Describable Nuclei, Subframe Logics and Negative Translations

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What do soundness/completeness of negative translations of intutionistic modal logics, extension stability of preservativity/provability logics and the use of nuclei on Heyting Algebra Expansions (HAEs) to form other HAEs have in common? As it turns out, in all those cases one deals with a certain kind of subframe property for a given logic, i.e., the question whether the logic in question survives a transition to at least some types of substructures of a specific (usually Kripke) semantics. The nucleic perspective on subframe logics has been introduced by Bezhanishvili and Ghilardi [2007] for the purely (super)intuitionistic syntax, *i.e.*, without modalities or other additional connectives. It has not been employed so much in the modal setting, mostly because the focus in the field tends to be on modal logics with classical propositional base, and nuclei are a rather trivial notion in the boolean setting. However, things are very different intuitionistically. Since the 1970's, nuclei have been studied in the context of point-free topologies (a.k.a. lattice-complete Heyting algebras), sheaves and Grothendieck topologies on toposes [Fourman and Scott, 1979], and finally arbitrary Heyting algebras [Macnab, 1981]. Other communities may know them as *lax modalities* [Fairtlough and Mendler, 1997] or *strong monads* (when algebras are understood as posets, and posets are understood as skeleton categories).

The presentation marries the nucleic view on subframe properties with the framework of *describable operations* introduced to study subframe logics in Wolter [1993]. Wolter's original setting was restricted to classical modal logics, but with minimal care his setup can be made to work intuitionistically and nuclei provide the missing ingredient to make it fruitful. From this perspective, we revisit our earlier syntactic studies of soundness and completeness of negative translations in modal logic [Litak et al., 2017] or *extension stability* for preservativity logics of Heyting Arithmetic (HA) based

constructive strict implication $\multimap$ [Litak and Visser, 2022]. Various characterization and completeness results can be obtained in a generic way. Further applications in progress include, *e.g.*, a joint study with Georg Struth of a describable nucleus on $(\mathsf{B})\mathsf{BI}$ yielding the class of intuitionistic (affine) assertions of separation logic [Ishtiaq and O'Hearn, 2001, §9] or nucleic perspective on algebraic cut elimination and algebraic proof theory [Belardinelli et al., 2004].

References

F. Belardinelli, P. Jipsen, and H. Ono. Algebraic aspects of cut elimination. *Studia Logica*, 77(2): 209–240, 2004. doi: 10.1023/B:STUD.0000037127.15182.2a. URL `https://doi.org/10.1023/B:STUD.0000037127.15182.2a`.

G. Bezhanishvili and S. Ghilardi. An algebraic approach to subframe logics. Intuitionistic case. *Annals of Pure and Applied Logic*, 147(1-2):84–100, 2007.

M. Fairtlough and M. Mendler. Propositional lax logic. *Information and Computation*, 137(1):1–33, 1997. doi: 10.1006/inco.1997.2627. URL `https://doi.org/10.1006/inco.1997.2627`.

M. P. Fourman and D. S. Scott. Sheaves and logic. In M. Fourman, C. Mulvey, and D. Scott, editors, *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977*, pages 302–401. Springer Berlin Heidelberg, Berlin, Heidelberg, 1979. doi: 10.1007/BFb0061824. URL `https://doi.org/10.1007/BFb0061824`.

S. S. Ishtiaq and P. W. O'Hearn. BI as an assertion language for mutable data structures. In C. Hankin and D. Schmidt, editors, *Proceedings of The 28th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL)*, pages 14–26, 2001. URL `http://dl.acm.org/citation.cfm?id=360204`.

Tadeusz Litak and Albert Visser. Lewisian fixed points I: Two incomparable constructions. *CoRR*, abs/1905.09450, 2022. URL `http://arxiv.org/abs/1905.09450`. To appear in the Outstanding Contributions volume dedicated to Dick de Jongh.

Tadeusz Litak, Miriam Polzer, and Ulrich Rabenstein. Negative Translations and Normal Modality. In D. Miller, editor, *2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017)*, volume 84 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 27:1–27:18, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi: 10.4230/LIPIcs.FSCD.2017.27. URL `http://drops.dagstuhl.de/opus/volltexte/2017/7741`.

D. S. Macnab. Modal operators on Heyting algebras. *Algebra Universalis*, 12(1):5–29, 1981. doi: 10.1007/BF02483860. URL `https://doi.org/10.1007/BF02483860`.

F. Wolter. *Lattices of Modal Logic*. PhD thesis, Fachbereich Mathematik, Freien Universität Berlin, 1993. URL `http://www.csc.liv.ac.uk/~frank/publ/Wolter_1993_PhD_Thesis.pdf`.