Sizes of Countable Sets

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Galieleo's paradox concerning the relation between collections of natural numbers and their squares may be the best illustration of the well-known fact (Mancosu 2009) that in comparing infinite collections one must choose one of two mutually exclusive principles:

1. *The Part-Whole Principle* (PW): "The whole is greater than its part."

2. *Cantor's Principle* (CP): "Two sets have the same size if and only if there is a one-to-one correspondence between their elements."

While Bolzano insisted on PW, according to Cantor, two sets have the same size if CP holds. Cantor's approach prevailed and is generally accepted as the only correct one. All infinite countable sets have one and the same size, namely $\aleph_0$, that is the cardinality of the set of natural numbers.

Bolzano's theory of infinite quantities preserving PW described primarily in *Paradoxes of the Infinite* is also meaningful and can be interpreted consistently in contemporary mathematics (Trlifajová 2018), (Bellomo & Massas 2021). Bolzano was aware of the existence of a one-to-one correspondence between some infinite multitudes, however, he writes: "Merely from this circumstance we can in no way conclude *that these multitudes are equal to one another if they are infinite* with respect to the plurality of their parts ... An equality of these multiplicities can only be concluded if some other reason is added, such as that both multitudes have exactly the same *determining ground*." (Bolzano 1851/2004, §21).

We introduce a theory of sizes of some countable sets based on Bolzano's ideas. The method is similar to that of Benci and Di Nasso's Numerosity Theory (NT) (Benci & Di Nasso 2003, 2019) but it differs in some substantial ways. Set sizes are determined constructively. They are unambiguous for they do not depend on the choice of an ultrafilter which is always partially arbitrary. Rules for determining are more rigorously justified, and so some results are more accurate. On the other side, sizes of countable sets are only partial and not linearly ordered. *Quid pro quo.*

Simultaneously, this is an answer to Matthew Parker, who argues in *Set Size and the Part-Whole Principle* (Parker 2013) that all Euclidean theories, i.e. theories satisfying PW, must be either very weak or arbitrary and misleading.

Canonically countable sets are those that can be arranged into mutually disjoint finite groups indexed by natural numbers according to its *determining ground*

$$A = \bigcup\{A_n, n \in \mathbb{N}\}.$$

Then a *size* of A is a sum of finite cardinalities $|A_n|$ expressed as a sequence of partial sums. We define a *size sequence* of A as the sequence $\sigma(A) = (\sigma_n(A))_{n\in\mathbb{N}}$ such that

$$\sigma_n(A) = |A_1| + \dots |A_n|.$$

The problem is the exact meaning of the *determining ground.* In some cases a *canonical arrangement* is evident, in other cases we will define it so that the following rules are satisfied.

A canonical arrangement of natural numbers $\mathbb{N} = \bigcup\{A_n, n \in \mathbb{N}\}$ is

$$A_n = \{n\}.$$

Let A, B be two canonically arranged sets, $A = \bigcup\{A_n, n \in \mathbb{N}\}$ and $B = \bigcup\{B_n, n \in \mathbb{N}\}$. Then

$$A \subseteq B \Rightarrow (\forall n \in \mathbb{N})(A_n \subseteq B_n).$$

A canonical arrngement of the Cartesian product $A \times B$ is defined for all $n \in \mathbb{N}$

$$(A \times B)_n = \bigcup\{A_i \times B_j, n = \max\{i, j\}\}.$$

Now, we can determine size sequences of integers, rational numbers and their subsets. If two intervals of rational numbers of have the same length then have the same size as well.

Theorem 1. Let A, B be two canonically countable sets.

1. If A is finite then $\sigma(A) =_{\mathcal{F}} |A|$
2. If A is a proper subset of B, $A \subset B$, then $\sigma(A) <_{\mathcal{F}} \sigma(B)$.
3. The size sequence of the union is $\sigma(A \cup B) = \sigma(A) + \sigma(B) - \sigma(A \cap B)$.
4. The size sequence of the Cartesian product is $\sigma(A \times B) = \sigma(A) \cdot \sigma(B)$.

Theorem 2. Let S be the set of size sequences, i.e. the set of non-decreasing sequences of natural numbers. Let *addition* and *multiplication* be defined componentwise, *equality* and *order* are also defined componentwise but from a sufficiently great index, i.e. modulo Fréchet filter Then the structure $(S, +, \cdot, =_{\mathcal{F}}, <_{\mathcal{F}})$ is a partial ordered non-Archimedean commutative semiring.

References

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