A Vopěnka-style principle for fuzzy mathematics

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Indistinguishability

Distinguish:

1. Indistinguishability in a mathematical structure

   = Indistinguishability of objects by some or all of the means available the structure

   Usually an equivalence relation

   Examples:

   - Topological indistinguishability = having the same neighborhoods
   - Indistinguishability of strings by a language L: $x \equiv_L y$ iff $(\forall z \in \Sigma^*)(xz \in L \Leftrightarrow yz \in L)$
   - Isomorphic objects in a category, Leibniz identity, ...

2. Indistinguishability as a phenomenon regarding perception (by humans or other agents, possibly equipped by some instruments)

3. Mathematical modeling of the latter (maps of the territory) = this talk

Pre-theoretical properties of indistinguishability

Observe:

- Indistinguishability can regard all or only some aspects of objects (position, size, color, ...)
- Indistinguishability depends on the means employed (naked eye, telescope, microscope, ...)

$\Rightarrow$ There is no single relation of indistinguishability, but rather many = a *kind* of relations (like, eg, orderings)

- Indistinguishability relations are
  - Reflexive, $x \sim x$
  - Symmetric, $x \sim y \to y \sim x$
  - Transitive, $x \sim y \sim z \to x \sim z$ ...or are they?

The Poincaré paradox

H. Poincaré (1902):

Indistinguishability relations should intuitively be transitive, but in a sufficiently long sequence of pairwise-indistinguishable objects,

$$x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{N-1} \sim x_N,$$

x_1 and x_N can well be distinguishable

$\Rightarrow$ Modeling indistinguishabilities as either

- equivalence relations (reflexive, symmetric, transitive) or
- proximity relations (reflexive, symmetric, but not necessarily transitive)

each addresses just one horn of the dilemma presented by the paradox

Another pre-theoretical property of indistinguishability

Observe:

- Indistinguishability is, generally, a graded notion:

  Some pairs of objects can be distinguished more easily than others

  Full indistinguishability is the limit of increasingly difficult distinguishability (note: indistinguishability is a negative notion)

$\Rightarrow$ Adequate models of indistinguishability should take its gradedness into account

Graded models of indistinguishability

Graded indistinguishability (or increasingly difficult distinguishability) can be mathematically modeled in several ways:

(a) $R = \bigcap_{d \in D} R_d$ for a directed set D
(where the elements of D represent, eg, distinction-difficulty levels, sharpness of the view, or instruments used for discrimination)

(b) Binary real-valued functions (eg, metrics, pseudometrics, …), representing the 'distance' of objects as regards their distinguishability
$$d\colon A^2 \to [0, +\infty]$$
Note: the reflexivity and symmetry of indistinguishability then correspond to the conditions $d(x, x) = 0$ and $d(x, y) = d(y, x)$

(c) Dually to the latter (and slightly more generally), fuzzy relations
$$R\colon A^2 \to L \quad \text{(for suitable structures of degrees } L\text{)}$$
satisfying appropriate conditions
(of fuzzy reflexivity, symmetry, and transitivity)

Fuzzy indistinguishability relations

Some merits of fuzzy relations as models of indistinguishability:

- They are fuzzily transitive (in fact, are fuzzy equivalences), but still avoid the Poincaré paradox
- They can be handled in first-order fuzzy logic similarly to classical equivalence relations
- When formalized in fuzzy logic, they admit more general degrees of indistinguishability besides reals (incl. abstract non-numerical degrees, hyperreal degrees, non-linearly ordered degrees, etc)
- Formalized in fuzzy logic, they interpret (graded) binary *predicates* $\Rightarrow$ no type-mismatch ("is indistinguishable from" is a binary predicate rather than a binary function)
- They are well and long studied (Zadeh 1971, Valverde 1985, ...)
- Their standard [0,1]-valued models are dual to (pseudo)metrics via simple functions ($1 - x$, $-\log x$ and such), so option (b) is included

Fuzzy logics (Łukasiewicz and others)

Axiomatics of Łukasiewicz logic (primitive language: &, $\to$, 0):

(BL1)	$((\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)))$
(BL4)	$(\varphi \mathbin{\&} (\varphi \to \psi)) \to (\psi \mathbin{\&} (\psi \to \varphi))$
(BL5a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \mathbin{\&} \psi) \to \chi)$
(BL5b)	$((\varphi \mathbin{\&} \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(BL6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(BL7)	$0 \to \varphi$
(Ł)	$((\varphi \to 0) \to 0) \to \varphi$
(MP)	$\varphi,\ \varphi \to \psi \ /\ \psi$

Defined connectives:

$$\neg\varphi \equiv_{\mathsf{df}} \varphi \to 0 \qquad 1 \equiv_{\mathsf{df}} \neg 0$$

$$\varphi \oplus \psi \equiv_{\mathsf{df}} \neg(\neg\varphi \mathbin{\&} \neg\psi) \qquad \varphi \leftrightarrow \psi \equiv_{\mathsf{df}} (\varphi \to \psi) \wedge (\psi \to \varphi)$$

$$\varphi \wedge \psi \equiv_{\mathsf{df}} \varphi \mathbin{\&} (\varphi \to \psi) \qquad \varphi \vee \psi \equiv_{\mathsf{df}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

Łukasiewicz fuzzy logic

Standard semantics on $[0,1]$:

$$\|\varphi \mathbin{\&} \psi\| = \max(0, \|\varphi\| + \|\psi\| - 1) \qquad \|\varphi \wedge \psi\| = \min(\|\varphi\|, \|\psi\|)$$
$$\|\varphi \oplus \psi\| = \min(1, \|\varphi\| + \|\psi\|) \qquad \|\varphi \vee \psi\| = \max(\|\varphi\|, \|\psi\|)$$
$$\|\varphi \to \psi\| = \min(1, 1 - \|\varphi\| + \|\psi\|) \qquad \|\varphi \leftrightarrow \psi\| = 1 - \big|\|\varphi\| - \|\psi\|\big|$$
$$\|\neg\varphi\| = 1 - \|\varphi\|$$

General semantics: MV-algebras (modulo signature)

= involutive divisible semilinear commutative bounded integral residuated lattices

(eg, Chang's algebra on $\{-\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$)

Optional expansion: $\text{Ł}_\triangle$ = Ł + the unary connective $\triangle$

- $\|\triangle\varphi\| = 1$ if $\|\varphi\| = 1$, otherwise 0 (in linear $\text{MV}_\triangle$-algebras)
- Axiomatized by adding 5 axioms + 1 rule (of necessitation) to Ł

Further fuzzy logics (of continuous t-norms)

Further fuzzy logics are obtained by changing the standard semantics of:

- $\&$ to another continuous commutative order-preserving monoidal operation (a continuous t-norm)
- $\rightarrow$ to its residuum, $x \rightarrow y =_{\mathrm{df}} \sup\{z \mid z \,\&\, x \leq y\}$
- $\neg, \leftrightarrow, \oplus$ accordingly

Prominent examples (besides Ł):

- Product fuzzy logic Π:
  - $x \,\&\, y = x \cdot y$
  - $x \rightarrow y = y/x$ if $x > y$, otherwise 1
- Gödel fuzzy logic G = intuitionistic + $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$:
  - $x \,\&\, y = \min(x, y)$
  - $x \rightarrow y = y$ if $x > y$, otherwise 1

Axiomatized by changing the axiom (Ł) appropriately

First-order fuzzy logics

Semantics: A model M over an L-algebra $\mathcal{A}$:

- $\|Px_1, \ldots, x_n\|, \|\varphi(x_1, \ldots, x_n)\| \colon M^n \to \mathcal{A}$
- Constants and functions as usual, $\|f(x_1, \ldots, x_n)\| \colon M^n \to M$
- $\|(\forall x)\varphi(x)\| =_{\mathrm{df}} \inf_{a \in M} \|\varphi\|(a), \quad \|(\exists x)\varphi(x)\| =_{\mathrm{df}} \sup_{a \in M} \|\varphi\|(a)$

M is safe $\equiv_{\mathrm{df}}$ the $\inf, \sup$ exist in $\mathcal{A}$ for all φ ($\mathcal{A}$ need not be lattice-complete)

Axiomatics of all safe L-models:

(models over a single L-algebra $\mathcal{A}$ typically not axiomatizable)

$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$	(t free for x in φ)
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$	(t free for x in φ)
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$	(x not free in χ)
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$	(x not free in χ)
$(\forall 3)$	$(\forall x)(\varphi \vee \chi) \to (\forall x)\varphi \vee \chi$	(optional, completness wrt lin L-alg)
(gen)	$\varphi \,/\, (\forall x)\varphi$	($(\forall 1)$–$(\exists 2)$+(gen) = Rasiowa's axioms)

Fuzzy indistinguishability relations

Fuzzy indistinguishability =

- a binary predicate $\sim$ in first-order fuzzy logic,
- ie, semantically, a binary fuzzy relation $R\colon M^2 \to \mathcal{A}$,

satisfying the following conditions:

Condition	Axiom $(\forall x)(\forall y)(\forall z)$	Semantics For all $a, b, c \in M$
Fuzzy reflexivity	$x{\sim}x$	$Raa = 1$
Fuzzy symmetry	$x{\sim}y \leftrightarrow y{\sim}x$	$Rab = Rba$
Fuzzy transitivity	$(x{\sim}y \mathbin{\&} y{\sim}z) \to x{\sim}z$	$Rab \mathbin{\&^{\mathcal{A}}} Rbc \leq Rac$

Optional (indistinguishabilities separating points):

Separation	$\triangle(x{\sim}y) \to x{=}y$	$Rab = 1$ only if $a = b$

Fuzzy indistinguishability relations

Examples:

- The Euclidean indistinguishability on $\mathbb{R}$: $Exy =_{\mathrm{df}} \max(1 - |x - y|, 0)$
  (over the standard MV-algebra $[0, 1]_{Ł}$)
- $Rxy =_{\mathrm{df}} 2^{-|x-y|}$ (over $[0, 1]_{\Pi}$)
- All classical ('crisp') equivalence relations are fuzzy indistinguishability relations too

Terminology: Fuzzy indistinguishability relations are also known as (fuzzy) indistinguishability operators, (fuzzy) similarity relations, fuzzy equivalences, or fuzzy equalities

Classical references:

- L. Zadeh: Similarity relations and fuzzy orderings. *Inf Sci* 1971
- L. Valverde: On the structure of F-indistinguishability operators. *FSS* 1985
- J. Recasens: *Indistinguishability Operators.* Springer 2010

Overcoming the Poincaré paradox

Fuzzy indistinguishability relations are fuzzy transitive ($Rxy \mathbin{\&} Ryz \leq Rxz$), but in most fuzzy logics (except G), & is not idempotent

(eg, $.99 \mathbin{\&} .99 = .98$ in $[0,1]_{Ł}$)

$\Rightarrow \|x_0 \sim x_i\|$ can decrease along the Poincaré sequence

$$x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{N-1} \sim x_N,$$

even if $\|x_i \sim x_{i+1}\| > 1 - \varepsilon$ for all i and a very small value $\varepsilon > 0$
(ie, even if the neighboring elements are practically indistinguishable)

$\Rightarrow$ For sufficiently large N, the guaranteed value $\|x_0 \sim x_N\|$ gets very small

(eg, $\|x_0 \sim x_N\| = 0$ for $N \geq 1/\varepsilon$ in $[0,1]_{Ł}$)

x_i:	x_0	$\sim$	x_1	$\sim$	x_2	$\sim$	x_3	$\sim$	$\dots$	$\sim$	x_{99}	$\sim$	x_{100}
$\|x_i \sim x_{i+1}\| =$		.99		.99		.99		.99		.99		.99	
$\|x_0 \sim x_i\| \geq$	1		.99		.98		.97		$\dots$		.01		0

Duality to (pseudo)metrics

Fuzzy indistinguishability relations valued in the standard algebras of a broad class of t-norm fuzzy logics are dual to (pseudo)metrics:

(Valverde 1985)

If R is a fuzzy indistinguishability valued in

- $[0,1]_\Pi$, then $d(x,y) = -\log(Rxy)$ is an (extended) pseudometric
- $[0,1]_{Ł}$, then $d(x,y) = 1 - Rxy$ is a bounded pseudometric
- $[0,1]_G$, then $d(x,y) = 1 - Rxy$ is a bounded pseudoultrametric

  $(d(x,z) \le \max(d(x,y), d(y,z)))$

(In all of these cases, d is a metric if R is separated)

The distance measures dissimilarity (so, distinguishability) of the objects

Duality to (pseudo)metrics

Vice versa, (pseudo)metrics give rise to fuzzy indistinguishability relations, namely, $Rxy = \mathrm{e}^{-d(x,y)}$ over $[0,1]_\Pi$ and $Rxy = 1 - d(x,y)$ over $[0,1]_{Ł,\mathrm{G}}$

This correspondence can be generalized to a broad class of continuous t-norms (Archimedean = with no idempotent elements) by means of their additive generators (ie, functions f such that $x \mathbin{\&} y = f^{(-1)}(f(x) + f(y))$):

R generates $d_R(x,y) = f(Rxy)$ and d generates $R_d xy = f^{(-1)}(d(x,y))$

$\Rightarrow$ Metric notions are applicable to fuzzy indistinguishability
eg, betweenness and one-dimensionality (Boixader–Jacas–Recasens 2017)

Another principle for perception-based indistinguishability

Arguably, perception-based indistinguishability satisfies another property, caused by the physical agents' limited ability of discernment:

One can never distinguish all elements of an infinite set from each other

Formally: $\neg\,\mathrm{Fin}(X) \rightarrow (\exists x, y \in X)(x \neq y \mathbin{\&} x \sim y)$

Trivial in classical logic (equivalences with only finitely many equivalence classes), but less so for a fuzzy indistinguishability R on M:

$$\text{If } A \subseteq M \text{ is infinite, then } \bigvee_{\substack{a,b \in A \\ a \neq b}} Rab = 1$$

Corresponds to the precompactness (ie, total boundedness) of the corresponding pseudometric

Relation to Vopěnka's treatment of infinity in AST

In his Alternative Set Theory (AST), Vopěnka equates infinity with unsurveyability:

In his approach, finite sets are those in which we can clearly discern all of their elements

> *"[Finite sets] can be construed as having the multitude of their elements present before the horizon that limits the clarity of the view."* *(Vopěnka 1989, p. 138)*

A set is considered infinite by Vopěnka iff some of its elements are not clearly distinguishable

$\Rightarrow$ Vopěnka's conception of infinity involves a kind of unavoidable indiscernibility between some elements

Infinity as indiscernibility

AST (which uses classical logic) treats the notions of infinity and indistinguishability in specific ways (left aside here):

- Infinity by means of semisets = proper subclasses of sets
- Indistinguishability along the lines of $R = \bigcap_{d \in D} R_d$ mentioned earlier

Precompact fuzzy indistinguishability relations

Observe:

If R is a precompact fuzzy indistinguishability on M and $A \subseteq M$ an infinite set, then for all $\alpha < 1$ there are $a, b \in A$ such that $Rab > \alpha$

$\Rightarrow$ In every infinite set, there can only be finitely many (n_α) elements indistinguishable from each other at most to degree α (ie, distinguishable at least to degree $\varepsilon = 1 - \alpha$), for each $\alpha < 1$ (n_α can increase with $\alpha \nearrow 1$)

Example: The Euclidean indistinguishability is not precompact on $\mathbb{R}$

(although it is precompact on bounded intervals)

Rather, α-cuts (for $\alpha < 1$) of a precompact indistinguishability relation on $\mathbb{R}$ must equate all elements in some half-bounded intervals

(α-cut $R_\alpha = \{\langle x, y \rangle \mid Rxy \geq \alpha\}$)

Existence of fuzzy minima

The precompactness of a fuzzy indistinguishability relation brings about many properties analogous to those of compact metric spaces (closedness is often non-essential under fuzzification)

Theorem (cf B., 2016): In an ordering fuzzified by a compatible precompact fuzzy indistinguishability relation, every fuzzy set has a non-empty fuzzy minimum

R is compatible with an ordering $\leq$ on A $\equiv_{\mathrm{df}}$
$a \leq b \leq c$ implies $Rab \geq Rac$ and $Rbc \geq Rac$, for all $a, b, c \in A$

S fuzzifies $\leq$ by a fuzzy relation R $\equiv_{\mathrm{df}}$ $Sab = (a \leq b) \vee Rab$

Fuzzy minimum of a fuzzy set A in a fuzzy ordering S:

$$(\mathrm{Min}_S\, A)a =_{\mathrm{df}} Aa \wedge \bigwedge_b (Ab \rightarrow^{\mathcal{A}} Sab)$$

An application: The non-triviality of fuzzy counterfactuals

The existence of fuzzy minima can be used in the recently proposed fuzzy semantics for counterfactuals (B.–Majer, Synthese 2021):

Lewis' Analysis 2: $A \mathrel{\Box\!\!\rightarrow} B$ is true in a possible world w if all A-worlds most similar to w are C-worlds (classically relies on the implausible Limit Assumption that there are closest A-worlds $\Rightarrow$ rejected)

Fuzzifying the similarity ordering of possible worlds (acknowledging its vagueness) by a precompact fuzzy indistinguishability on the distances of possible worlds from the actual world (precompact, since our ability to discern distances of worlds is not infinite $\Rightarrow$ a natural assumption),

we can show that the fuzzified semantics of $A \mathrel{\Box\!\!\rightarrow} C$ can retain Lewis' (simpler and intuitive) Analysis 2 (all of the closest A-worlds are C-worlds) without the implausible Limit Assumptions

(In particular, precompactness guarantees the existence of minimal A-worlds to non-zero degrees by the previous theorem)