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From Semantic Games to Analytic Calculi

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Motivation

From a TLS review of

Hugo Mercier and Dan Sperber:
THE ENIGMA OF REASON –
A new theory of human understanding

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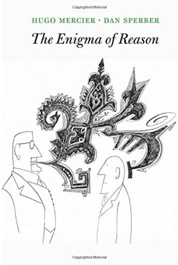

> *"Reasoning is not meant to be done alone in a room [. . .] [R]easoning, like sex, works better when another person is involved."*
>
> Cecilia Heyes, TLS, July 28, 2017

Overview

- the most basic logic game:
  Hintikka's game for classical logic
- from Hintikka's game to sequent calculus via disjunctive states
- Hintikka's game and many truth values:
  - many-valued truth tables, Nmatrices
  - Giles's game for Łukasiewicz logic
- analyzing a hypersequent calculus using games
- some hints on current developments

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The meaning of connectives is encoded in a game:

– players I and You, acting in role **P** (proponent) or **O** (opponent)

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Rules of the game refer to the form of the current formula:

$F \wedge G \Rightarrow$ **O** chooses F or G, **P** asserts F or G, accordingly

$F \vee G \Rightarrow$ **P** asserts F or G, according to her own choice

$\neg F \Rightarrow$ after switching roles **P** (the other player) asserts F

Winning condition:

If an atom A is reached, **P** wins if A is true in $\mathcal{I}$, otherwise **O** wins

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Central Fact: (characterization of Tarski's "truth in a model")

I have a winning strategy iff F is true in $\mathcal{I}$

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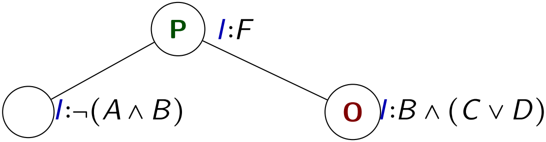

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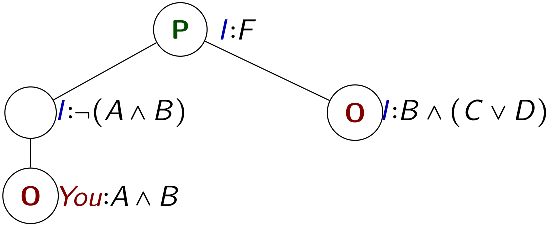

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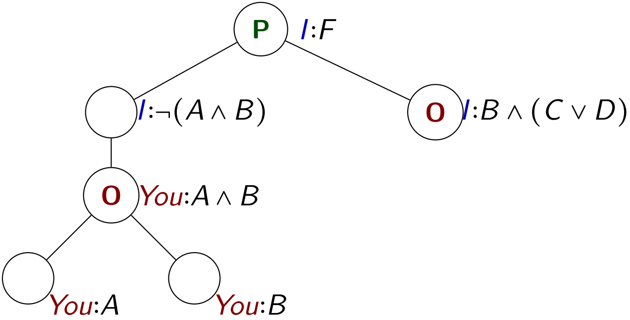

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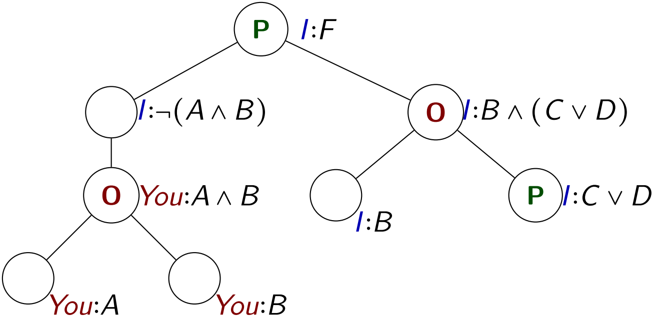

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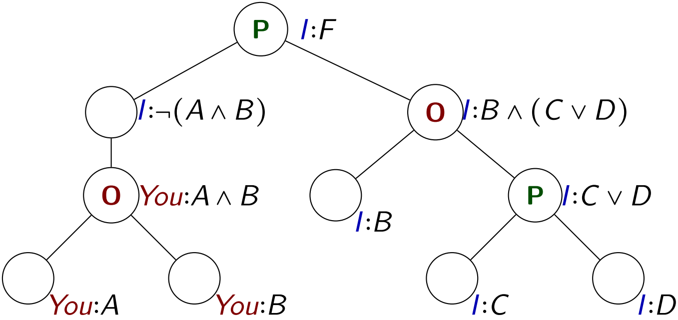

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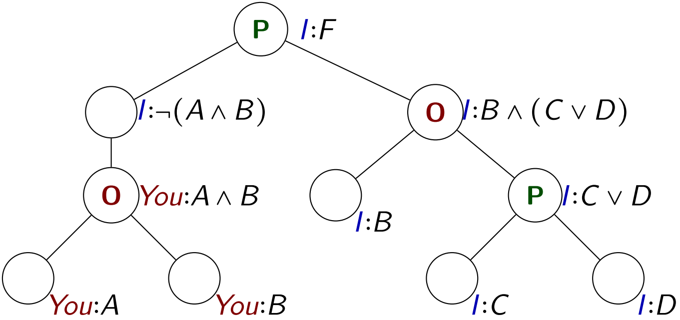

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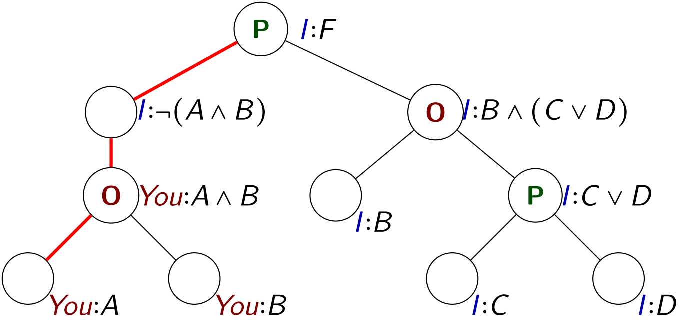

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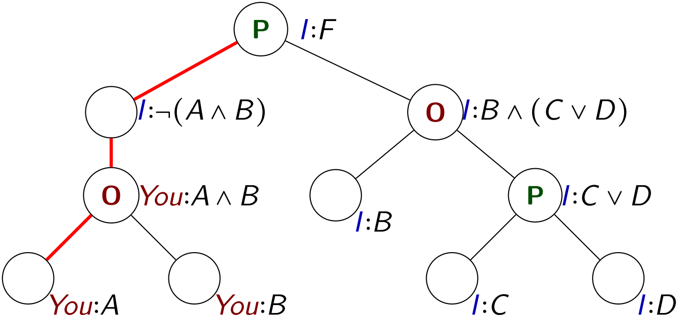

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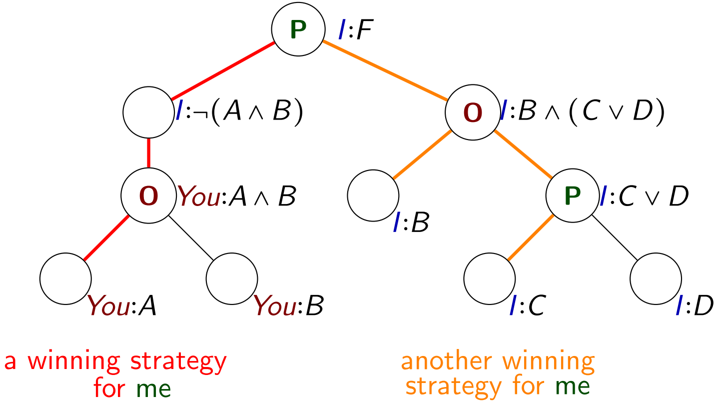

Extracting a classical sequent calculus in 3 easy steps:

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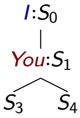
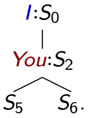


or (my choice)



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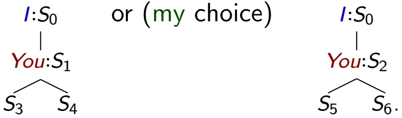
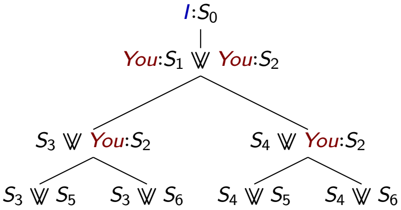


then I have a corresponding disjunctive strategy:



Example (ctd.) – a disjunctive strategy for player I

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$$You{:}A \mathbin{\vee\!\!\vee} You{:}B \mathbin{\vee\!\!\vee} I{:}C \mathbin{\vee\!\!\vee} I{:}D$$

NB:

– if we replace, e.g., C by A, then the formula becomes tautological

– correspondingly, every final disjunctive state contains some atom with both labels ('You' as well as 'I')

$\Longrightarrow$ we can find a winning strategy for any interpretation!

Extracting a classical sequent calculus (ctd.):

Step 2: Formulate the game rules from *I*/*You* perspective

Remember that there 2 kinds of choices involved:

– use meta-level disjunction ($\mathbin{\rotatebox[origin=c]{180}{$\mathbb{\Lambda}$}}$) for I-choices

– use branching for You-choices

$$\frac{You{:}A_1 \wedge A_2}{You{:}A_1 \Vvee You{:}A_2}\ \wedge\text{-You} \qquad \frac{I{:}A_1 \wedge A_2}{I{:}A_1 \qquad I{:}A_2}\ \wedge\text{-I}$$

$$\frac{I{:}A_1 \vee A_2}{I{:}A_1 \Vvee I{:}A_2}\ \vee\text{-I} \qquad \frac{You{:}A_1 \vee A_2}{You{:}A_1 \qquad You{:}A_2}\ \vee\text{-You}$$

$$\frac{I{:}\neg A}{You{:}A}\ \neg\text{-I} \qquad \frac{You{:}\neg A}{I{:}A}\ \neg\text{-You}$$

Extracting a classical sequent calculus (ctd.):

Step 3: sequents as meta-disjunctions of *I*/*You*-signed formulas

– Put *I*-signed formulas to the right of the sequent arrow '$\vdash$' and put *You*-signed formulas to the left of '$\vdash$'
– add 'side formulas' (Γ, Δ) to the main (exhibited) ones
– write the rules upside down

$$\frac{A_1, A_2, \Gamma \vdash \Delta}{A_1 \wedge A_2, \Gamma \vdash \Delta}\ \wedge\text{-l} \qquad \frac{\Gamma \vdash \Delta, A_1 \qquad \Gamma \vdash \Delta, A_2}{\Gamma \vdash \Delta, A_1 \wedge A_2}\ \wedge\text{-r}$$

$$\frac{A_1, \Gamma \vdash \Delta \qquad A_2, \Gamma \vdash \Delta}{A_1 \vee A_2, \Gamma \vdash \Delta}\ \vee\text{-l} \qquad \frac{\Gamma \vdash \Delta, A_1, A_2}{\Gamma \vdash \Delta, A_1 \vee A_2}\ \vee\text{-r}$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg\text{-l} \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg\text{-r}$$

NB: These are the logical rules for $\wedge/\vee/\neg$ for **LK** (additively)!

Extracting a classical sequent calculus (ctd.):

What about structural rules and initial sequents?

- permutation corresponds to commutativity of $\mathbb{W}$
- contraction corresponds to idempotency of $\mathbb{W}$
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  which are now those containing an atomic formula appearing with both labels (I and You = 'left' and 'right')

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What about implication?

- remember: $A \rightarrow B$ can be defined as $\neg A \vee B$
- a corresponding rule arises by combining **P**'s choice with a role switch, when **P** chooses $\neg A$
- further connectives call for combining **P** and **O** choices (two-level rules, instead of a single choice or role switch)

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What about quantifiers?

- more tricky – involves Herbrand's theorem

Hintikka's game and (many) truth values

Observation: Hintikka's game looks strictly classical (bivalent):

- winning/loosing corresponds to atomic truth/falsehood
- there are just three types of moves:

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  NB: combinations of such moves (as, e.g., in $\rightarrow$-rule)

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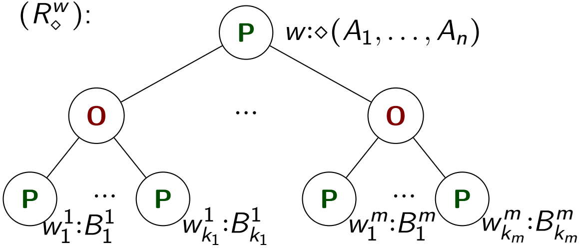
(2) Taking role switch as the clue:
The "role assignment" can be seen as truth value
P asserts "**t**:F" before, but "**f**:F" after role switch.

Corresponding generalization:
P always asserts (and **O** always denies) a statement of the form "F takes value w", denoted as w:F

General format of a game rule for connective $\diamond$:

$(R^w_\diamond)$:



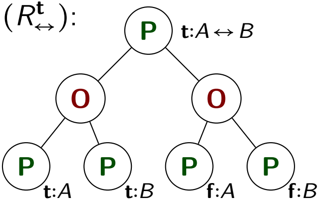
where $w^i_j \in TV$ and $B^i_j \in \{A_1, \ldots, A_n\}$ for $1 \leq i \leq m$, $1 \leq j \leq k_i$.

Remark:

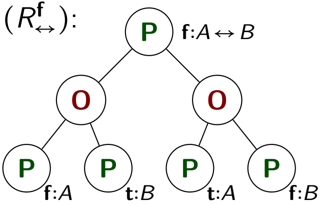
A dual form, where **O** chooses first, leads to equivalent results

Concrete rule instances (classical equivalence):

$(R^{\mathbf{t}}_{\leftrightarrow})$:



$(R^{\mathbf{f}}_{\leftrightarrow})$:



Note: These rules correspond to external disjunctive normal forms (the dual form corresponds to conjuctive normal forms)

$$\mathbf{t}{:}A \leftrightarrow B \equiv ((\mathbf{t}{:}A \mathbin{\wedge\!\!\wedge} \mathbf{t}{:}B) \mathbin{\vee\!\!\vee} (\mathbf{f}{:}A \mathbin{\wedge\!\!\wedge} \mathbf{f}{:}B))$$

$$\mathbf{f}{:}A \leftrightarrow B \equiv ((\mathbf{f}{:}A \mathbin{\wedge\!\!\wedge} \mathbf{t}{:}B) \mathbin{\vee\!\!\vee} (\mathbf{t}{:}A \mathbin{\wedge\!\!\wedge} \mathbf{f}{:}B))$$

Normal forms can be directly read off from arbitrary truth tables: rules corresponding to arbitrary finite truth tables arise!

From truth tables to semantic games

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Definition:

A matrix semantics $\mathcal{M}$ for a propositional language $\mathcal{L}$ specifies a truth table $\widehat{\diamond}$ over truth values $\mathcal{V}$ for each connective $\diamond$ in $\mathcal{L}$.
This induces a (deterministic) valuation $v_{\mathcal{M}}^{\alpha} : \mathrm{FORM}_{\mathcal{L}} \to \mathcal{V}$ over an assignment $\alpha : \mathrm{PV} \to \mathcal{V}$, as usual.

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Definition:

Given a matrix semantics $\mathcal{M}$, a corresponding $\mathcal{M}$-game (played under an assignment α) is obtained from external disjunctive normal forms for every pair $w{:}\diamond(F_1, \ldots, F_n)$, as outlined before. Ending in $w{:}A$, **P** wins if $v^{\alpha}_{\mathcal{M}}(A) = w$, otherwise **O** wins.

Theorem:

For every matrix semantics $\mathcal{M}$ and assignment α t.f.a.e.:

(1) **P** has a winning strategy for the $\mathcal{M}$-game starting with $w{:}F$.

(2) $v^{\alpha}_{\mathcal{M}}(F) = w$, where $v^{\alpha}_{\mathcal{M}}$ is the valuation over α.

Inverting the direction:

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Generalization to Avron's Nmatrices

Claim:

Starting with semantic games to obtain truth-functional semantics straightforwardly leads to Nmatrices!

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An Nmatrix semantics $\mathcal{N}$ specifies a nondeterministic truth table $\widehat{\diamond} : \mathcal{V}^n \mapsto (2^{\mathcal{V}} - \emptyset)$ for each n-ary connective $\diamond$.

At least two possible forms nondeterministic valuations arise:

Dynamic valuation:

- $\vec{v}^{\alpha}_{\mathcal{N}}(F) = \alpha(F)$ if $F \in \mathrm{PV}$
- $\vec{v}^{\alpha}_{\mathcal{N}}(\diamond(F_1, \dots, F_n)) \in \widetilde{\diamond}(\vec{v}^{\alpha}_{\mathcal{N}}(F_1), \dots, \vec{v}^{\alpha}_{\mathcal{N}}(F_n))$ for n-ary $\diamond$

Static valuation:

A static valuation $\check{v}^{\alpha}_{\mathcal{N}}$ is a dynamic valuation satisfying

$\check{v}^{\alpha}_{\mathcal{N}}(\diamond(G_1, \dots, G_n)) = \check{v}^{\alpha}_{\mathcal{N}}(\diamond(F_1, \dots, F_n))$ if $\check{v}^{\alpha}_{\mathcal{N}}(G_i) = \check{v}^{\alpha}_{\mathcal{N}}(F_i)$

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Caveat: While static and dynamic valuations can modeled, a new type of valuation ('liberal valuation') is more natural

Game rules and Nmatrices

Observation:

Arbitrary collections of rules $R^w_\diamond$ (one for each pair $\langle w, \diamond \rangle$) do not correspond to truth functional finite valued logics. (Just consider identical external normal forms for different truth values.)

However:

Every collection of rules $R^w_\diamond$ does determine a particular form of valuation over some finite-valued Nmatrix semantics $\mathcal{N}$: namely one, where different occurrences of the same subformula might be evaluated differently. (Otherwise like dynamic valuation.)

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Connections to analytic proof systems

- The translation from ordinary game states to disjunctive states can be applied to applied to $\mathcal{M}$-games just as well.

  This results in signed sequent calculi (also known as ‘many-sided sequent systems’).

- Nmatrices were invented for the analysis of sequent systems!

  There is tight connection between Nmatrix-based semantics and so-called canonical proof systems.

  General criteria for cut-eliminability arise in this manner.

- Nmatrices arise naturally if certain logical logical rules are missing in canonical proof systems.

Robin Giles about reasoning in theories of physics

Robin Giles 1974/77: 'A non-classical logics for physics'

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Principles of Giles's analysis of reasoning:

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Principles of Giles's analysis of reasoning:

- All assertions have to be tested with respect to concrete (instances of) binary experiments:
  Each atomic assertion $P(t_1, \ldots, t_n)$ is connected to a parameterized experiment $E_P^{t_1,\ldots,t_n}$ that may fail or succeed.

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  NB: since experiments are dispersive, assertions are risky!

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  NB: since experiments are dispersive, assertions are risky!
- A tenet collects all assertions of a player (me or you).
  Repetita juvant: Tenets are multisets of interpreted sentences.

Important observations:

- I can quantify the expected loss for my tenet $\{q_1, \ldots, q_n\}$ of atomic assertions by assigning a subjective failure probability $\langle q_i \rangle$ to the experiment E_{q_i}.
- While these probabilities may have some objective grounds they are still subjective in the sense that I don't care which values you associate with the same experiments.
- Events are (unrepeatable) instances of (repeatable) elementary experiments. In other words:
  experiments are event types, such that the same probabilities are assigned to events of the same type.
- Final (or: atomic) game states of are denoted by $[p_1, \ldots, p_n \,\|\, q_1, \ldots, q_m]$, where $\{p_1, \ldots, p_n\}$ is your tenet and $\{q_1, \ldots, q_m\}$ is my tenet of assertions.
  My corresponding risk, i.e., my expected loss of money is

$$\sum_{1 \leq i \leq m} \langle q_i \rangle € - \sum_{1 \leq j \leq n} \langle p_j \rangle €$$

What about logically complex statements?

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For the reduction of logically complex assertions to atomic ones, Giles suggests a game referring to Lorenzen's dialogue game, rather than to Hintikka. semantic game, introduced roughly at the same time.)

Giles states the rules in the following — old fashioned — way:

- *He who asserts $A \to B$ agrees to assert B if his opponent will assert A.*
- *He who asserts $A \vee B$ undertakes to assert either A or B at his own choice.*
- *He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice.*

Defining $\neg A = A \to \bot$ leads to

- *He who asserts $\neg A$ agrees to pay 1$ to his opponent if he will assert A.*

Observations about the dialogue part of Giles's game

(1) Assertions are attacked at most once: *'repetita juvant'*.

(2) Principle of limited liability for attacks:
The players may explicitly choose not to attack an assertion.

(3) In contrast to Lorenzen:

- no regulations on the succession of moves!
- no restrictions on who can attack what!

(4) Giles defends the $\wedge$-rule by reference to the above principle of limited liability: each assertion carries a maximal risk of 1\$.

Giles has no rule for strong conjunction (&)!
By extending the principle to defense move we obtain:

- *If a player asserts $A \,\&\, B$ she has to assert either both, A and B, or else has to assert $\bot$ (i.e., to pay 1€).*

Adequateness of Giles's game for propositional Ł

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I always have a strategy for avoiding expected loss for precisely those initial statements that are valid in Łukasiewicz logic.

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Theorem (coarse version):

I always have a strategy for avoiding expected loss for precisely those initial statements that are valid in Łukasiewicz logic.

Theorem (refined version):

Suppose we play the game starting with my assertion of F with respect to given assignment $\langle\cdot\rangle$ of risk values to atomic assertions.

The following are equivalent:

- F evaluates to $1-x$ in Łukasiewicz logic under the interpretation that assigns $1 - \langle p\rangle$ to each atom p.
- My best strategy guarantees that the play ends in a state, where my risk is at most x€, while You have a strategy enforcing that my risk is at least x€.

Can games help to analyze analytic proof systems?

Suppose I'd announce that I want to talk about the logic given by the following Hilbert-style system:

1	$(A \to B) \to C) \to ((((B \to A) \to C) \to C)$
2	$(A \to B) \to ((B \to C) \to (A \to C))$
3	$\bot \to A$
4	$((A \to \bot) \to \bot)) \to A$
5	$(A \,\&\, B) \to B$
6	$(A \,\&\, B) \to (B \,\&\, A)$
7	$(A \,\&\, (A \to B)) \to (B \,\&\, (B \to A))$
8	$((A \,\&\, B) \to C) \to (A \to (B \to C))$
9	$(A \to (B \to C)) \to ((A \,\&\, B) \to C)$

Modus Ponens is the only inference rule

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4 & ((A \to \bot) \to \bot)) \to A \\
5 & (A \,\&\, B) \to B \\
6 & (A \,\&\, B) \to (B \,\&\, A) \\
7 & (A \,\&\, (A \to B)) \to (B \,\&\, (B \to A)) \\
8 & ((A \,\&\, B) \to C) \to (A \to (B \to C)) \\
9 & (A \to (B \to C)) \to ((A \,\&\, B) \to C)
\end{array}
$$

Modus Ponens is the only inference rule

You were justified to loose interest in my presentation, because of this inadequate presentation of a logic!

An improvement?

Suppose I replace the above list of axioms by

1. $A \to (B \to A)$
2. $(A \to B) \to ((B \to C) \to (A \to C))$
3. $(\neg A \to \neg B) \to (B \to A)$
4. $((A \to B) \to B) \to ((B \to A) \to A)$

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\end{array}
$$

A much more reasonable start:

I want to talk about

- one of three fundamental fuzzy logics
- the logic based on the t-norm $\max(0, x + y - 1)$
- the logic of all MV-algebras
- the logic where formulas represent McNaughton functions
- the logic of Mundici's Ulam-Renyi game semantics
- the only fuzzy logic where *all* truth functions are continuous
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In other words: Łukasiewicz logic **Ł**!

Formal reasoning

The above remarks about 'false starts' seem to suggest:

- proof theoretic (syntactic) presentations are uninformative
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- proof theoretic (syntactic) presentations are uninformative
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But what if we focus on formal reasoning (within the logic)?!

Hilbert style systems are indeed problematic for this purpose!
But:
think of Gentzen's characterization of classic vs. intuitionistic inference in terms of the cut-free sequent calculus!

The following analytic proof systems are related in this respect:

- analytic tableaux
- natural deduction
- calculus of structures
- ...

HŁ – A hypersequent system for Łukasiewicz logic:

Initial sequents:

$$A \vdash A \ (ID) \qquad \vdash \ (EMPTY) \qquad \bot, \Gamma \vdash A \ (\bot, l)$$

Logical rules:

$$\frac{B, \Gamma \vdash \Delta, A \mid \Gamma \vdash \Delta \mid \mathcal{H}}{A \to B, \Gamma \vdash \Delta \mid \mathcal{H}} \ (\to, l) \qquad \frac{A, \Gamma \vdash \Delta, B \mid \mathcal{H} \qquad \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta, A \to B \mid \mathcal{H}} \ (\to, r)$$

$$\frac{A, \Gamma \vdash \Delta \mid B, \Gamma \vdash \Delta \mid \mathcal{H}}{A \wedge B, \Gamma \vdash \Delta \mid \mathcal{H}} \ (\wedge, l) \qquad \frac{\Gamma \vdash \Delta, A \mid \mathcal{H} \qquad \Gamma \vdash \Delta, B \mid \mathcal{H}}{\Gamma \vdash \Delta, A \wedge B \mid \mathcal{H}} \ (\wedge, r)$$

$$\frac{A, B, \Gamma \vdash \Delta \mid \mathcal{H} \qquad \bot, \Gamma \vdash \Delta \mid \mathcal{H}}{A \,\&\, B, \Gamma \vdash \Delta \mid \mathcal{H}} \ (\&, l) \qquad \frac{\Gamma \vdash \Delta, A, B \mid \Gamma \vdash \Delta, \bot \mid \mathcal{H}}{\Gamma \vdash \Delta, A \,\&\, B \mid \mathcal{H}} \ (\&, r)$$

Structural rules:

$$\frac{\mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}} \ (EW) \qquad \frac{\Gamma \vdash \Delta \mid \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}} \ (EC) \qquad \frac{\Gamma \vdash \Delta \mid \mathcal{H}}{A, \Gamma \vdash \Delta \mid \mathcal{H}} \ (IW)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}}{\Gamma_1 \vdash \Delta_2 \mid \Gamma_2 \vdash \Delta_1 \mid \mathcal{H}} \ (SPLIT) \qquad \frac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{H} \qquad \Gamma_2 \vdash \Delta_2 \mid \mathcal{H}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}} \ (MIX)$$

Although unusual, **HŁ** has nice properties:

- sound and complete for **Ł**
- (potentially much shorter, but hard to find) proofs using

$$\frac{\mathcal{H} \mid \Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2 \mid \mathcal{H}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}} \; (CUT)$$

  can be stepwise transformed into cut-free proofs [CM03]
- applications of structural rules can be limited to atomic hypersequents (except (EW) for trivial reasons)
- the 'purely logical' version of **HŁ** reduces all complex hypersequents to atomic hypersequents, for which validity can be checked in PTIME

Nevertheless:

is **HŁ** a really convincing analysis of actual reasoning?!

Dialogue games and the meaning of connectives

Lorenzen/Giles Idea (similar to Hintikka):
The meaning of a logical connective is given by dialogue game rules, like the following:

Let **P**/**O** stand for me/you or for you/me

P asserts	'attack' by **O**	answer by **P**
$A \to B$	A	B
$A \vee B$	'?'	A or B (**P** chooses)
$A \wedge B$	'l?' or 'r?' (**O** chooses)	A or B (accordingly)
$A \& B$	'?'	A and B

Note: $\neg A$ abbreviates $A \to \bot$.
The assertion '$\bot$' is always false.

The rules in sequent-style format

State of the game: $[A_1, \ldots, A_n \,\|\, B_1, \ldots, B_m]$

I assert $B_1, \ldots, B_m$, while you assert $A_1, \ldots, A_n$

The rules from my point of view (for brevity, only $\rightarrow$ and $\&$):

$$\frac{[B, \Gamma \,\|\, \Delta, A]}{[A \rightarrow B, \Gamma \,\|\, \Delta]} \ (\rightarrow, \mathit{me}) \qquad \frac{[A, \Gamma \,\|\, \Delta, B]}{[\Gamma \,\|\, \Delta, A \rightarrow B]} \ (\rightarrow, \mathit{you})$$

$$\frac{[A, B, \Gamma \,\|\, \Delta]}{[A \,\&\, B, \Gamma \,\|\, \Delta]} \ (\&, \mathit{me}) \qquad \frac{[\Gamma \,\|\, \Delta, A, B]}{[\Gamma \,\|\, \Delta, A \,\&\, B]} \ (\&, \mathit{you})$$

Note: the labels refer to the attacking player

- complex statements are decomposed exactly once
- no ‘hedging’ or ‘refuse to attack’ is allowed
- arbitrary states are reduced to atomic states
- no winning conditions formulated yet!

Dialogues as evaluation games

NB: If we add an evaluation function – assigning real numbers to atomic states – to the dialogue rules we obtain an evaluation game

A simple, but interesting example:

1. assign an arbitrary pay-off value $v(p) \in \mathbb{R}$ to each atom p
2. define $v([p_1, \ldots, p_n \parallel q_1, \ldots, q_m]) = \sum_i v(q_i) - \sum_j v(p_j)$
3. $\Longrightarrow$ finite 2-person game with perfect information: guaranteed pay-off for me can be calculated using induction following the max-min strategy for finite game trees

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The resulting logic is Slaney's Abelian logic (which coincides with one of Casari's logics of comparison):

- 'truth value set' is $\mathbb{R}$
- truth function for conjunction: addition
- truth function for implication: subtraction
- validity: value ≥ 0 under all assignments

Dialogues as evaluation games (ctd.)

To obtain Łukasiewicz logic we have to do three things:

(1) restrict to $v(p) \in [0,1]$ for atoms p; $v(\bot){=}0$

(2) allow refusion to attack (no player is forced to attack)

(3) allow hegding of maximal loss: instead of defending my(your) assertion I(you) can replace it by asserting $\bot$

A simplification:

(2) is only relevant for implication ($\rightarrow$).

(3) is only relevant for strong conjunction (&).

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A simplification:
(2) is only relevant for implication ($\to$).
(3) is only relevant for strong conjunction ($\&$).

The resulting rules are:

$$\frac{[B,\Gamma \parallel \Delta, A]}{[A \to B, \Gamma \parallel \Delta]}\ (\to, me) \quad \frac{[\Gamma \parallel \Delta]}{[A \to B, \Gamma \parallel \Delta]}\ (\to, me) \quad \frac{[A,\Gamma \parallel \Delta, B] \qquad [\Gamma \parallel \Delta]}{[\Gamma \parallel \Delta, A \to B]}\ (\to, you)$$

$$\frac{[A, B,\Gamma \parallel \Delta] \quad [\bot,\Gamma \parallel \Delta]}{[A \& B, \Gamma \parallel \Delta]}\ (\&, me) \quad \frac{[A,\Gamma \parallel \Delta, B]}{[\Gamma \parallel \Delta, A \& B]}\ (\&, you) \quad \frac{[\Gamma \parallel \Delta, \bot]}{[\Gamma \parallel \Delta, A \& B]}\ (\&, you)$$

What is the relation to Giles's game?

Remember: Giles talks about:

- payments to the opponent for each false assertion
- dispersive experiments that decide about the truth/falsity of atomic assertions
- probabilities associated with experiments
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- probabilities associated with experiments
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Have we lost the connection to Giles's approach?!

No!

Giles's story about dispersive experiments etc. is only a proposal to attach tangible meaning to $v(p)$ and to $v([p_1, \ldots, p_n \parallel q_1, \ldots, q_m])$

My expected loss in such a final state can be calculated to be $\sum_i \langle q_i \rangle - \sum_j \langle p_j \rangle$€, where $\langle p \rangle$ is short for the risk associated with the corresponding experiment E_p: $\langle p \rangle = 1 - \pi(E_p)$

Minimizing my expected payment to You amounts to maximizing v

From evaluation games to hypersequent systems

Giles's game – and its variants – are semantic games, i.e., interactive forms of determining truth values (Giles: risk values), given particular assignments.

While the rules can be presented in sequent format we still seem to be far from a hypersequent calculus like **HŁ** for checking validity.

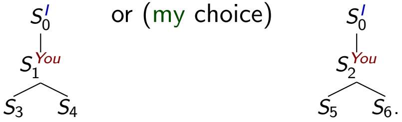
However, we can use the same generic way as for Hintikka's games to turn the semantic game into a provability game:
Keep all choices available: states $\longrightarrow$ disjunctive states

Resulting disjunctive strategies can be seen as
– either referring to a generalized, parallel version of the game
– or simply a bookkeeping device that collects all relevant ordinary strategies into one combined structure (tree)

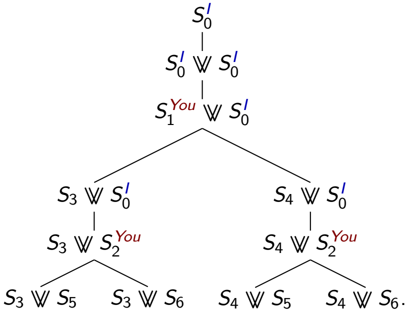
Evaluation of atomic disjunctive states:
winning means: at least one component state is winning (for me)

From strategies to disjunctive strategies

Suppose players me and you have the following choices:



corresponding disjunctive strategy:



Disjunctive winning strategy for $(p \to q) \vee (q \to p)$

$$[\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel (p \to q) \vee (q \to p)]^{You} \mathbb{W} [\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel \underline{(p \to q) \vee (q \to p)}]^{I} \mathbb{W} [\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel p \to q]^{You} \mathbb{W} [\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel p \to q]^{You} \mathbb{W} [\ \parallel \underline{(p \to q) \vee (q \to p)}]^{I}$$

$$[\ \parallel p \to q]^{You} \mathbb{W} [\ \parallel q \to p]^{You}$$

$$[\ \parallel \underline{p \to q}]^{You} \mathbb{W} [\ \parallel q \to p]^{You}$$

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$$[p \parallel q] \mathbb{W} [q \parallel p] \qquad [p \parallel q] \mathbb{W} [\ \parallel\ ] \qquad [\ \parallel\ ] \mathbb{W} [q \parallel p] \qquad [\ \parallel\ ] \mathbb{W} [\ \parallel\ ].$$

Disjunctive game rules <u>are</u> hypersequent rules!

Rules of the disjunctive game:

$$\frac{[B, \Gamma \parallel \Delta, A] \Cup [\Gamma \parallel \Delta] \Cup \mathcal{H}}{[A \to B, \Gamma \parallel \Delta] \Cup \mathcal{H}} \; (\to, l) \qquad \frac{[A, \Gamma \parallel \Delta, B] \Cup \mathcal{H} \quad [\Gamma \parallel \Delta] \Cup \mathcal{H}}{[\Gamma \parallel \Delta, A \to B] \Cup \mathcal{H}} \; (\to, r)$$

$$\frac{[A, \Gamma \parallel \Delta] \Cup [B, \Gamma \parallel \Delta] \Cup \mathcal{H}}{[A \wedge B, \Gamma \parallel \Delta] \Cup \mathcal{H}} \; (\wedge, l) \qquad \frac{[\Gamma \parallel \Delta, A] \Cup \mathcal{H} \quad [\Gamma \parallel \Delta, B] \Cup \mathcal{H}}{[\Gamma \parallel \Delta, A \wedge B] \Cup \mathcal{H}} \; (\wedge, r)$$

$$\frac{[A, B, \Gamma \parallel \Delta] \Cup \mathcal{H} \quad [\bot, \Gamma \parallel \Delta] \Cup \mathcal{H}}{[A \,\&\, B, \Gamma \parallel \Delta] \Cup \mathcal{H}} \; (\&, l) \qquad \frac{[\Gamma \parallel \Delta, A, B] \Cup [\Gamma \parallel \Delta, \bot] \Cup \mathcal{H}}{[\Gamma \parallel \Delta, A \,\&\, B] \Cup \mathcal{H}} \; (\&, r)$$

Disjunctive game rules are hypersequent rules!

Logical rules of **HŁ**:

$$\frac{B, \Gamma \vdash \Delta, A \mid \Gamma \vdash \Delta \mid \mathcal{H}}{A \to B, \Gamma \vdash \Delta \mid \mathcal{H}} \ (\to, l) \qquad \frac{A, \Gamma \vdash \Delta, B \mid \mathcal{H} \qquad \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta, A \to B \mid \mathcal{H}} \ (\to, r)$$

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What happened to structural rules and initial sequents?

Initial sequents: $A \vdash A$ (ID) $\quad \vdash$ $(EMPTY)$ $\quad \bot, \Gamma \vdash A$ $(\bot, l)$

Structural rules:

$$\frac{\mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}}\ (EW) \qquad \frac{\Gamma \vdash \Delta \mid \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}}\ (EC) \qquad \frac{\Gamma \vdash \Delta \mid \mathcal{H}}{A, \Gamma \vdash \Delta \mid \mathcal{H}}\ (IW)$$

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Remember: the structural rules of **HŁ** are only needed at the atomic level. (For proving sequents (EW) is redundant.)

If we are satisfied with more complex initial sequents then the structural rules are redundant!

Should we be satisfied with complex initial sequents?

In this case: yes!
Reason: it can be checked in PTIME whether a given atomic hypersequent is valid or not.

Other t-norm based fuzzy logics

t-norms are binary operations on $[0,1]$ that are associative, commutative, non-decreasing (in both arguments) with 1 as unit.

Three fundamental logics based on t-norms $\circ$ and their residua:

Logic	$x \circ y$	$x \Rightarrow y$, for $x > y$
Łukasiewicz:	$\max(0, x+y-1)$	$1 - x + y$
Gödel:	$\min(x, y)$	y
Product:	$x \cdot y$	y/x

Gödel logic G: hypersequents [Avron 91], sequents-of-relation [Baaz/F 99], parallellized dialogue games [F 02], ...

Łukasiewizc logic Ł: hypersequents [Metcalfe et.al. 02]

Product logic P: hypersequent calculus [Metcalfe et.al. 03]

Other ways of combining elementary claims

(We translate "loosing when false" into "winning when true")

Basic idea: $[p_1, \ldots, p_n \,\|\, q_1, \ldots, q_m]$ denotes my expect gain when betting for positive results of the q_i's against your bet for positive results of the p_i's.

This is ambigous!

"Beting for $B_1, \ldots, B_m$" can mean (at least) one of the following

- betting separately: $\langle B_1, \ldots, B_m \rangle =_{df.} \sum_i \langle B_i \rangle$ ($\Rightarrow$ logic **Ł**)
- betting jointly: $\langle B_1, \ldots, B_m \rangle =_{df.} \prod_i \langle B_i \rangle$ ($\Rightarrow$ logic **P** [**CHL**])
- worst case bet: $\langle B_1, \ldots, B_m \rangle =_{df.} \min_i \langle B_i \rangle$ ($\Rightarrow$ logic **G**)

Underlying dialogue rules (i.e., 'meaning postulates' for connectives) remain unchanged! Only the axioms change!

However:

The rules for constructing strategies must be made more explicit.

(Other logics ctd.):

"Making the rules for constructing strategies more explicit" means: making the (implicit) case distinction $A \leq B / B < A$ explicit, at least in the rules for implication.

This obviously requires to consider "$<$" in addition to "$\leq$" in denoting (disjunctive) states and corresponding (hyper)sequents.

With hindsight, "$<$" should have been there from the beginning!
Observe:
With $\leq$ the game is not zero-sum: both players can 'win'
(or possibly none, if we require a positive expected gain).

A case where we don't have to change anything except axioms:
Cancellative hoop logic CHL: like product, but over $(0, 1]$
(Another case is classical logic!)

Some hints on current developments (1)

Hintikka & Sandu considered imperfect information games:

Interesting already on the propositional level, leading to the interpretation of Nash equilibria as intermediary truth vaules.

[Mayer, F 2015/18]: every $r \in [0,1]_{\mathbb{Q}}$ can be represented!

Ongoing research:
Which truth functions can be represented by imp.inf.-games?
How can Hintikka-style games be combined with Giles-style games?

Some hints on current developments (2)

Truth comparison game:

Systematic reduction of **P** s claims of the form $F < G$ or $F \leq G$ using rules like

$$\frac{A_1 \wedge A_2 < B}{A_1 < B \text{ or } A_2 < B} \wedge\text{<-l} \qquad \frac{A < B_1 \wedge B_2}{A < B_1 \text{ and } A < B_2} \wedge\text{<-r}$$

Gödel logic **G** can be easily characterized in this manner.

Lifting to disjunctive states yields a 'sequents-of-relations' calculus.

Some hints on current developments (3)

Current FWF project:
From semantic games to analytic calculi – and back

Some results:

- Alexandra Pavlova, Robert Freiman, Timo Lang
  From Semantic Games to Provability:
  The Case of Gödel Logic
  Studia Logica, 2022
- Robert Freiman:
  Games for Hybrid Logic
  WoLLIC 2021
- Alexandra Pavlova:
  Provability Games for Non-classical Logics
  – Mezhirov's Game for MPC, KD!, and KD
  WoLLIC 2021

Summary and Conclusion

- Analytic ('Gentzen style') proof systems are needed for effective proof search, but also for analyzing reasoning within a logic like Łukasiewicz logic **Ł**.
- Hypersequents enable useful analytic systems, but seem problematic as formal models of reasoning.
- Dialogue games, like Giles's for **Ł**, model reasoning from first principles, but seem only to refer to truth evaluation.
- We have shown:
  Constructing disjunctive strategies for Giles-style games corresponds directly to logical hypersequent rules.
  Structural rules are only needed to reduce valid atomic hypersequents into simple sequents.

  This principle generalizes to other fuzzy logics.

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