# Derivability of rules of $\beta$-conversion in partial type theory

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For motivation consider a familiar mathematical problem: *find the domain of the function*

$$f(x) = \frac{3}{x^2 - 3x + 2}$$

$f$'s domain – i.e. the set of real numbers for which $f$ is defined – contains any number $n$ if and only if the result of substituting $n$'s name for $x$ in $\frac{3}{x^2-3x+2}$ is a *valid expression*, i.e. a name of some number $m$; $m$ is then the value of $f$ at $n$. Since no fraction with zero denominator represents a number, and the original expression is equivalent to $\frac{3}{(x-1)\times(x-2)}$, it can be readily seen that $f$'s domain contains all numbers except 1 and 2, i.e. $\mathbb{R}\backslash\{1;2\}$.

When solving the above problem, one in fact employs the pivotal rules of *type theory* ($TT$) (i.e. a *higher-order logic* with a hierarchy of functions sorted in interpretations (sets) $\mathscr{D}_\tau$ of *types* $\tau$) namely the *rules of $\beta$-conversion* (i.e. *$\beta$-contraction*: $\vdash$; *$\beta$-expansion*: $\dashv$):

$$[\lambda \tilde{x}_m.C](\bar{D}_m) \dashv\vdash C_{(\bar{D}_m/\bar{x}_m)}$$

where $\tilde{X}_m$ is short for $X_1X_2...X_m$; $\bar{X}_m$ is short for $X_1, X_2, ..., X_m$; but $C_{(\bar{D}_m/\bar{x}_m)}$ is short for $C_{(D_1/x_1)...(D_m/x_m)}$, where $C_{(D/x)}$ is the result of substituting $D$ for all free occurrences of $x$ in $C$ (interpreted in our approach as $[\![Sub(\ulcorner D\urcorner, \ulcorner x\urcorner, \ulcorner C\urcorner)]\!]^{\mathscr{M},v}$, where $v$ is an *assignment*, $\mathscr{M}$ is a *model* that is built, inter alia, from a frame $\mathscr{F} = \{\mathscr{D}_\tau \mid \tau \in \mathscr{T}\}$, where $\mathscr{T}$ is the set of all relevant types; $\ulcorner X\urcorner$ presents $X$ as such, not $X$'s value).

However, within *partial TT*, i.e. a TT that embraces both total and partial functions,[1] the above classical formulation of $\beta$-contraction is not valid. For example,

$$[\lambda x.\lambda y. \div (x,x)](\div(3,0)) \not\Rightarrow_\beta \lambda y. \div (\div(3,0), \div(3,0)),$$

for $[\lambda x.\lambda y. \div (x,x)](\div(3,0))$ is *non-denoting* (because $D := \div(3,0)$ is non-denoting), but $\lambda y. \div (\div(3,0), \div(3,0))$ denotes a certain partial function. This is why Tichý 1982, Moggi 1988, Farmer 1990, Feferman 1995, Beesson 2004 and others conditioned the rule by requiring that $D$ entering $\beta$-reduction must be *denoting*.

[1] A *total/partial function[-as-graph]* maps all/some-but-not-all members of its domain $\mathscr{D}$ to some members of its range $\mathscr{D}'$. Note that such functions differ from *functions-as-computations*.

In Tichý's 1982 convenient 'two-dimensional' *natural deduction ND* for his *simple TT* (STT) with total and partial (multiargument) functions, his safe $\beta$-contraction rule *by-name* reads[2]

$$(\beta{-}\mathrm{CON}) \qquad [\lambda\tilde{x}_m.C](\bar{D}_m){:}\mathbf{a} \vdash C_{(\bar{D}_m/\bar{x}_m)}{:}\mathbf{a}$$

in which terms $C$ are 'signed' by $:\mathbf{a}$, which is a terse variant of $\cong \mathbf{a}$, where $\cong$ is a symbol of *congruence*; $\mathbf{a}$ is either a variable $a$ or a constant A or an acquisition $\ulcorner A \urcorner$. This requires here the whole $\beta$-redex, i.e. the application written on the left-hand side of $\vdash$, and thus also its parts being *denoting*.

However, Tichý's proposal is too restrictive. For example,

$$[\lambda x. \div(x, 0)](3) \Rightarrow_\beta (\div(3, 0))$$

is not handled by ($\beta$-CON). To capture also such examples we propose the '*negative*' variant of the above '*positive*' rule ($\beta$-CON) (i.e. ($\beta$-CON$^+$)),

$$(\beta{-}\mathrm{CON}^-)\ \Gamma \longrightarrow [\lambda\tilde{x}_m.C](\bar{D}_m){:}\_; \Gamma \longrightarrow D_1{:}\mathbf{x}_1; ...; \Gamma \longrightarrow D_m{:}\mathbf{x}_m \vdash \Gamma \longrightarrow C_{(\bar{D}_m/\bar{x}_m)}{:}\_$$

where each $D_i{:}x_i$ says that $D_i$ is denoting an object in the range of $x_i$, and $X{:}\_$ represents that $X$ is *non-denoting* ($\_$ stands for any type-theoretically appropriate non-denoting term).

Our further main contribution (see reference below) is a derivation of rules of $\beta$-conversion *by-value*, both in 'positive' and 'negative' variants, from the primitive rules (e.g. ($\beta$-CON)) of the natural deduction ND for partial TT. Notation $\pm$ covers both $+$-and $-$-variants and $V$ indicates that one substitutes the *value* of $D$, which is directly 'named' by $\mathbf{d}$ – while it is $\mathbf{d}$ (not $D$) what is substituted for $x$ through $C$, cf. $C_{(\bar{\mathbf{d}}_m/\bar{x}_m)}$:

$$(\beta{-}\mathrm{CON}^{V\pm})\ \Gamma \longrightarrow [\lambda\tilde{x}_m.C](\bar{D}_m){:}\underline{\mathbf{a}}, \Gamma \longrightarrow D_1{:}\mathbf{d}_1; ...; \Gamma \longrightarrow D_m{:}\mathbf{d}_m \vdash \Gamma \longrightarrow C_{(\bar{\mathbf{d}}_m/\bar{x}_m)}{:}\underline{\mathbf{a}}$$

# References


Beeson, Michael (2004): Lambda Logic. In: *Automated Reasoning: Second International Joint Conference, IJCAR 2004, Cork, Ireland, July 4–8, 2004, Proceedings.* Ed. by David Basin and Michael Rusinowitch. Lecture Notes in Artificial Intelligence. Springer, 2004, pp. 460–474.

Farmer, William M. (1990): A Partial Functions Version of Church's Simple Theory of Types. In: *Journal of Symbolic Logic* 55.3, pp. 1269–1291.

Feferman, Solomon (1995): Definedness. In: *Erkenntnis* 43.3, pp. 295–320.

Moggi, Eugen (1988): *The Partial Lambda-Calculus*, Ph.D. thesis, University of Edinburgh.

Kuchyňka, Petr; Raclavský, Jiří (2021): $\beta$-reduction-by-name, by-value and $\eta$-reduction in partial type $\lambda$-calculus/partial type theory. *Logic Journal of the IGPL (conditionally accepted).*

Tichý, Pavel (1982): Foundations of Partial Type Theory. In: *Reports on Mathematical Logic* 14, pp. 57–72.


[2]In this abstract, we omit $\beta$-expansion rule(s). Further, let $C$ etc. be *typed* by types $\tau_{(i)}$ as follows: $C, \mathbf{a}/\tau; D_1, x_1/\tau_1; ...; D_m, x_m/\tau_m$, so $\lambda\tilde{x}_m.C/\langle\bar{\tau}_m\rangle{\to}\tau$ (the type of functions from $\mathscr{D}_{\tau_1} \times ... \times \mathscr{D}_{\tau_m}$ to $\mathscr{D}_\tau$), where $\bar{\tau}_m$ is short for $\tau_1, \tau_2, ..., \tau_m$.