Abstract Evolution Systems

Wiesław Kubiś

Institute of Mathematics, CAS

Czech Gathering of Logicians, 16 – 17.6.2022

Joint work with Paulina Radecka.

Abstract rewriting systems

Abstract rewriting systems

Definition

A rewriting system is a pair $(X, \to)$, where $\to$ is a binary (typically irreflexive) relation on X. The elements of X are called states and the elements of $\to$ are called transitions. A sequence of transitions

$$x = x_0 \to x_1 \to \cdots \to x_{n-1} = y$$

is called a *path* from x to y.

Abstract rewriting systems

Definition

A rewriting system is a pair $(X, \to)$, where $\to$ is a binary (typically irreflexive) relation on X. The elements of X are called states and the elements of $\to$ are called transitions. A sequence of transitions

$$x = x_0 \to x_1 \to \cdots \to x_{n-1} = y$$

is called a *path* from x to y.

Definition

A rewriting system $(X, \to)$ is locally confluent if for every state z, for every transitions $z \to x$, $z \to y$ there exist a state w and paths $x \to \cdots \to w$, $y \to \cdots \to w$.

Abstract rewriting systems

Definition

A rewriting system is a pair $(X, \to)$, where $\to$ is a binary (typically irreflexive) relation on X. The elements of X are called states and the elements of $\to$ are called transitions. A sequence of transitions

$$x = x_0 \to x_1 \to \cdots \to x_{n-1} = y$$

is called a *path* from x to y.

Definition

A rewriting system $(X, \to)$ is locally confluent if for every state z, for every transitions $z \to x$, $z \to y$ there exist a state w and paths $x \to \cdots \to w$, $y \to \cdots \to w$.
The system $(X, \to)$ is confluent if for every state z, for every paths $z \to \cdots \to x$, $z \to \cdots \to y$ there exist a state w and paths $x \to \cdots \to w$, $y \to \cdots \to w$.

Definition

A state v in a rewriting system $(X, \to)$ is normalized if there is no transition from v to any other state.

Definition

A state v in a rewriting system $(X, \to)$ is normalized if there is no transition from v to any other state.
A state a is an origin if there is a path from a to any other state.

Definition

A state v in a rewriting system $(X, \to)$ is normalized if there is no transition from v to any other state.
A state a is an origin if there is a path from a to any other state.

Proposition

A confluent rewriting system with an origin has at most one normalized state.

Definition

A state v in a rewriting system $(X, \to)$ is normalized if there is no transition from v to any other state.
A state a is an origin if there is a path from a to any other state.

Proposition

A confluent rewriting system with an origin has at most one normalized state.

Remark

In general, local confluence does not imply confluence.

Newman's Lemma

Definition

A rewriting system $(X, \to)$ is terminating if there is no infinite path

$$x_0 \to x_1 \to \cdots .$$

In other words, the inverse of $\to$ is a well-founded relation.

Newman's Lemma

Definition

A rewriting system $(X, \to)$ is terminating if there is no infinite path

$$x_0 \to x_1 \to \cdots .$$

In other words, the inverse of $\to$ is a well-founded relation.

Lemma (Newman 1942)

A locally confluent terminating rewriting system is confluent.

Newman's Lemma

Definition

A rewriting system $(X, \to)$ is terminating if there is no infinite path

$$x_0 \to x_1 \to \cdots .$$

In other words, the inverse of $\to$ is a well-founded relation.

Lemma (Newman 1942)

A locally confluent terminating rewriting system is confluent.

Remark

Huet 1980 provided a proof using induction with respect to $\leftarrow$.

Newman's Lemma

Definition

A rewriting system $(X, \to)$ is terminating if there is no infinite path

$$x_0 \to x_1 \to \cdots .$$

In other words, the inverse of $\to$ is a well-founded relation.

Lemma (Newman 1942)

A locally confluent terminating rewriting system is confluent.

Remark

Huet 1980 provided a proof using induction with respect to $\leftarrow$.

Corollary

A locally confluent terminating rewriting system with an origin has a unique normalized state.

Finite homogeneous structures

Finite homogeneous structures

The setup

Let $\mathcal{F}$ be a class of finite structures, closed under isomorphisms.
We say that $\mathcal{F}$ has the amalgamation property if for every $Z \in \mathcal{F}$, for every embeddings $f \colon Z \to X$, $g \colon Z \to Y$ with $X, Y \in \mathcal{F}$, there exist $W \in \mathcal{F}$ and embeddings $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f = g' \circ g$.
Let us assume $\mathcal{F}$ has the origin, namely, a "trivial" structure that embeds into any other one.

Finite homogeneous structures

The setup

Let $\mathcal{F}$ be a class of finite structures, closed under isomorphisms.
We say that $\mathcal{F}$ has the amalgamation property if for every $Z \in \mathcal{F}$, for every embeddings $f\colon Z \to X$, $g\colon Z \to Y$ with $X, Y \in \mathcal{F}$, there exist $W \in \mathcal{F}$ and embeddings $f'\colon X \to W$, $g'\colon Y \to W$ such that $f' \circ f = g' \circ g$.
Let us assume $\mathcal{F}$ has the origin, namely, a "trivial" structure that embeds into any other one.

Theorem

Assume $\mathcal{F}$ is as above, and there is a natural number N such that $|X| \leq N$ for every $X \in \mathcal{F}$.

Finite homogeneous structures

The setup

Let $\mathcal{F}$ be a class of finite structures, closed under isomorphisms.
We say that $\mathcal{F}$ has the amalgamation property if for every $Z \in \mathcal{F}$, for every embeddings $f \colon Z \to X$, $g \colon Z \to Y$ with $X, Y \in \mathcal{F}$, there exist $W \in \mathcal{F}$ and embeddings $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f = g' \circ g$.
Let us assume $\mathcal{F}$ has the origin, namely, a "trivial" structure that embeds into any other one.

Theorem

Assume $\mathcal{F}$ is as above, and there is a natural number N such that $|X| \leq N$ for every $X \in \mathcal{F}$.
Then there exists a unique, up to isomorphism, structure $V \in \mathcal{F}$ satisfying

(1) *Every $X \in \mathcal{F}$ embeds into V.*

(2) *Every isomorphism between $\mathcal{F}$-substructures of V extends to an automorphism of V.*

Claim

Confluent terminating rewriting systems and finite homogeneous structures are two faces of the same theory.

Abstract evolution systems

Definition

An evolution system is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object, called the origin, and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows, called transitions.

Abstract evolution systems

Definition

An evolution system is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object, called the origin, and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows, called transitions.

The only requirements are:

1. All identities are in $\mathcal{T}$.
2. $h \circ t \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an isomorphism.

Abstract evolution systems

Definition

An evolution system is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object, called the origin, and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows, called transitions.

The only requirements are:

1. All identities are in $\mathcal{T}$.
2. $h \circ t \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an isomorphism.

We are interested in evolutions, namely, sequences of the form

$$\Theta \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition.

Abstract evolution systems

Definition

An evolution system is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object, called the origin, and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows, called transitions.
The only requirements are:

1. All identities are in $\mathcal{T}$.
2. $h \circ t \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an isomorphism.

We are interested in evolutions, namely, sequences of the form

$$\Theta \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition.

A path is a finite composition of transitions.

Finite objects

Definition

An object X is finite if there exists a path from the origin Θ to X.

Confluence

Definition

We say that $\mathcal{E}$ is confluent if for every finite object Z, for every **paths** $f\colon Z \to X$, $g\colon Z \to Y$ there exist paths f', g' such that $f' \circ f = g' \circ g$.

Confluence

Definition

We say that $\mathcal{E}$ is confluent if for every finite object Z, for every **paths** $f\colon Z \to X$, $g\colon Z \to Y$ there exist paths f', g' such that $f' \circ f = g' \circ g$.

Definition

We say that $\mathcal{E}$ is locally confluent if for every finite object Z, for every **transitions** $f\colon Z \to X$, $g\colon Z \to Y$ there exist paths f', g' such that $f' \circ f = g' \circ g$.

Amalgamation

Definition

We say that $\mathcal{E}$ has the finite amalgamation property if for every finite object Z, for every transitions $f\colon Z \to X$, $g\colon Z \to Y$ there exist transitions f', g' such that $f' \circ f = g' \circ g$.

Amalgamation

Definition

We say that $\mathcal{E}$ has the finite amalgamation property if for every finite object Z, for every transitions $f\colon Z \to X$, $g\colon Z \to Y$ there exist transitions f', g' such that $f' \circ f = g' \circ g$.

Proposition

Finite amalgamation property implies confluence.

Termination

Definition

An evolution system $\mathcal{E}$ is terminating if there is no evolution consisting of nontrivial transitions.
A transition is nontrivial if it is not an isomorphism.

Termination

Definition

An evolution system $\mathcal{E}$ is terminating if there is no evolution consisting of nontrivial transitions.
A transition is nontrivial if it is not an isomorphism.

Definition

An object N is normalized if it is finite and every path starting from N consists of trivial transitions.

Finite Fraïssé theory

Proposition

Assume $\mathcal{E}$ is a confluent terminating evolution system. Then there exists a unique, up to isomorphism normalized object N. Furthermore:

1. *Every finite object has a path into N.*
2. *For every paths $f_0, f_1 \colon A \to N$ with A finite, there exists an automorphism $h \colon N \to N$ such that $f_1 = h \circ f_0$.*

Newman's Lemma revisited

Definition

An evolution system is regular if $t \circ h \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an iso.

Newman's Lemma revisited

Definition

An evolution system is regular if $t \circ h \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an iso.

Theorem

A locally confluent terminating regular evolution system is confluent.

Newman's Lemma revisited

Definition

An evolution system is regular if $t \circ h \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an iso.

Theorem

A locally confluent terminating regular evolution system is confluent.

Given a rewriting system $(X, \to)$, define $x \leq y$ if there is a path

$$x = x_0 \to x_1 \to \cdots \to x_n = y.$$

Fix $\Theta \in X$. Then $\mathbb{X} = \langle (X, \leq), \to, \Theta \rangle$ becomes an evolution system.

Applications in incidence geometry

Theorem (P. Nowakowski & W.K. 2021)

The smallest projective planes $\mathbb{Z}_2\mathbf{P}^2$, $\mathbb{Z}_3\mathbf{P}^2$ are homogeneous. No other projective plane is homogeneous.

Fraïssé theory revisited

Theorem

Assume $\mathcal{E}$ is an essentially countable evolution system with the finite amalgamation property. Then:

Theorem

Assume $\mathcal{E}$ is an essentially countable evolution system with the finite amalgamation property. Then:

1. *There exists an evolution*

$$\vec{u}\colon \Theta = U_0 \to U_1 \to \cdots$$

with the absorption property, namely, for every n, for every transition $f\colon U_n \to Y$ there are $m > n$ and a path $g\colon Y \to U_m$ with $g \circ f = u_n^m$, where u_n^m is the path from U_n to U_m extracted from $\vec{u}$.

Fraïssé theory revisited

Theorem

Assume $\mathcal{E}$ is an essentially countable evolution system with the finite amalgamation property. Then:

1. *There exists an evolution*

$$\vec{u}\colon \Theta = U_0 \to U_1 \to \cdots$$

   with the absorption property, namely, for every n, for every transition $f\colon U_n \to Y$ there are $m > n$ and a path $g\colon Y \to U_m$ with $g \circ f = u_n^m$, where u_n^m is the path from U_n to U_m extracted from $\vec{u}$.

2. *The colimit $U_\infty = \lim \vec{u}$ is defined uniquely (up to isomorphism) by the absorption property.*

Fraïssé theory revisited

Theorem

Assume $\mathcal{E}$ is an essentially countable evolution system with the finite amalgamation property. Then:

1. *There exists an evolution*

$$\vec{u}\colon \Theta = U_0 \to U_1 \to \cdots$$

   with the absorption property, namely, for every n, for every transition $f\colon U_n \to Y$ there are $m > n$ and a path $g\colon Y \to U_m$ with $g \circ f = u_n^m$, where u_n^m is the path from U_n to U_m extracted from $\vec{u}$.
2. *The colimit $U_\infty = \lim \vec{u}$ is defined uniquely (up to isomorphism) by the absorption property.*
3. *U_∞ is homogeneous with respect to paths between finite objects.*

Theorem

Assume $\mathcal{E}$ is an essentially countable evolution system with the finite amalgamation property. Then:

1. *There exists an evolution*

$$\vec{u}\colon \Theta = U_0 \to U_1 \to \cdots$$

   with the absorption property, namely, for every n, for every transition $f\colon U_n \to Y$ there are $m > n$ and a path $g\colon Y \to U_m$ with $g \circ f = u_n^m$, where u_n^m is the path from U_n to U_m extracted from $\vec{u}$.
2. *The colimit $U_\infty = \lim \vec{u}$ is defined uniquely (up to isomorphism) by the absorption property.*
3. *U_∞ is homogeneous with respect to paths between finite objects.*

$$\begin{tikzcd} \Theta = U_0 \arrow[r] & U_1 \arrow[r] & \cdots \arrow[r] & U_n \arrow[r] \arrow[d] & \cdots \arrow[r] & U_m \arrow[r] & \cdots \\ & & & Y & & & \end{tikzcd}$$

Fraïssé theory revisited

Theorem

Assume $\mathcal{E}$ is an essentially countable evolution system with the finite amalgamation property. Then:

1. *There exists an evolution*

$$\vec{u} \colon \Theta = U_0 \to U_1 \to \cdots$$

   with the absorption property, namely, for every n, for every transition $f \colon U_n \to Y$ there are $m > n$ and a path $g \colon Y \to U_m$ with $g \circ f = u_n^m$, where u_n^m is the path from U_n to U_m extracted from $\vec{u}$.

2. *The colimit $U_\infty = \lim \vec{u}$ is defined uniquely (up to isomorphism) by the absorption property.*

3. *U_∞ is homogeneous with respect to paths between finite objects.*

$$\begin{tikzcd}
\Theta = U_0 \arrow[r] & U_1 \arrow[r] & \cdots \arrow[r] & U_n \arrow[r] \arrow[d] & \cdots \arrow[r] & U_m \arrow[r] & \cdots \\
& & & Y \arrow[urr, dashed] & & &
\end{tikzcd}$$

Classical Fraïssé limits

Example

Let $\mathcal{F}$ be a class of finite structures in a fixed language $\mathcal{L}$ consisting of finitely many relations. Assume $\mathcal{F}$ is closed under isomorphisms and substructures.

Example

Let $\mathcal{F}$ be a class of finite structures in a fixed language $\mathcal{L}$ consisting of finitely many relations. Assume $\mathcal{F}$ is closed under isomorphisms and substructures.
We can convert $\mathcal{F}$ to an evolution system, where the origin is $\emptyset$ and transitions are one-point extensions. The category $\mathfrak{V}$ consists of all structures in the language $\mathcal{L}$, where the arrows are all homomorphisms.

Example

Let $\mathcal{F}$ be a class of finite structures in a fixed language $\mathcal{L}$ consisting of finitely many relations. Assume $\mathcal{F}$ is closed under isomorphisms and substructures.
We can convert $\mathcal{F}$ to an evolution system, where the origin is $\emptyset$ and transitions are one-point extensions. The category $\mathfrak{V}$ consists of all structures in the language $\mathcal{L}$, where the arrows are all homomorphisms.
Assume $\mathcal{F}$ has the amalgamation property. Then the generic evolution leads to the Fraïssé limit of $\mathcal{F}$, the unique homogeneous countable structure U_∞ whose age is $\mathcal{F}$.

W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, arXiv:2109.12600

W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, arXiv:2109.12600

Further research:

W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, arXiv:2109.12600

Further research:

- Uncountable evolutions (with Antonio Avilés and Ivan Di Liberti)

W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, arXiv:2109.12600

Further research:

- Uncountable evolutions (with Antonio Avilés and Ivan Di Liberti)
- Abstract stochastic processes

W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, arXiv:2109.12600

Further research:

- Uncountable evolutions (with Antonio Avilés and Ivan Di Liberti)
- Abstract stochastic processes
- ???

W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, arXiv:2109.12600

Further research:

- Uncountable evolutions (with Antonio Avilés and Ivan Di Liberti)
- Abstract stochastic processes
- ???

THE END