A category-theoretic language for metric Fraïssé theory

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Classical Fraïssé theory [3, 7.1] studies countable (ultra)homogeneous first-order structures. Irwin and Solecki [4] introduced projective Fraïssé theory of topological structures, where an extension of a structure is a quotient from a larger structure instead of an embedding into a larger structure, as in the classical case. Both versions of the theory can be unified and can go beyond first-order structures – using the language of category theory, which captures the essence of structural constructions and abstracts from irrelevant details. The idea to use the language of category theory goes back to Droste and Göbel [1] [2], Pech and Pech [7], and Kubiś [6], who introduced the notion of a Fraïssé sequence.

The core of the discrete abstract Fraïssé theory can be summarized in the following theorems. The setup of the first theorem consists of a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ satisfying several conditions essentially saying that $\mathcal{L}$ arises by freely adding limits of sequences to $\mathcal{K}$. (Ultra)homogeneity and the extension property are the key desired properties of Fraïssé limits.

Theorem 1. Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be and $\mathcal{L}$-object. The following conditions are equivalent.

(i) U is homogeneous and cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$.

(ii) U has the extension property and is cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(iii) U is an $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

Such object U is unique up to isomorphism and is cofinal in $\mathcal{L}$. It is called the *Fraïssé limit* in $\langle \mathcal{K}, \mathcal{L} \rangle$.

Theorem 2. A category $\mathcal{K} \neq \emptyset$ has a Fraïssé sequence if and only if

(i) $\mathcal{K}$ is directed,

(ii) $\mathcal{K}$ has the amalgamation property,

(iii) $\mathcal{K}$ is dominated by a countable subcategory.

The first application of the projective Fraïssé theory by Irwin and Solecki was taking $\mathcal{K}$ to be the category of all connected finite linear graphs and quotient maps, and obtaining a pre-space of the pseudo-arc as a Fraïssé limit. This way the authors obtained a new characterization of the pseudo-arc, which looks like an approximate form of homogeneity. We extend the language of category theory by adding approximate equalities $f \approx_\varepsilon g$ of maps for $\varepsilon > 0$, turning every hom-set $\mathcal{L}(X, Y)$ into a metric space, and we view the pseudo-arc itself as a Fraïssé limit in this context. This was done by Kubiś [5] in the context of metric-enriched categories. We further extend the framework to so-called *MU-categories*, and as an application, we realize pseudo-solenoids directly as (approximate) Fraïssé limits in the category of circle-like continua and continuous surjections.

In the talk we review the discrete abstract Fraïssé theory, and continue by extending it to MU-categories.

References

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