid j(a) = a\}$ (the collection of fixpoints of j)

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- Any n-ary operation $\heartsuit : A^n \to A$ can be turned into $\heartsuit_j : A_j^n \to A_j$ by $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(c_1, \ldots, c_n))$
  (or, strictly speaking, $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(\iota_j(c_1), \ldots, \iota_j(c_n)))$ if the identity embedding $\iota_j : A_j \to A$ made visible)

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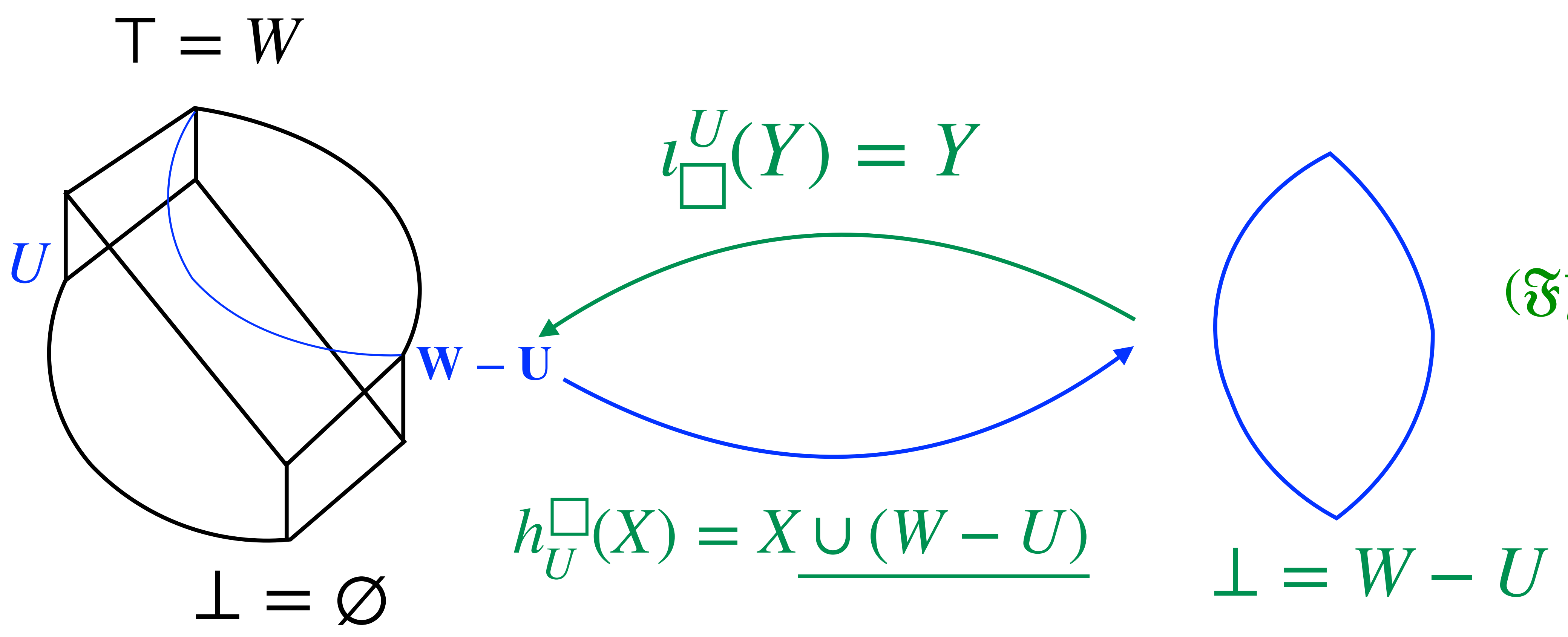
$(\mathfrak{F}^+)_\square = \langle \mathscr{P}(W), \square_R \rangle$



$(\mathfrak{F}^+_U)_\square = \langle \uparrow_{\mathscr{P}(W)} (W - U), \square^U_R \rangle$

$\square^U_R Y = W - \Diamond_R(U - Y)$
$= h^U_\square \square_R (\iota^U_\square Y)$

Subframe logics in the BAO setting

- We started with normal modal logic (with $\Box$ as primitive) over CPC
- Abstract algebraic logic (AAL) perspective:
  a logic Λ as a set of theorems $\iff$ an equational theory $\mathsf{Var}(\Lambda)$
- Def: Λ is a subframe logic if $\mathsf{Var}(\Lambda)$ is closed under nucleic quotients
  That is, for any $\mathfrak{A} \in \mathsf{Var}(\Lambda)$ and any nucleus $j : A \to A$, $\mathfrak{A}_j \in \mathsf{Var}(\Lambda)$
  (this definition follows G. Bezhanishvili & S. Ghilardi rather than Wolter)
- Theorem: Kripke-subframe logics are subframe in this sense. (Wolter, I guess)
  For transitive normal modal logics, not only does the converse holds as well,
  but they do enjoy the finite model property (essentially Fine)
  (G. Bezhanishvili & S. Ghilardi & M. Jibladze: still holds for weak transitivity,
  F. Wolter: ... but not for 2-transitivity)

Algebraic semantics of IPC

- Heyting algebras: bounded lattices where $\wedge$ has right adjoint $\to$ (hence distributive)
- G. Bezhanishvili & Ghilardi 2007: nuclei on Heyting algebras capture descriptive/Priestley/Esakia subframe constructions

- Recall the construction of $\mathfrak{A}_j$, i.e., the nucleic quotient of $\mathfrak{A}$ via j:
  For any $\mathfrak{A}$ and any nucleus $j : A \to A$, we can define
  $A_j = \{a \in A \mid j(a) = a\}$ (the collection of fixpoints of j)

- Any n-ary operation $\heartsuit : A^n \to A$ is turned into $\heartsuit_j : A_j^n \to A_j$ by
  $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(c_1, \ldots, c_n))$
  (or, strictly speaking, $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(\iota_j(c_1), \ldots, \iota_j(c_n)))$ if the identity embedding $\iota_j : A_j \to A$ made visible

- While we explicitly see the "extensional" connectives ($\wedge$, $\vee$, $\to$, $\top$, $\bot$) of $\mathfrak{A}_j$ as obtained this way ...

- As $j(\top) = \top$, $\quad j(j(a) \wedge j(b)) = j(a) \wedge j(b)$ and $j(j(a) \to j(b)) = j(a) \to j(b)$,
  $\mathfrak{A}_j$ is an implicative subsemilattice of $\mathfrak{A}$: we only need to prefix j in front of $\vee$ and $\bot$

- Furthermore, $\mathfrak{A}_j$ obtained this way is a Heyting algebra in its own right!
  But not necessarily satisfying the same equational axioms as the original $\mathfrak{A}$:
  the subframe ones are precisely the safe ones

- Also, as for preservation of $\perp$:
  nuclei satisfying $j(\perp) = \perp$ are called dense

- G. Bezhanishvili & S. Ghilardi show that the (pre-existing) notion of (superintuitionistic) cofinal subframe logics corresponds to preservation by dense nuclei

- Furthermore, this is in turn equivalent to a seemingly stronger property: preservation by locally dense nuclei: those satisfying $j(\neg j(\perp)) = \top$ (correspond to strict lax modalities of Aczel 2001)

Pleasant results in the pure Heyting signature

- Theorem (Fine, Zakharyaschev):
  - A (locally dense) nuclear superintuitionistic logic/variety has the finite frame/algebra property (in the modal setting, true only in the presence of wK4!)
  - A logic/variety is nuclear iff it is axiomatized by $(\wedge , \rightarrow)$-formulas/identities
  - A logic/variety is (locally) dense nuclear iff it is axiomatized by $(\wedge , \rightarrow , \bot)$-formulas/identities
- Theorem (quite a few good people):
  TFAE for a superintuitionistic logic Λ:
  - $\mathrm{Var}(\Lambda)$ is nuclear
  - Λ is axiomatized by NNIL formulas (No Nesting of Implication to the Left)
    "NNIL" is pronounced as "NIL", where the first "N" is pronounced with some slight hesitation - Visser et al. 1995
  - Λ is axiomatized by formulas preserved by submodels of Kripke models

But we also begin to see first problems

- Nucleic quotient of a perfect BAO ($\mathscr{CAV}$-BAO or simply a Kripke algebra) is again the dual of a Kripke frame
- This does not hold anymore in the Heyting setting!
- More issues to follow …

What happens when more connectives present?

- Intuitionistic modal logics: with box only …? With diamond(s) too?
- Most broadly: extensions of Weiss's ICK?
  (Basic Intutionistic Conditional Logic, JPL 2019: arrow distributing over $\wedge$ in the 2nd coord.
  Chellas-Weiss semantics or generalized Routley-Meyer semantics)
- More narrow: constructive strict implication/Lewis arrow? Heyting-Lewis Calculus:
  $\mathsf{HLC}^\flat$ (formerly known as iA^-) = ICK + arrow transitive, admitting gen. necessitation, antimonotone in the 1st coord.
  $\mathsf{HLC}^\sharp$: $\mathsf{HLC}^\flat$ + Di (formerly known as iA) = $\wedge$ of arrows with the same consequent $\leftrightarrow$ an arrow with $\vee$ in the antecedent
  ($\mathsf{HLC}^\flat$ includes, e.g., the logic of type inhabitation of Haskell arrows)
- Superlogics of HLC capturing preservativity in Heyting Arithmetic and its extension?
  variants of the Löb axiom + more?
  (generalized Veltman semantics)
- The logic of bunched implications BI?
  (commutative and associative) $\star$ adjoint to $-\!\!\star$
  (variants of partial monoid semantics, also topological ones)

(Term-definable) nuclei on Heyting Algebra Expansions (HAEs): Towards general theory

Classical subframization is describable I

- Here we want to introduce another item from Wolter's toolbox (in a suitably generalized form)

- Namely, we focus on his notion of a describable operation

- Let us start in even more generality: given any class of algebras V and a set-valued operation $C : V \to \mathscr{P}(V)$, we extend it to subclasses $K \subseteq V$: $C(K) = \bigcup \{C(\mathfrak{A}) \mid \mathfrak{A} \in K\}$ (note that I don't want $C(K)$ to be closed under isomorphism)

- Say that C is delimited if for any $K \subseteq V$, $C(C(K)) \subseteq C(K)$
  Say that C is extensive if for any $K \subseteq V$, $K \subseteq C(K)$ (inflationary?)

- The operation of forming all subframes/nucleic quotients $\mathbb{J}$ is both delimited and extensive (on Boolean or Heyting algebras, or their expansions)

- But on Boolean algebras it has yet another property, more problematic in the Heyting case

Classical subframization is describable II

- Let $\mathfrak{A}$ be a Boolean modal algebra. Consider the situation when $\mathbb{J}(\mathfrak{A}) \vDash \varphi$

  (I assume it is clear what it means for φ to hold in a class of algebras)

  Does it boil down to $\mathfrak{A} \vDash \varphi^j$ for some suitable translation $(\,\cdot\,)^j$?

  (jumping ahead a bit, we can speak of nucleic Gödel-Gentzen or generalized negative translation)

  Fix a fresh variable p. Define a recursive translation:

  $q^{\mathsf{u},p} = q \vee p \qquad (\neg\varphi)^{\mathsf{u},p} = \neg\varphi^{\mathsf{u},p} \vee p \qquad (\varphi \wedge \psi)^{\mathsf{u},p} = \varphi^{\mathsf{u},p} \wedge \psi^{\mathsf{u},p} \qquad (\Box\varphi)^{\mathsf{u},p} = \Box(\varphi^{\mathsf{u},p}) \vee p$

  (not exactly how this is presented by Wolter, as he focuses on the diamond-relativization)

  we could also use the open translation instead of the closed one

- Theorem (essentially Kracht/Wolter): For any Boolean modal algebra $\mathfrak{A}$, any φ and any fresh p,

  $\mathbb{J}(\mathfrak{A}) \vDash \varphi$ iff $\mathfrak{A} \vDash \varphi^{\mathsf{u},p}$

- Whenever there is $(\,\cdot\,)^c$ s.t. $C(\mathfrak{A}) \vDash \varphi$ iff $\mathfrak{A} \vDash \varphi^c$, C is a weakly describable operation

  with the describing translation made explicit, we can use this notion for the pair $\langle C, (\,\cdot\,)^c \rangle$

Problems even in the pure Heyting signature

- The lattice of nuclei on a Heyting algebra is quite complex
- Describability is thus a subtle (or messy) business
- Let us look at several standard examples of nuclei, taken from
  - "Sheaves and Logic", Fourman and Scott 1977
  - "Modal operators on Heyting algebras", Macnab 1981

- $\mathsf{J}_a\varphi = a \vee \varphi$ (Macnab writes u_a): the closed quotient, dense (and identity) for a = ⊥.

- $\mathsf{J}^a\varphi = a \to \varphi$ (Macnab writes v_a): the open quotient, dense (and identity) for a = ⊤.

- $\mathsf{B}_a\varphi = (\varphi \to a) \to a$ (Macnab writes w_a): the boolean quotient,
  dense for a = ⊥; even then identity not a special case.
  Denote the dense case as $\mathsf{B}_\perp\varphi = \neg\neg\varphi$ ($\mathsf{w}_\perp$): the double-negation quotient.
  (this one may collapse duals of Kripke structures to pointless/atomless algebras)

- $(\mathsf{J}_a \wedge \mathsf{J}^b)\varphi = (a \vee \varphi) \wedge (b \to \varphi)$: the forcing quotient,
  dense (and identity) for a = ⊥.

- $(\mathsf{B}_a \wedge \mathsf{J}^a)\varphi = (\varphi \to a) \to \varphi$: a mixed quotient; dense (and identity) for a = ⊤.

- For each of these (definable) lax modalities (Aczel terminology), given an algebra $\mathfrak{A}$, we can consider the corresponding class of nucleic quotients of $\mathfrak{A}$ ($\mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{W}_{\perp}$ ...) obtained by varying a, b... across the carrier of $\mathfrak{A}$
- Clearly, each of them is (at least) weakly describable
- How to describe the class of all nucleic quotients though?

Basics facts about describability

- In our generalizations, we have to make finer distinctions than Wolter did

- A weakly describable operation is stabilizing if $C(\mathfrak{A}) \vDash \varphi$ implies $C(\mathfrak{A}) \vDash \varphi^c$
  A weakly describable operation is subsuming if $C(\mathfrak{A}) \vDash \varphi$ implies $\mathfrak{A} \vDash \varphi$

- Fact: Being stabilizing is equivalent to $(\varphi^c)^c \in \mathsf{Ded}(\varphi^c)$ for all φ
  Fact: Being subsuming implies that $\varphi \in \mathsf{Ded}(\varphi^c)$ for all φ
  Fact: Being delimited (recall it's $C(C(K)) \subseteq C(K)$) and weakly describable implies being stablizing
  Fact: Being extensive (recall it's $K \subseteq C(K)$) and weakly describable implies being subsuming

- Definition: A weakly describable operation is Wolterian or fully describable when delimited and extensive
  Theorem (Wolter): The subframization/nucleization operation $\mathbb{J}$ on Boolean algebra expansions (BAEs) is fully describable (Wolterian)

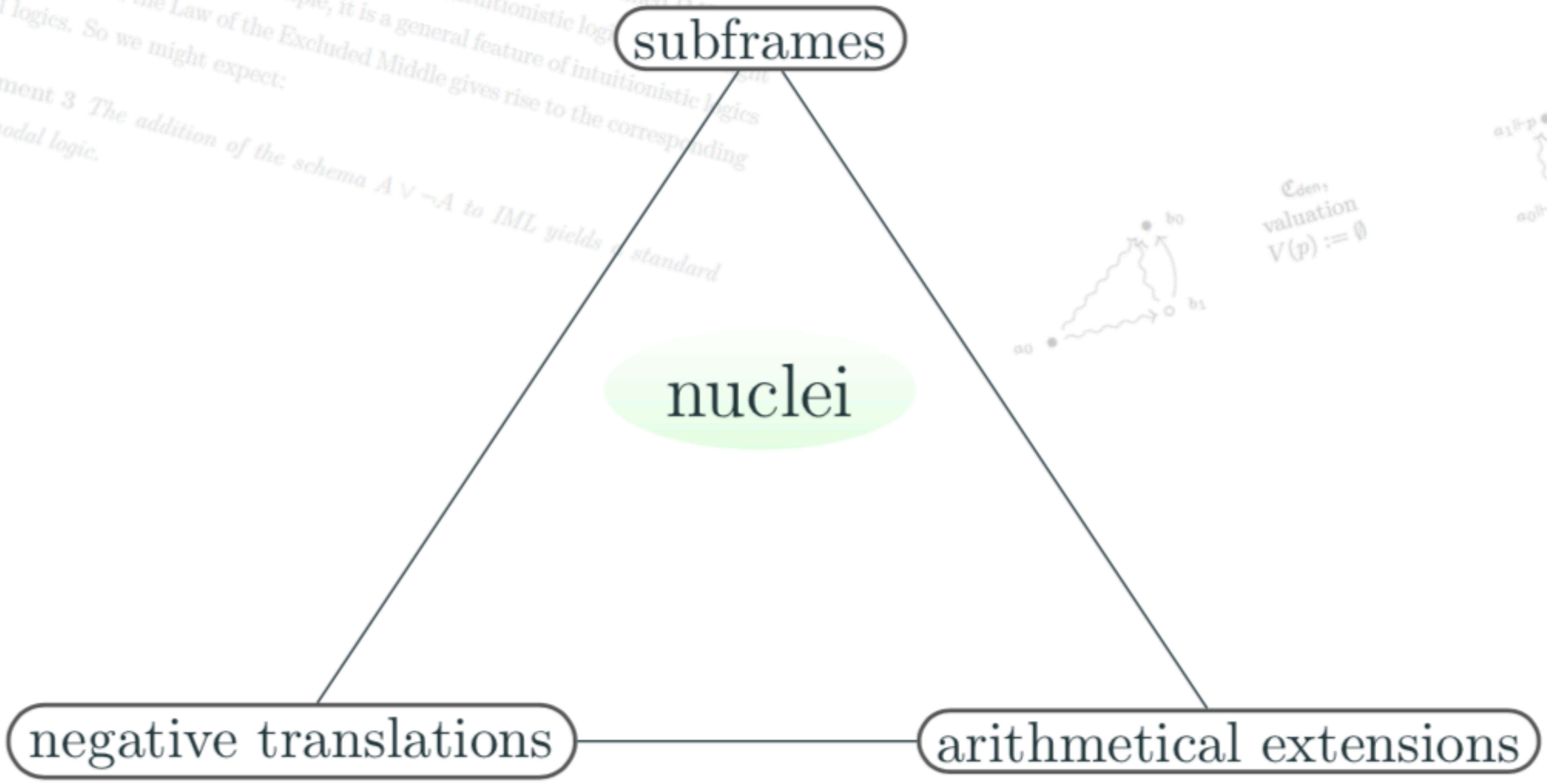
  Meaningful weakly describable operations should be stabilizing, but not all of them will be Wolterian-describable

  Actually, finding workable general criteria for being stabilizing turned out to be non-trivial!

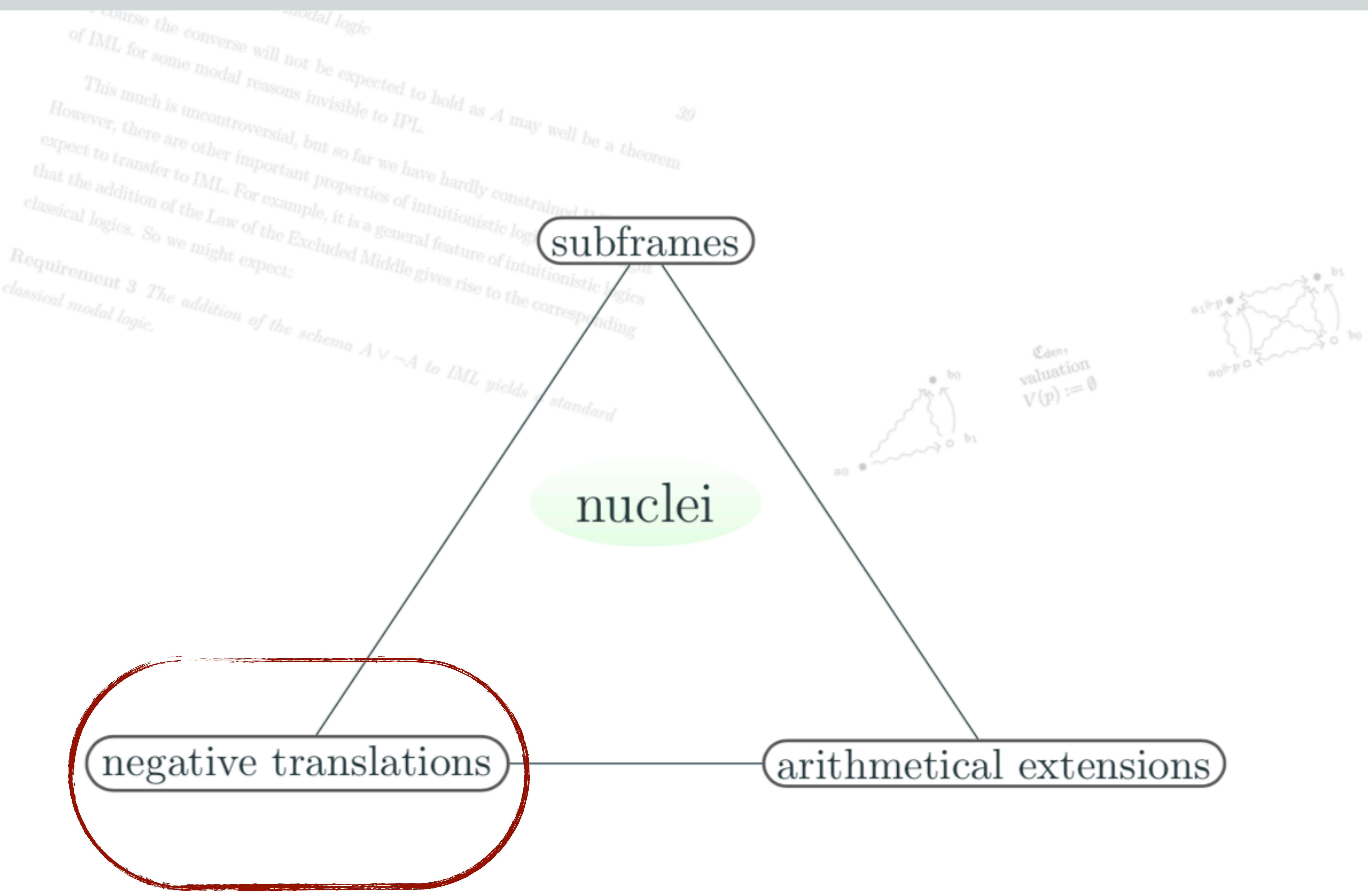
Case study I:
The subframe property as a form of completeness

- Given a set-valued operation C on some ambient variety V, a subvariety $W \subseteq V$ (or its corresponding logic $\mathsf{Log}(W)$) is *C-complete* if $C(W) \subseteq W$
- For extensive operations, this obviously equivalent to $C(W) = W$
- In particular, subframe logics = $\mathbb{J}$-complete ones

stability of a logic under ...



stability of a logic under . . .



Case study II:
negative translations for Heyting Algebra Expansions (HAEs)

- In a FSCD 2017 paper (jointly with M. Polzer and U. Rabenstein)
  (also CMCS 2019: Going negative in Prague)
  $\neg\neg$-completeness of intuitionistic normal $\Box$-modal logics

- By this we meant: for which such logic Λ, it is the case that
  $\varphi^{\neg\neg} \in \Lambda$ iff $\varphi \in \Lambda \oplus (p \vee \neg p)$, where $(\,\cdot\,)^{\neg\neg}$ is a suitably generous negative translation?

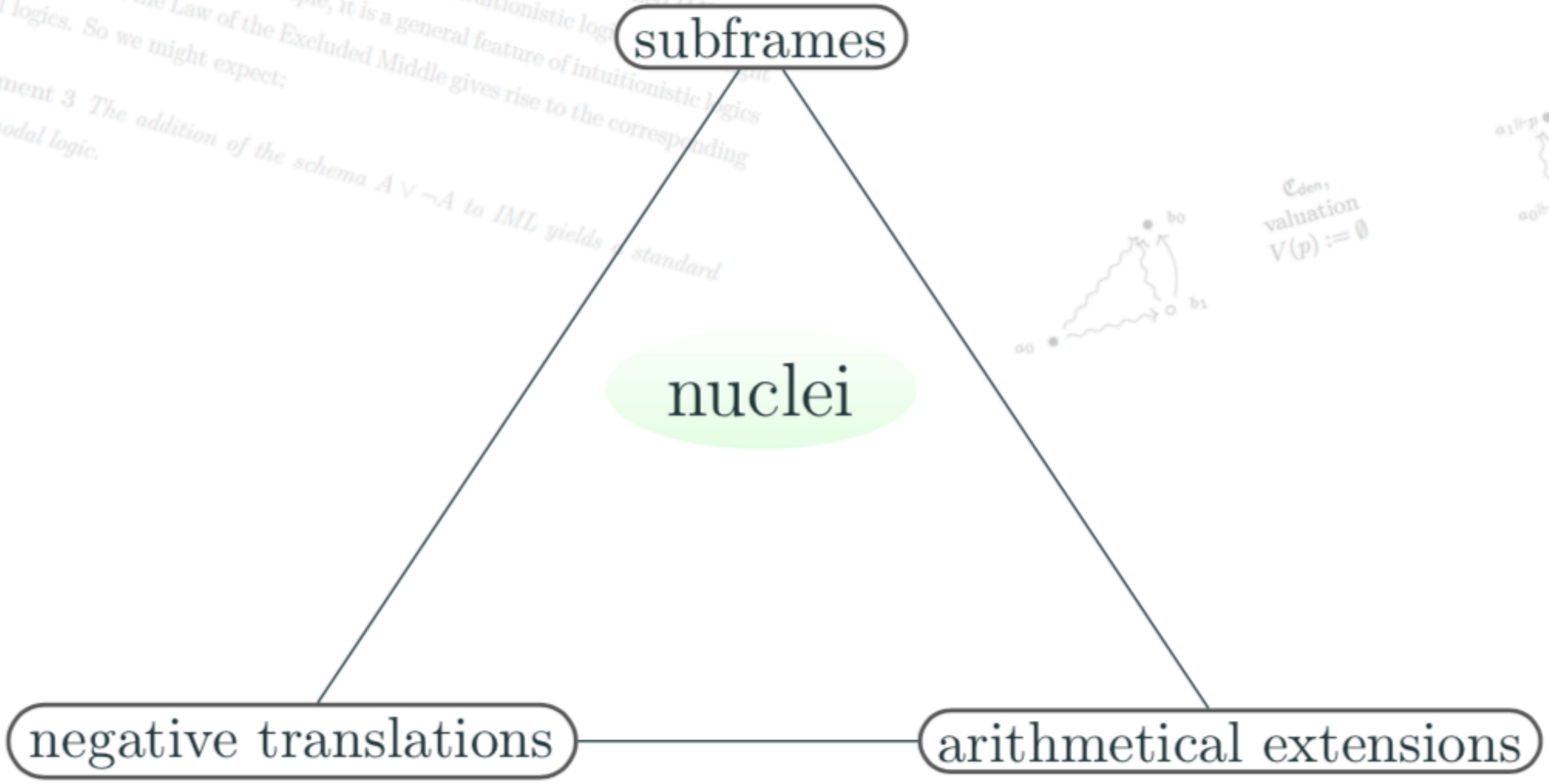
- As turns out, this is precisely the study of $\mathbb{W}_{\perp}$-completeness!

- A non-extensive operation, but nevertheless very natural
  (main motivation for me to remove extensiveness from Wolter's axioms)

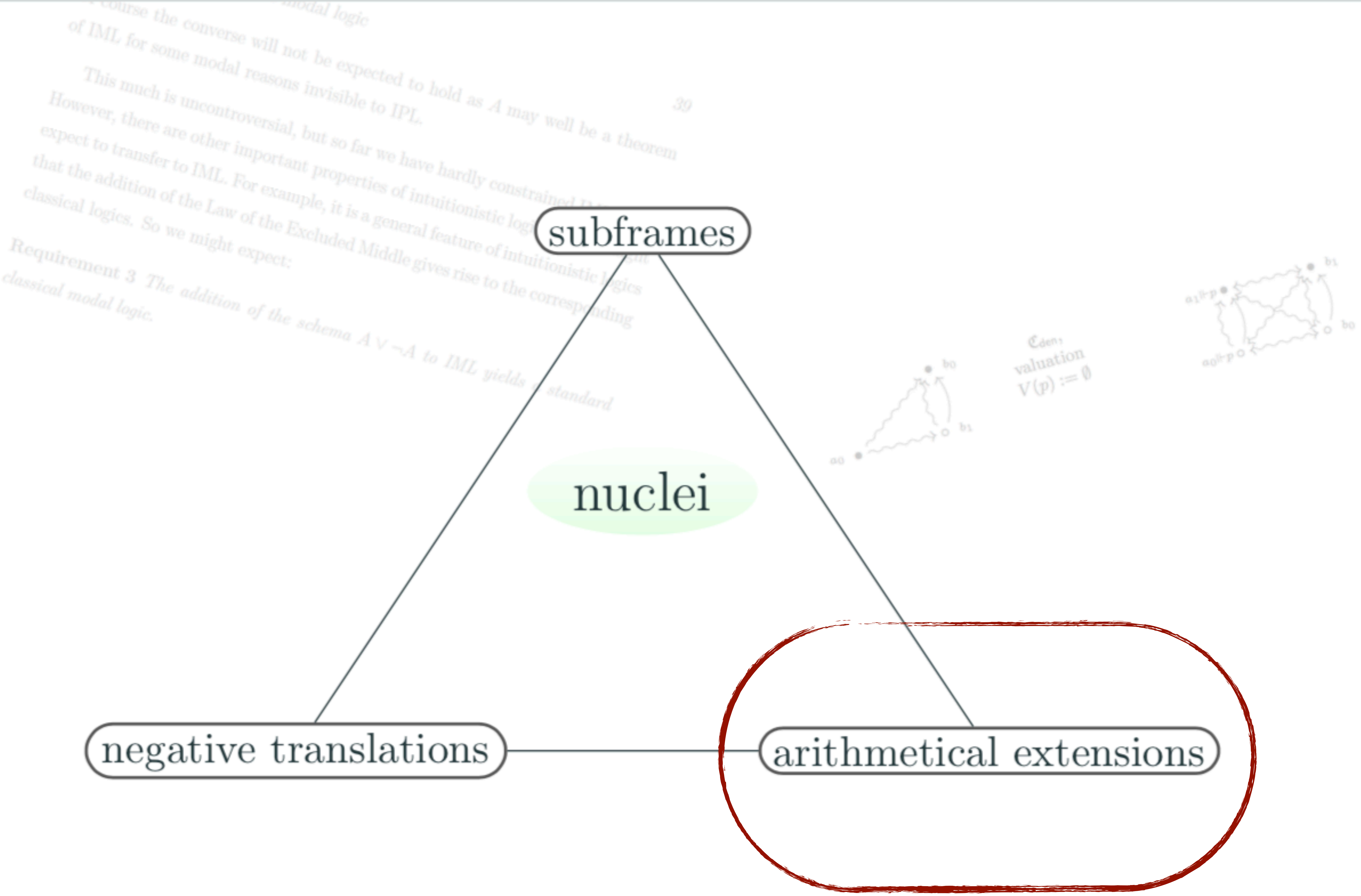
- Many of our results were special cases of those in "Towards general theory" above

- On the other hand, it is interesting how far our FSCD "enveloped implications"
  completeness criterion generalizes to other settings

stability of a logic under . . .



stability of a logic under ...



Case study III (with Albert Visser): Extension stability for logics of arithmetical interpretations

- Motivation comes from the metatheory of Heyting Arithmetic
- We study signatures extending the provability signature e.g., the constructive strict implication connective $\dashv$
- It is most commonly interpreted as preservativity wrt a fixed aritmetical theory T and a fixed set of sentences Δ (most commonly $\Delta = \Sigma^0_1$):
  A $\dashv_{\Delta, \mathrm{T}}$ B if for every Δ-sentence S, $T \vdash S \to A$ implies $T \vdash S \to B$
  But many other interpretations are possible
- We say that F is an elegant interpretation if for any recursively axiomatizable arithmetical theory U and any arithmetical sentences A, B, C, we have that
  $$U \vdash (B \dashv_{\mathrm{F,U+A}} C) \leftrightarrow (A \to B) \dashv_{\mathrm{F,U}} (A \to C)$$

- Now when we look at the propositional logic $\Lambda_F(T)$ determined by a given arithmetical interpretation F and a given theory T, and an arithmetic sentence A, does it hold that $\Lambda_F(T) \subseteq \Lambda_F(T+A)$ (?)

  Must a principle valid in a base theory hold in all its finite extensions?

- As it turns out, this is not the case: Σ^0_1-preservativity logic of HA (Markov's Principle can be axiomatized by a single sentence)

- $\Lambda_F(T)$ is called extension stable if (?) holds (for an elegant F)

- The study of extension stability = the study of $\mathbb{V}$-completeness

- Challenge: logics which do not always allow a Kripke-style semantics!

Stabilizing and Wolterian quotients

- Theorem: For any weakly describable operation, C-completeness is equivalent to admissibility of the rule "from φ, infer φ^c". Furthermore, restricting the attention to axioms is enough.

- Theorem: For any weakly describable operation, C-complete logics form a complete $\bigwedge$-subsemilattice of the lattice of all logics

- Theorem: If a weakly describable operation is stabilizing ($\mathsf{J}_a, \mathsf{J}^a, \mathsf{B}_a, \mathsf{B}_\perp$), then for any set of formulas Γ

  * Γ^c axiomatizes a C-complete logic

  * The least C-complete logic containing Γ is obtained as $\mathsf{Ded}(\Gamma \cup \Gamma^c)$

- Theorem: If an operation is Wolterian ($\mathsf{J}_a, \mathsf{J}^a$), then

  * The least C-complete logic containing Γ is obtained as $\mathsf{Ded}(\Gamma^c)$

  * The greatest C-complete logic contained in Γ is $\mathsf{Ded}(\{\gamma^c \mid \Gamma \vdash \gamma^c\})$

  * C-complete logics form a complete $\bigwedge\bigvee$-sublattice of the lattice of all logics

Aside: good operations

- Wolter has noted that nucleization on Boolean modal algebras is good:
  - ✴ the corresponding translation is computable and decidable
  - ✴ a finite algebra yields a finite set of finite ones
  - ✴ when $\mathfrak{A}$ is the dual of a Kripke frame, so are all algebras in $\mathbb{J}(\mathfrak{A})$
- The last condition is, as we noted, non-trivial in a Heyting setting (boolean quotients)
- In the presence of additional operators, "Kripkeanity" can go wrong even in cases when the Heyting reduct remains unproblematic:
  the open nucleus does not preserve $\mathrm{HLC}^{\sharp}$
  (recall Di? but $\mathrm{HLC}^{\flat}$ is safe for open nuclei)
- Dual perspective to be developed (most interesting for specific signatures and axioms allowing "natural" interpretations… such as $\mathrm{HLC}^{\flat}$ and $\mathrm{HLC}^{\sharp}$ indeed)

General completeness criterion

- (already mentioned)
  Generalization of the criterion of "enveloped implications" from our FSCD 2017 paper

- Much more to be done by generalizing modal & intuitionistic theory of subframe formulas

More examples?

- Georg Struth: A (B)BI nucleus defined by $\mathrm{I}(p) = \top \star p$ yields the collection of intuitionistic (affine) assertions …
- The use of (a subclass of) nuclei in algebraic cut elimination and Ono-style completeness proofs for substructural logics …
(OTOH, a different notion of "nucleus": preserves fusion)