Czech Gathering of Logicians 2022

Ládví Academy Campus, June 16 & 17, 2022

Organization and scientific sponsorship

This event is organized by the Institute of Computer Science of the Czech Academy of Sciences and co-organized by the Institute of Information Theory and Automation of the Czech Academy of Sciences. The conference benefits from the scientific sponsorship of the Czech Society for Cybernetics and Informatics.

Zuzana Haniková, Vítězslav Švejdar, Jamie Wannenburg

editors

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Call for Abstracts

Czech Gathering of Logicians 2022

June 16-17, 2022
Institute of Computer Science and
Institute of Information Theory and Automation
Czech Academy of Sciences
Prague, Ládví Academy Campus
http://uivty.cs.cas.cz/~clog2022/

Invited speakers

Libor Běhounek (University of Ostrava)
Chris Fermüller (Vienna University of Technology)
Elías Fuentes Guillén (Czech Academy of Sciences)
Vít Punčochář (Czech Academy of Sciences)
Šárka Stejskalová (Charles University)

Contributed talks

Czech Gathering of Logicians is an annual regional event that brings together researchers in all areas of logic. We encourage researchers working in a field relevant to the conference to submit an abstract of 1-2 pages in pdf format, including references. New and recent research work is welcome for presentation. All submissions will be evaluated by the programme committee. Accepted submissions will be presented at the meeting in 30 minute talks including discussion. The conference language is English, both for submission and for presentation.

Submission deadline: April 30 2022, AoE.
Please submit by email to: clog2022@cs.cas.cz.
We will confirm receipt with each submission.
Notification of acceptance: May 20, 2022.

Programme committee

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Organized by Institute of Computer Science and
Institute of Information Theory and Automation
of the Czech Academy of Sciences

Conference series

This is the 9th Gathering of Logicians. Earlier instalments were organized by the

- Institute of Computer Science, CAS (2013),
- Institute of Philosophy, CAS (2014),
- Institute of Computer Science, CAS (2015),
- Institute of Mathematics, CAS (2016),
- Institute of Philosophy, CAS (2017),
- Institute of Computer Science, CAS (2018),
- Institute of Computer Science, CAS (2019), and
- Masarykova univerzita (2021).

There was no Gathering in 2020.

Programme

Czech Gathering of Logicians 2022

Ládví Academy Campus, June 16 & 17, 2022

Thursday, June 16

9:00	*registration*	
9:30	*opening*	
9:45–10:45	Chris Fermüller	From semantic games to analytic calculi
10:45–11:00	*coffee break*	
11:00–11:30	Jamie Wannenburg	One-sorted program algebras
11:30–12:00	Zuzana Rybaříková	Specification of tenses in Tichý's Transparent Intensional Logic and Prior's Temporal Logic
12:00–12:30	Jiří Raclavský	Derivability of rules of β-conversion in partial type theory
14:00–15:00	Šárka Stejskalová	Compactness principles for uncountable trees
15:00–15:30	Pavel Arazim	The limits of expressing logic according to both early and later Wittgenstein
15:30–15:45	*coffee break*	
15:45–16:15	Kentarô Yamamoto	The small index property of the Fraïssé limit of finite Heyting algebras
16:15–16:45	Adam Bartoš	A category-theoretic language for metric Fraïssé theory
16:45–17:15	Wieslaw Kubiś	Abstract evolution systems, homogeneity and termination
18:00–21:00	Evening in the Library	of Petr Hájek & banquet

All talks take place in the *Institute of Information Theory and Automation* (ÚTIA) building, big lecture room on the ground floor, facing the main entrance. The *Evening in the Library of Petr Hájek* takes place in the *Institute of Computer Science* building (ÚI), first floor.

Programme

Czech Gathering of Logicians 2022

Ládví Academy Campus, June 16 & 17, 2022

Friday, June 17

9:30–10:30	Libor Běhounek	A Vopěnka-style principle for fuzzy mathematics
10:30–11:00	Carles Noguera	The general algebraic framework for Mathematical Fuzzy Logic
11:00–11:15	*coffee break*	
11:15–12:15	Elías Fuentes Guillén	A hitherto unknown text by Bolzano on his *Beyträge*
12:15–12:45	Kateřina Trlifajová	Sizes of countable sets
14:00–15:00	Vít Punčochář	First-order logic of questions
15:00–15:15	*coffee break*	
15:15–15:45	Ludovica Conti	Arbitrary abstraction and logicality
15:45–16:15	Tadeusz Litak	Describable nuclei, subframe logics and negative translations
16:15–16:45	Emil Jeřábek	Basic analytic functions in VTC^0
16:45	*closing*	

All talks take place in the *Institute of Information Theory and Automation* (ÚTIA) building, big lecture room on the ground floor, facing the main entrance.

Programme

Evening in the Library of Petr Hájek

Institute of Computer Science, June 16, 2022, 6 p.m.

18:00	–	18:05	Petr Cintula	Opening of the Evening
18:05	–	18:10	Zuzana Haniková	Highlights of Petr Hájek Library
18:10	–	18:20	Vítězslav Švejdar	Valuables in Petr Hájek archive
18:20	–	18:25	Tereza Šírová	Introduction to Petr Hájek Library
18:25	–	21:00	*Banquet*	
18:30		onwards	*Library excursions*	

The Evening takes place in the *Institute of Computer Science* building, first floor (from lift: right; left; straight). If not an ICS member, please consider signing your name in the attendance list downstairs. Thank you.

Invited Abstracts

From Semantic Games to Analytic Calculi

Chris Fermüller
Technische Universität Wien

It is well known that Tarski's notion of truth in a model can be characterized in terms of a game between a *Verifier* (or *Proponent*) and a *Falsifier* (or *Opponent*). Jaakko Hintikka referred to this fact as *game theoretic semantics* and proposed a generalization of the semantic game for classical first-order logic featuring imperfect information, leading to Independence Friendly (IF) logic. Here, however, we will look at another application of semantics games. Starting with the simplest semantic game, namely that for propositional classical logic, we will show how a systemic search for winning strategies for *Proponent* corresponds to Gentzen's sequent calculus for classical logic if we abstract away from concrete models. The central ingredient in this transformation of games into proof systems consists in lifting individual states of the semantic game to so-called disjunctive states, where all possible moves of *Proponent* are recorded. It turns out that classical sequents can be viewed as representations of disjunctive states and that Gentzen's logical rules directly correspond to Hintikka's rules for the semantic game.

The interest in the indicated method of lifting states to disjunctive states and then mapping those disjunctive states into objects of inference in an appropriate calculus, lies in its flexibility. Starting, e.g., with many-valued models or with Kripke models for modal logics, one may turn corresponding semantic games into various forms of analytic calculi. The term 'analytic' can be read in two (related) ways, here. The resulting calculi are analytic in the sense of avoiding the cut-rule. But they are also analytic in the wider sense of relating inference rules to the intended semantics in a systematic manner. The paradigmatic case for this method – revisited in some details in this presentation – is that of propositional Łukasiewiz logic, where one transforms Giles's semantic game into a corresponding hypersequent calculus.

The purpose of this talk is not to present new results or technical details that are needed to prove corresponding adequateness theorems, but rather to provide a gentle introduction, guided by examples, that is intended to make the overall methodology accessible without appealing to specific background knowledge. It also allows us to advertise an ongoing research project.

Compactness principles for uncountable trees

Šárka Stejskalová

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We will discuss some results related to compactness principles at small infinite cardinals which extend the usual compactness of first-order logic. We will primarily focus on compactness principles related to trees, in particular as regards generalizations of König's lemma (that narrow trees of height ω have always a cofinal branch).

In the first part of the talk, we define basic concepts to make the lecture accessible also for non-specialists in set theory. We review the most common compactness principles and their basic properties. In the second part of the talk, we survey recent results in this area and state some open questions.

A Vopěnka-style principle for fuzzy mathematics

Libor Běhounek
University of Ostrava

The indistinguishability of objects can be mathematically modeled in many different ways—e.g., using classical equivalence or proximity relations, via metrics or topology, using rough sets, or using fuzzy similarity relations. Modeling indistinguishability by means of fuzzy similarity relations (or fuzzy equivalences, [1]) is elegant in that it provides a solution to Poincaré's paradox [2]—namely, the contradiction consisting in the fact that indistinguishability should intuitively be transitive, yet in a sufficiently long series whose every two neighboring elements are mutually indistinguishable, the extremal elements may be distinguished: fuzzy equivalences allow the latter, but are still transitive in the sense of fuzzy logic. Fuzzy equivalences happen to be dual to (generalized) metrics, so many metric and topological notions carry over to fuzzy equivalences.

Vopěnka's Alternative Set Theory (AST, [3]) has its own intriguing model of indistinguishability, construed as a non-standard equivalence relation arising by discrimination via progressively sharpened perspectives towards the horizon. Here, however, I will draw on another fundamental idea of Vopěnka's AST, namely his characterization of finite sets in terms of the surveyability and clear discernibility of all of their elements by the limited human means (even if idealized). One way of interpreting Vopěnka's principle of infinity-as-indiscernibility is that in any infinite set, some elements are inevitably indistinguishable from each other.

If we abstract away from the specifics of AST and apply the latter principle to the model of indistinguishability in fuzzy mathematics, it amounts to the requirement of (metric) precompactness, or the total boundedness of the generalized metric dual to the fuzzy indistinguishability relation. This requirement can be easily expressed by means of formal fuzzy logic and investigated by the methods of formal fuzzy mathematics. In the talk, I will show some consequences of this principle, such as the existence of fuzzy minima in fuzzy orderings compatible with a precompact fuzzy equivalence relation, and the use of this general fact in the recently proposed fuzzy semantics of counterfactual conditionals [4].

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A hitherto unknown text by Bolzano on his *Beyträge*

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In 1810, Bolzano, who at the time taught "religious doctrine" at the University of Prague, published his *Beyträge zu einer begründeteren Darstellung der Mathematik* (*Contributions to a Better-Grounded Presentation of Mathematics*). This work consisted of two parts (one on the concept of mathematics and its division; another one that provided an attempt at a new logic) and an appendix (on the Kantian theory of the construction of concepts), and it was planned to be the first in a series of *contributions* in which, beginning from its foundations, he planned to address all theoretical disciplines of mathematics. Bolzano in fact worked on this project for some years but, in view of the few and "superficial" reviews of its first instalment, in 1817, in the preface to his *Rein analytischer Beweis* (*Purely Analytic Proof*), he announced that he had decided to postpone the publication of any subsequent *contributions*. However, he never published any further instalments of such a project and, not being entirely satisfied with the logic included in his *Beyträge*, on the importance of which he insisted in two of his works published in the mid-1810s, he began to work on what eventually became his *Wissenschaftslehre* (*Theory of Science*), published in 1837.

In this talk I will present a short text written by Bolzano in 1810-11 on his *Beyträge* that was hitherto unknown to his scholars and which I recently discovered. This text was published anonymously in 1811 and a draft of it is preserved at the Literární archiv Památníku národního písemnictví (LA PNP), which, however, was assumed to be Bolzano's transcription of a review that would have been published at the time but the provenance of which was unknown. I will explain the history of this text and the evidence that allowed me to elucidate its authorship. But, in addition to this, I will discuss the *Beyträge* in the light of two reviews of it published in 1810-11 (Bolzano's transcriptions of which are held at the LA PNP) and of the hitherto unknown text on it, which provides an unusual insight into what we must take Bolzano himself to have considered most noteworthy about his 1810 work, namely his "outline of a new logic". By contrast, while over the years the general aim of this work to improve the 'grounds' of mathematics has been praised, the part on logic has usually been considered, as his contemporaries did, as non-innovative and even deficient. And yet, as Bolzano's 1811 text points out and as I will show, this part did contain some "new and fruitful views".

First-Order Logic of Questions

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In my talk I will present an overview of a research area known as inquisitive semantics (Ciardelli, 2016; Ciardelli et al., 2019; Grilletti, 2020; Punčochář, 2016, 2019). I will explain philosophical motivations behind this framework and its basic mathematical features. The focus will be mainly on the first-order version of the theory. The main results and open questions in this area will be presented.

Inquisitive semantics is a framework that allows us to represent questions and statements in a uniform way. This would not be possible in the standard semantics based on the notion of truth. Unlike statements, questions are not true or false. For this reason inquisitive semantics employs an "information-based" semantics in which logical connectives are not characterized by truth conditions but rather in terms of informational support. While truth can be captured as a relation between first-order models and formulas, support is a relation between information states and formulas. An information state can be intuitively conceived of as a (typically incomplete) representation of a structure. Formally, it can be defined as a set of first-order models—those models that are compatible with the representation. (We restrict ourselves to the cases where the models forming an information states share a common domain.) Hence, support is formally defined as a relation between sets of models and formulas (with respect to an evaluation of variables).

Let us define the basic framework more precisely. For the sake of simplicity, we just consider a standard first-order language without functional symbols and identity. As the basic logical symbols we can take $\bot, \wedge, \rightarrow, \forall$. The symbols $\neg, \vee$ and $\exists$ can be defined in terms of the basic symbols in the usual way.

Let s be a set of first-order models with a common domain U. An evaluation in s is a function that assigns to every variable an element from U. If e is an evaluation, x a variable and o an element from U, then $e(o/x)$ will denote, as expected, the evaluation that assigns o to x and $e(y)$ to every other variable y. The relation of support is then defined as follows:

$s \vDash_e Pt_1, \dots, t_n$ iff for all $\mathcal{M} \in s$, $Pt_1, \dots, t_n$ is true in $\mathcal{M}$ w.r.t. e,

$s \vDash_e \bot$ iff s is empty,

$s \vDash_e \varphi \wedge \psi$ iff $s \vDash_e \varphi$ and $s \vDash_e \psi$,

$s \vDash_e \varphi \rightarrow \psi$ iff for every $t \subseteq s$, if $t \vDash_e \varphi$, then $t \vDash_e \psi$,

$s \vDash_e \forall x \varphi$ iff for every $o \in U$, $s \vDash_{e(o/x)} \varphi$.

It can be easily shown that for every formula φ of this language, φ is supported by a state s (w.r.t. e) iff φ is true in every model of s (w.r.t. e) in the sense of the standard semantics for classical logic. As a consequence, for the basic language, the logic determined by this semantics based on support conditions coincides with classical first-order logic. However, the merit of this setting is that it allows us to extend the language with questions and equip them with a suitable semantics. Questions are introduced into the language via two question-forming operators: inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and inquisitive existential quantifier $\exists\!\!\!\exists$. For example,

- $Pa \mathbin{\vee\mkern-13mu\vee} Qa$ represents the question *whether a has the property P or the property Q*,
- $\exists\mkern-11mu\exists xPx$ represents the question that asks *what is an object that has the property P*.

Note that while it does not make sense to ask whether a question is true in a structure, it makes a perfect sense to ask whether a question is resolved by an information state. The semantic support clauses for questions specify under what conditions are questions resolved:

- $s \vDash_e \varphi \mathbin{\vee\mkern-13mu\vee} \psi$ iff $s \vDash_e \varphi$ or $s \vDash_e \psi$,
- $s \vDash_e \exists\mkern-11mu\exists x\varphi$ iff for some $o \in U$, $s \vDash_{e(o/x)} \varphi$.

This looks like the usual clauses for disjunction and existential quantifier but note that the defined symbols $\vee$ and $\exists$ behave differently in the information-based semantics. For example, the difference between $\exists\mkern-11mu\exists$ and $\exists$ is illustrated when we spell out the semantic clauses:

- $s \vDash_e \exists\mkern-11mu\exists xPx$ iff there is $o \in U$ s.t. for every $\mathcal{M} \in s$, Px is true in $\mathcal{M}$ w.r.t. $e(o/x)$,
- $s \vDash_e \exists xPx$ iff for every $\mathcal{M} \in s$ there is $o \in U$ s.t. Px is true in $\mathcal{M}$ w.r.t. $e(o/x)$.

We can define first-order inquisitive logic as the set of formulas that are supported by every information state. Despite some serious effort to resolve this problem, it is still an open question whether inquisitive logic is recursively axiomatizable. It is also not known whether the related consequence relation is compact. In my talk these central problems of inquisitive semantics will be discussed together with some positive results obtained in the area (for example completeness results for various fragments of the language). I will also present an algebraic approach to these issues based on (Punčochář, 2021) that I believe might be helpful in the solution of the main problems.

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Contributed Abstracts

One-sorted Program Algebras*

Igor Sedlár[1] and Johann J. Wannenburg[1]

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Kleene algebra with tests [6], KAT, is a simple algebraic framework for verifying properties of propositional while programs. KAT subsumes Propositional Hoare logic (PHL) [7] and it has been applied in a number of verification tasks. KAT is PSPACE-complete [2], has computationally attractive fragments [9], and its extensions have been applied beyond while programs, for instance in network programming languages [1].

KAT is two-sorted, featuring a Boolean algebra of tests embedded into a Kleene algebra of programs. For various reasons, a one-sorted alternative to KAT may be desirable. For instance, "one-sorted domain semirings are easier to formalise in interactive proof assistants and apply in program verification and correctness" [4, p. 576]. A one-sorted alternative called *Kleene algebra with antidomain* was introduced in [3]. The idea of KAA is to expand Kleene algebra with an *antidomain* operator a, such that $d(x) = a(a(x))$ is a *domain* operation, where the set of images of elements of the algebra under d forms a Boolean algebra in which the complement of $d(x)$ is $a(x)$. Hence, one obtains a Boolean algebra of tests in a one-sorted setting. Consequently, the equational theory of KAT embeds into the equational theory of KAA.

It is known that KAA is decidable in EXPTIME [8], and KAA can be used to create modal operators that invert the sequential composition rule of PHL. Such inversions are derivable from KAA but not KAT [10]. However, KAA has certain features that may be undesirable depending on the application. First, if $\mathbf{K}$ is a KAA, $d(\mathbf{K})$ is necessarily the maximal Boolean subalgebra of the negative cone of $\mathbf{K}$; see Thm. 8.5 in [3]. In a sense, then, every "proposition" is considered a test, contrary to some of the intuitions expressed in [6]. These intuitions also collide with the approach of taking KAT as KA with a Boolean negative cone [4, 5]. Second, not every Kleene algebra expands to a KAA, not even every finite

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one; see Prop. 5.3 in [3]. This is in contrast to the fact that every Kleene algebra expands to a KAT.

In this talk we generalize KAA to a framework we'll call *one-sorted Kleene algebra with tests*, KAt. We start by assuming equations that essentially state nothing more than that each KAt has a Boolean subalgebra in the negative cone. Already in this case KAt has most of the desired features of KAA: every KAt contains a Boolean subalgebra of tests and the equational theory of KAT embeds into the equational theory of KAt. In addition, every Kleene algebra expands into a KAt (ensuring that it is a conservative expansion), and the subalgebra of tests in KAt is not necessarily the maximal Boolean subalgebra of the negative cone. We then consider various extensions of KAt with axioms known from KAA to show which properties of the domain operator are still consistent with the desired features of KAt. In addition, we consider a variant of the KAt framework where test complementation is defined using a residual of Kleene algebra multiplication.

References

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Specification of Tenses in Tichý's Transparent Intensional Logic and Prior's Temporal Logic

Zuzana Rybaříková

Department of Philosophy, University of Ostrava

In his paper 'The Logic of Temporal Discourse', Pavel Tichý (1980) pointed out that contemporary systems of logic were unable to sufficiently formalise temporal discourse. He therefore suggested temporal specification in Transparent Intensional Logic (TIL), a system of logic that he developed. Discussing contemporary systems of logic, Tichý also took into account the system of Arthur N. Prior, who is considered a founding father of modern temporal logic, and his criticism was also addressed to Prior. Tichý only focused, however, on Prior's early systems of temporal logic. Patrick Blackburn (2006) recently raised awareness that Prior also developed systems of hybrid logic in his latest periods (see e.g. Prior 2003a; Prior 2003b). From the point of view of temporal specification, this system is particularly interesting as the system has greater expressive power than Prior's early systems of temporal logic. Hence it could also deal with the problematic specifications of tenses that Pavel Tichý pointed out (see Blackburn and Jørgensen 2016). The aim of my talk is to demonstrate that the temporal propositions that Tichý introduced as problematic could be formalised in Prior's hybrid logic. I will also compare formalisations in TIL and hybrid logic and Tichý's and Prior's views that influenced their systems of logic.

The challenging features of temporal discourse that Tichý pointed out are, for instance, the difference between Past Simple and Present Perfect. Let us imagine we have a friend in common whose name is Nick. Nick was happy on Christmas Eve in 2021 and has been happy ever since. We could say:

1. Nick was happy on Christmas Eve in 2021.

and

2. Nick has been happy since Christmas Eve in 2021.

Tichý proposed a formalisation of these two propositions as:

1. $\lambda w\ \lambda t\ P_t\ [\mathbf{Onc}_w\ \lambda w\ \lambda t H_{wt} X]\ \lambda t.t = T^0$[1]

respectively

2. $\lambda w\ \lambda t\ Pf_t\ [\mathbf{Thr}_w\ \lambda w\ \lambda t H_{wt} X]\ \lambda t.\mathbf{Aft}\ t = T^0$[2]

Although Prior did not discuss the difference between Past Simple and Present Perfect in his system of hybrid logic, these propositions could also be formalised in it, namely, as:

1. $P(a \wedge p)$[3]

and

2. $P(a \wedge p) \wedge \forall b[TaFb \rightarrow (b \wedge p)]$[4]

Although Prior's formalisation is by no means as detailed as Tichý's, it is able to grasp the basic difference between these two tenses. There are, however, more challenging temporal specifications that could be compared.

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Prior A. N. (2003a). Tense Logic and the Logic of Earlier and Later. In Prior A. N., *Papers on Time and Tense*. Hasle, P., Øhrstrøm, P., Braüner, T., & Copeland, J. (eds.), Oxford University Press, 117-138.

[1] *Gloss*. In any possible world w and time t ($\lambda w\ \lambda t$) it was the case (Pt) once ($\mathbf{Onc}_w$) that in any possible world w and time t Nick (X) is happy in this possible world and in this time (H_{wt}) which is Christmas Eve in 2021 ($\lambda t.t = T^0$).

[2] *Gloss*. In any possible world w and time t ($\lambda w\ \lambda t$) it has been the case (Pf) that throughout ($\mathbf{Thr}_w$) in any possible world w and time t Nick (X) is happy in this possible world and in this time (Hwt) which is any time after Christmas Eve in 2021 ($\lambda t.\mathbf{Aft}\ t = T^0$).

[3] *Gloss*. It has been the case (P) on Christmas Eve in 2021 (a) that Nick is happy (p).

[4] Gloss. It has been the case (P) on Christmas Eve in 2021 (a) that Nick is happy (p) and for every instant 'b', if the instant 'b' is later than Christmas Eve in 2021 (TaFb), it is the case at 'b' that Nick is happy at the instant ($b \wedge p$).

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Derivability of rules of β-conversion in partial type theory

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For motivation consider a familiar mathematical problem: *find the domain of the function*

$$f(x) = \frac{3}{x^2 - 3x + 2}$$

f's domain – i.e. the set of real numbers for which f is defined – contains any number n if and only if the result of substituting n's name for x in $\frac{3}{x^2-3x+2}$ is a *valid expression*, i.e. a name of some number m; m is then the value of f at n. Since no fraction with zero denominator represents a number, and the original expression is equivalent to $\frac{3}{(x-1)\times(x-2)}$, it can be readily seen that f's domain contains all numbers except 1 and 2, i.e. $\mathbb{R}\backslash\{1;2\}$.

When solving the above problem, one in fact employs the pivotal rules of *type theory* (*TT*) (i.e. a *higher-order logic* with a hierarchy of functions sorted in interpretations (sets) $\mathscr{D}_\tau$ of *types* τ) namely the *rules of* β*-conversion* (i.e. β*-contraction*: $\vdash$; β*-expansion*: $\dashv$):

$$[\lambda \tilde{x}_m.C](\bar{D}_m) \dashv\vdash C_{(\bar{D}_m/\bar{x}_m)}$$

where $\tilde{X}_m$ is short for $X_1X_2...X_m$; $\bar{X}_m$ is short for $X_1, X_2, ..., X_m$; but $C_{(\bar{D}_m/\bar{x}_m)}$ is short for $C_{(D_1/x_1)...(D_m/x_m)}$, where $C_{(D/x)}$ is the result of substituting D for all free occurrences of x in C (interpreted in our approach as $[\![Sub(\ulcorner D \urcorner, \ulcorner x \urcorner, \ulcorner C \urcorner)]\!]^{\mathscr{M},v}$, where v is an *assignment*, $\mathscr{M}$ is a *model* that is built, inter alia, from a frame $\mathscr{F} = \{\mathscr{D}_\tau \mid \tau \in \mathscr{T}\}$, where $\mathscr{T}$ is the set of all relevant types; $\ulcorner X \urcorner$ presents X as such, not X's value).

However, within *partial TT*, i.e. a TT that embraces both total and partial functions,[1] the above classical formulation of β-contraction is not valid. For example,

$$[\lambda x.\lambda y. \div (x,x)](\div(3,0)) \not\Rightarrow_\beta \lambda y. \div (\div(3,0), \div(3,0)),$$

for $[\lambda x.\lambda y. \div (x,x)](\div(3,0))$ is *non-denoting* (because $D := \div(3,0)$ is non-denoting), but $\lambda y. \div (\div(3,0), \div(3,0))$ denotes a certain partial function. This is why Tichý 1982, Moggi 1988, Farmer 1990, Feferman 1995, Beesson 2004 and others conditioned the rule by requiring that D entering β-reduction must be *denoting*.

[1] A *total/partial function[-as-graph]* maps all/some-but-not-all members of its domain $\mathscr{D}$ to some members of its range $\mathscr{D}'$. Note that such functions differ from *functions-as-computations*.

In Tichý's 1982 convenient 'two-dimensional' *natural deduction ND* for his *simple TT* (STT) with total and partial (multiargument) functions, his safe β-contraction rule *by-name* reads[2]

$$(\beta\text{−CON}) \qquad [\lambda\tilde{x}_m.C](\bar{D}_m){:}\mathbf{a} \vdash C_{(\bar{D}_m/\bar{x}_m)}{:}\mathbf{a}$$

in which terms C are 'signed' by :$\mathbf{a}$, which is a terse variant of $\cong \mathbf{a}$, where $\cong$ is a symbol of *congruence*; $\mathbf{a}$ is either a variable a or a constant A or an acquisition $\ulcorner A \urcorner$. This requires here the whole β-redex, i.e. the application written on the left-hand side of $\vdash$, and thus also its parts being *denoting*.

However, Tichý's proposal is too restrictive. For example,

$$[\lambda x. \div (x, 0)](3) \Rightarrow_\beta (\div(3, 0))$$

is not handled by (β-CON). To capture also such examples we propose the '*negative*' variant of the above '*positive*' rule (β-CON) (i.e. (β-CON$^+$)),

$$(\beta\text{−CON}^-)\ \Gamma \longrightarrow [\lambda\tilde{x}_m.C](\bar{D}_m){:}\_; \Gamma \longrightarrow D_1{:}\mathbf{x}_1; ...; \Gamma \longrightarrow D_m{:}\mathbf{x}_m \vdash \Gamma \longrightarrow C_{(\bar{D}_m/\bar{x}_m)}{:}\_$$

where each $D_i{:}x_i$ says that D_i is denoting an object in the range of x_i, and $X{:}\_$ represents that X is *non-denoting* ($\_$ stands for any type-theoretically appropriate non-denoting term).

Our further main contribution (see reference below) is a derivation of rules of β-conversion *by-value*, both in 'positive' and 'negative' variants, from the primitive rules (e.g. (β-CON)) of the natural deduction ND for partial TT. Notation $\pm$ covers both +- and −-variants and V indicates that one substitutes the *value* of D, which is directly 'named' by $\mathbf{d}$ – while it is $\mathbf{d}$ (not D) what is substituted for x through C, cf. $C_{(\bar{\mathbf{d}}_m/\bar{x}_m)}$:

$$(\beta\text{−CON}^{V\pm})\ \Gamma \longrightarrow [\lambda\tilde{x}_m.C](\bar{D}_m){:}\underline{\mathbf{a}}, \Gamma \longrightarrow D_1{:}\mathbf{d}_1; ...; \Gamma \longrightarrow D_m{:}\mathbf{d}_m \vdash \Gamma \longrightarrow C_{(\bar{\mathbf{d}}_m/\bar{x}_m)}{:}\underline{\mathbf{a}}$$

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[2]In this abstract, we omit β-expansion rule(s). Further, let C etc. be *typed* by types $\tau_{(i)}$ as follows: $C, \mathbf{a}/\tau; D_1, x_1/\tau_1; ...; D_m, x_m/\tau_m$, so $\lambda\tilde{x}_m.C/\langle\bar{\tau}_m\rangle{\rightarrow}\tau$ (the type of functions from $\mathscr{D}_{\tau_1} \times ... \times \mathscr{D}_{\tau_m}$ to $\mathscr{D}_\tau$), where $\bar{\tau}_m$ is short for $\tau_1, \tau_2, ..., \tau_m$.

The limits of expressing logic according to both early and later Wittgenstein

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In his Tractatus, Wittgenstein dedicates some of the most fascinating, yet also most enigmatic passages to the sphere of the mystical. One of the characteristics of this sphere is supposed to be its ineffability. Any attempts to describe it force us to maim the expressive powers of the language we use. Surprisingly enough, Wittgenstein treats logic in a very similar way in Tractatus. Logic, then, can only be shown, not expressed. Or, to be more precise, logic can only show itself. This view is sometimes, for instance by Stekeler-Weithofer, seen as refuted by the later development of logic, particularly by the development of the plurality of non-classical logics which purport to study various kinds of reasoning. I will present a perspective from which Wittgenstein is right even in face of the rich plurality of logical systems.

Besides being ineffable, the mystical, as well as logic, is also supposed to be fundamental, in fact much more important than what lies outside it. Therefore, logic also deserves this honourable status, according to Wittgenstein. Nevertheless, logicians today purport to be making explicit all kinds of logical laws which hold in variegated areas, which causes the unprecedented plurality of logics. On the other hand, it is not clear what the import of all this intellectual work is. Is there a lesson to be learned from Wittgenstein for the contemporary philosophy of logic? In order to access this possible lesson, we have to pay attention not only to early Wittgenstein but also to his later development where the notion of game and language game became prominent. I will show that taking seriously Wittgenstein´s motivation - which originates in his discussions with Moritz Schlick and his conception of games - to treat our linguistic activities as games, which are partly playful and unserious, shows us the limits of formal logical systems. They are language games themselves but do not understand themselves properly which causes them to be unsatisfying and turns the plurality of logics into a curse rather than into a blessing, getting us close to the positions of logical nihilists, such as G. Russell, rather than to those of logical pluralists.

A further perspective from which logical systems fail to properly describe laws of reasoning is provided by Wittgenstein in his On Certainty. Nevertheless, just as in Philosophical investigations, he does not address logic directly and therefore his argument must be extracted from his writing in a non-trivial way. He sees particularly certain sentences as fundamental for the working of our language games and vice versa. Certainty therefore equals fundamentality. Yet precisely by enabling the language games, these sentences cannot properly enter these language games. Using these sentences in the context of any specific conversation fails to convey their real meaning. When I am looking at my hands and try to formulate the skepticist question whether these are indeed my hands, then, according to Wittgenstein, my interlocutor would typically doubt whether I understand the meaning of the word *hand*. If we apply this reasoning to logic, this would mean that logical laws cannot be expressed. A lot might have changed in Wittgenstein's transition from Tractatus to his later thought. Nevertheless, my argumentation suggests that his view of formal logic as a scientific discipline has not changed very much. And the reasons for Wittgenstein's position are still of interest.

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The small index property of the Fraïssé limit of finite Heyting algebras

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Consider a countable class of isomorphism types of finitely generated structures with the amalgamation property, the joint embedding property, and the hereditary property. By the *Fraïssé limit* of such a class $\mathcal{K}$, we mean the unique countable structure M whose age, i.e., the class of finitely generated substructures of M, is $\mathcal{K}$ up to isomorphism.

One pervasively studied aspect of ultrahomogenous structures—the Fraïssé limit of some classes of structures—is their automorphism groups (see, e.g., Macpherson [3]). Many studies on the automorphism groups of concrete ultrahomogeneous structures involved uniformly locally finite ones, which are necessarily ω-categorical. (For instance, the simplicity of the automorphism group of the countable atomless Boolean algebra, which is ultrahomogeneous and uniformly locally finite, was established by Anderson [1].) The present author offered in an article under review a case study on the automorphism group of a natural non-uniformly locally ultrahomogeneous structure: the Fraïssé limit L of finite Heyting algebras, whose existence follows from Maksimova's result [4] on the Craig interpolation theorem for intuitionistic logic. One main result there was that $\mathrm{Aut}(L)$ was simple.

In the present work, we show yet another important property of $\mathrm{Aut}(L)$. We equip $\mathrm{Aut}(M)$ for an arbitrary countable structure M with the so-called pointwise convergence topology, which is the topology induced as a subset of the Baire space ${}^{\omega}\omega$ if the domain of L is ω. Under this topology, every open subgroup of $\mathrm{Aut}(M)$, which is now a topological group, has countable indices. With this in mind, a topological group G is said to have the *small index property* if, conversely, every subgroup of G with a countable index is open. The topology of $\mathrm{Aut}(M)$ with the small index property is, therefore, completely determined just from its abstract group structure.

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The small index property of $\mathrm{Aut}(M)$ has been shown for many ultrahomogeneous structures M. Examples relevant to the present conference include the countable atomless Boolean algebra [5] and the Fraïssé limit of finite distributive lattices [2]. By using the simplicity of $\mathrm{Aut}(L)$ and adapting Truss's argument, we obtain the following:

Theorem. The topological group $\mathrm{Aut}(L)$ has the small index property.

Following Truss, we prove this by studying the action of the automorphism group on the dual topological space of the structure. In our case, this space will be an Esakia space. Unlike the case of the countable atomless Boolean algebra, this action will not be transitive; the proof must take care of this appropriately.

Finally, we can further show the *strong* small index property, which has model-theoretic consequences on L.

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A category-theoretic language for metric Fraïssé theory

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(joint work with Wiesław Kubiś)

Classical Fraïssé theory [3, 7.1] studies countable (ultra)homogeneous first-order structures. Irwin and Solecki [4] introduced projective Fraïssé theory of topological structures, where an extension of a structure is a quotient from a larger structure instead of an embedding into a larger structure, as in the classical case. Both versions of the theory can be unified and can go beyond first-order structures – using the language of category theory, which captures the essence of structural constructions and abstracts from irrelevant details. The idea to use the language of category theory goes back to Droste and Göbel [1] [2], Pech and Pech [7], and Kubiś [6], who introduced the notion of a Fraïssé sequence.

The core of the discrete abstract Fraïssé theory can be summarized in the following theorems. The setup of the first theorem consists of a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ satisfying several conditions essentially saying that $\mathcal{L}$ arises by freely adding limits of sequences to $\mathcal{K}$. (Ultra)homogeneity and the extension property are the key desired properties of Fraïssé limits.

Theorem 1. Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be and $\mathcal{L}$-object. The following conditions are equivalent.

(i) U is homogeneous and cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$.

(ii) U has the extension property and is cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(iii) U is an $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

Such object U is unique up to isomorphism and is cofinal in $\mathcal{L}$. It is called the *Fraïssé limit* in $\langle \mathcal{K}, \mathcal{L} \rangle$.

Theorem 2. A category $\mathcal{K} \neq \emptyset$ has a Fraïssé sequence if and only if

(i) $\mathcal{K}$ is directed,

(ii) $\mathcal{K}$ has the amalgamation property,

(iii) $\mathcal{K}$ is dominated by a countable subcategory.

The first application of the projective Fraïssé theory by Irwin and Solecki was taking $\mathcal{K}$ to be the category of all connected finite linear graphs and quotient maps, and obtaining a pre-space of the pseudo-arc as a Fraïssé limit. This way the authors obtained a new characterization of the pseudo-arc, which looks like an approximate form of homogeneity. We extend the language of category theory by adding approximate equalities $f \approx_\varepsilon g$ of maps for $\varepsilon > 0$, turning every hom-set $\mathcal{L}(X, Y)$ into a metric space, and we view the pseudo-arc itself as a Fraïssé limit in this context. This was done by Kubiś [5] in the context of metric-enriched categories. We further extend the framework to so-called *MU-categories*, and as an application, we realize pseudo-solenoids directly as (approximate) Fraïssé limits in the category of circle-like continua and continuous surjections.

In the talk we review the discrete abstract Fraïssé theory, and continue by extending it to MU-categories.

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Abstract evolution systems*

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We introduce the concept of an abstract evolution system, which provides a convenient framework for studying generic mathematical structures and their properties. Roughly speaking, an *evolution system* is a category endowed with a selected class of morphisms called *transitions*, satisfying certain natural conditions. It can also be viewed as a generalization of abstract rewriting systems, where the partially ordered set is replaced by a category. In our setting, the *process* of rewriting plays a nontrivial role, whereas in rewriting systems only the result of a reduction/rewriting is relevant. An analogue of Newman's Lemma holds in our setting, although the proof is a bit more delicate, nevertheless, still based on Huet's idea using well founded induction.

Formally, an *evolution system* is a structure of the form $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object (called the *origin*) and $\mathscr{T}$ is a class of $\mathfrak{V}$-arrows (its elements are called *transitions*). An *evolutions* is a sequence of the form

$$\Theta \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition. The category $\mathfrak{V}$ serves as the universe of discourse. Given a $\mathfrak{V}$-object X, we denote $\mathscr{T}(X) = \{f \in \mathscr{T} \colon \operatorname{dom}(f) = X\}$, that is, the set of all transitions with domain X. Two transitions $f, g \in \mathscr{T}(X)$ are *isomorphic* if there is an isomorphism h in $\mathfrak{V}$ such that $g = h \circ f$. The system is *regular* if transitions commute with isomorphisms, that is, $f \circ h$ is a transition whenever f is a transition and h is an isomorphism. An object X will be called *finite* if there exist transitions $f_0, \dots, f_{n-1}$ such that $f_i \colon X_i \to X_{i+1}$ for $i < n$, $X_0 = \Theta$ and $X_n = X$. We say $\mathscr{E}$ has the *finite amalgamation property* if for every finite object C, for every transitions $f \colon C \to A$, $g \colon C \to B$ there are paths $f' \colon A \to D$, $g' \colon B \to D$ with $f' \circ f = g' \circ g$. An evolution system $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$ is *essentially countable* if for every finite object X there is a countable set of transitions $\mathscr{F}(X) \subseteq \mathscr{T}(X)$ such that every transition in $\mathscr{T}(X)$ is isomorphic to a transition in $\mathscr{F}(X)$.

Below are two natural motivating examples of evolution systems.

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Example 1. Let $\mathscr{F}$ be a class of finite structures in a fixed first-order language consisting of relations only. It is convenient to assume $\mathscr{F}$ is closed under isomorphisms. Let $\sigma\mathscr{F}$ denote the class of all structures of the form $\bigcup_{n\in\omega} X_n$, where $\{X_n\}_{n\in\omega}$ is a chain in $\mathscr{F}$. Let $\mathfrak{V}$ be the category of all embeddings between structures in $\sigma\mathscr{F}$. Let $\mathscr{T}$ consist of all embeddings of the form $f\colon X \to Y$, where $Y \setminus f[X]$ is a singleton or the empty set. In other words, transitions are one-point extensions and isomorphisms. Finally, Θ might be the empty structure. Clearly, $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$ is an evolution system.

Example 2. Let $\mathscr{F}$ be a fixed class of finite nonempty relational structures and consider it as a category where the arrows are epimorphisms. Define transitions to be epimorphisms $f\colon X \to Y$ such that either f is an isomorphism (a bijection) or else there is a unique $y \in Y$ with a nontrivial f-fiber and moreover $f^{-1}(y)$ consists of precisely two points. Define $\mathfrak{V}$ to be the opposite category, so that $f \in \mathfrak{V}$ is an arrow from Y to X if it is an epimorphism from X onto Y. Then $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, \Theta \rangle$ is an evolution system, where Θ is a prescribed finite structure in $\mathscr{F}$.

We say that an evolution $\vec{u}$ has the *absorption property* if for every $n \in \omega$, for every transition $t\colon U_n \to Y$ there are $m > n$ and a path $g\colon Y \to U_m$ such that $g \circ t = u_n^m$.

Theorem 3. *Assume $\mathscr{E}$ is an essentially countable evolution system that has the finite amalgamation property. Then there exists a unique, up to isomorphism, evolution with the absorption property.*

A system is *terminating* if every evolution is eventually trivial, namely, from some point on all transitions are isomorphisms. The following result is an extension of Newman's Lemma [3]; the proof is based on the idea of Huet [1], using well founded induction.

Theorem 4. *A locally confluent regular terminating evolution system is confluent.*

Confluent terminating systems provide a good framework for studying finite homogeneous structures.

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The general algebraic framework for Mathematical Fuzzy Logic

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Originating as an attempt to provide solid logical foundations for fuzzy set theory [19], and motivated also by philosophical and computational problems of vagueness and imprecision [16], Mathematical Fuzzy Logic (MFL) has become a significant subfield of mathematical logic [17]. Throughout the years many particular many-valued logics and families of logics have been proposed and investigated by MFL and numerous deep mathematical results have been proven about them (see the three volumes of handbook of MFL [5]). In the early years, there was, however, a great deal of repetition in the papers published on this topic; it was common to encounter articles that studied slightly different logics by repeating the same definitions and essentially obtaining the same results by means of analogous proofs. This unnecessary ballast was delaying the development of MFL while obscuring the reasons behind the main results. Therefore, MFL was an area of science screaming for systematization through the development and application of uniform, general, and abstract methods.

Abstract algebraic logic presented itself as the ideal toolbox to rely on; indeed, this general theory is applicable to all non-classical logics and provides an abstract insight into the fundamental (meta)logical properties at play. However, the existing works in that area (summarized in excellent monographs [2, 14, 15]) did not readily give the desired answers. Despite their many merits, these texts live at a level of abstraction a little too far detached from the intended field of application in MFL. They are indeed great sources of knowledge and inspiration, but there is still a lot of work to be done in order to bring the theory closer to the characteristic particularities of MFL, in particular in first-order logics.

These considerations led us, the authors of this contribution, to writing an extensive series of papers (e.g., [1, 3, 4, 6–8, 10–12, 18] to name the most important ones) in which we have developed various aspects of the general theory of MFL at different levels of generality and abstraction.

Our first attempt at systematizing this bulk of research was a chapter published in 2011 in the Handbook of Mathematical Fuzzy Logic [9] where we provided rudiments of a well rounded theory constituting solid foundations sufficient (and necessary!) for a rapid development of new particular fuzzy logics demanded by emerging applications. The goal of this talk is to summarize the subsequent 10 years of development and refinements of this theory and present its now matured state of the art as described in our recent monograph [13].

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Sizes of Countable Sets

Kateřina Trlifajová

Galieleo's paradox concerning the relation between collections of natural numbers and their squares may be the best illustration of the well-known fact (Mancosu 2009) that in comparing infinite collections one must choose one of two mutually exclusive principles:

1. *The Part-Whole Principle* (PW): "The whole is greater than its part."
2. *Cantor's Principle* (CP): "Two sets have the same size if and only if there is a one-to-one correspondence between their elements."

While Bolzano insisted on PW, according to Cantor, two sets have the same size if CP holds. Cantor's approach prevailed and is generally accepted as the only correct one. All infinite countable sets have one and the same size, namely $\aleph_0$, that is the cardinality of the set of natural numbers.

Bolzano's theory of infinite quantities preserving PW described primarily in *Paradoxes of the Infinite* is also meaningful and can be interpreted consistently in contemporary mathematics (Trlifajová 2018), (Bellomo & Massas 2021). Bolzano was aware of the existence of a one-to-one correspondence between some infinite multitudes, however, he writes: "Merely from this circumstance we can in no way conclude *that these multitudes are equal to one another if they are infinite* with respect to the plurality of their parts ... An equality of these multiplicities can only be concluded if some other reason is added, such as that both multitudes have exactly the same *determining ground*." (Bolzano 1851/2004, §21).

We introduce a theory of sizes of some countable sets based on Bolzano's ideas. The method is similar to that of Benci and Di Nasso's Numerosity Theory (NT) (Benci & Di Nasso 2003, 2019) but it differs in some substantial ways. Set sizes are determined constructively. They are unambiguous for they do not depend on the choice of an ultrafilter which is always partially arbitrary. Rules for determining are more rigorously justified, and so some results are more accurate. On the other side, sizes of countable sets are only partial and not linearly ordered. *Quid pro quo.*

Simultaneously, this is an answer to Matthew Parker, who argues in *Set Size and the Part-Whole Principle* (Parker 2013) that all Euclidean theories, i.e. theories satisfying PW, must be either very weak or arbitrary and misleading.

Canonically countable sets are those that can be arranged into mutually disjoint finite groups indexed by natural numbers according to its *determining ground*

$$A = \bigcup\{A_n, n \in \mathbb{N}\}.$$

Then a *size* of A is a sum of finite cardinalities $|A_n|$ expressed as a sequence of partial sums. We define a *size sequence* of A as the sequence $\sigma(A) = (\sigma_n(A))_{n\in\mathbb{N}}$ such that

$$\sigma_n(A) = |A_1| + \ldots |A_n|.$$

The problem is the exact meaning of the *determining ground.* In some cases a *canonical arrangement* is evident, in other cases we will define it so that the following rules are satisfied.

A canonical arrangement of natural numbers $\mathbb{N} = \bigcup\{A_n, n \in \mathbb{N}\}$ is

$$A_n = \{n\}.$$

Let A, B be two canonically arranged sets, $A = \bigcup\{A_n, n \in \mathbb{N}\}$ and $B = \bigcup\{B_n, n \in \mathbb{N}\}$. Then

$$A \subseteq B \Rightarrow (\forall n \in \mathbb{N})(A_n \subseteq B_n).$$

A canonical arrngement of the Cartesian product $A \times B$ is defined for all $n \in \mathbb{N}$

$$(A \times B)_n = \bigcup\{A_i \times B_j, n = \max\{i, j\}\}.$$

Now, we can determine size sequences of integers, rational numbers and their subsets. If two intervals of rational numbers of have the same length then have the same size as well.

Theorem 1. Let A, B be two canonically countable sets.

1. If A is finite then $\sigma(A) =_{\mathcal{F}} |A|$
2. If A is a proper subset of B, $A \subset B$, then $\sigma(A) <_{\mathcal{F}} \sigma(B)$.
3. The size sequence of the union is $\sigma(A \cup B) = \sigma(A) + \sigma(B) - \sigma(A \cap B)$.
4. The size sequence of the Cartesian product is $\sigma(A \times B) = \sigma(A) \cdot \sigma(B)$.

Theorem 2. Let S be the set of size sequences, i.e. the set of non-decreasing sequences of natural numbers. Let *addition* and *multiplication* be defined componentwise, *equality* and *order* are also defined componentwise but from a sufficiently great index, i.e. modulo Fréchet filter Then the structure $(S, +, \cdot, =_{\mathcal{F}}, <_{\mathcal{F}})$ is a partial ordered non-Archimedean commutative semiring.

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Arbitrary Abstraction and Logicality

Ludovica Conti

In this talk, I will discuss a criterion (*general weak invariance*) that has been recently suggested in order to argue for the logicality of abstraction operators, when they are understood as arbitrary expressions (cf. Boccuni Woods 2020).

Abstractionist theories are systems composed by a logical theory augmented with one or more abstraction principles (AP), of form: $f_R\alpha = f_R\beta \leftrightarrow R(\alpha, \beta)$ – that introduce, namely rule and implicitly define, the corresponding term-forming operators f_R. Thus, the logicality of these theories plainly depends on the logicality of the abstraction principles. This issue was originally raised into the seminal abstractionist program, Frege's Logicism – proposed with the foundational purpose to derive arithmetical laws as logical theorems and to define arithmetical expressions by logical terms. The inconsistency of this project (i.e. a theory equivalent to second-order logic augmented with Basic Law V) seemed to determine the inconsistency and, then (in a classical logic) the non-logicality of Basic Law V and – *a fortiori* – of any other abstraction principle[1].

Recently, the issue of the logicality has been resumed regarding the consistent abstraction principles, in order to clarify that conclusion in light of the intervening studies about logicality and represents, still today, an open question of the abstractionist debate. Briefly, a standard account of logicality has been provided, in semantical terms, by means of the Tarskian notions of invariance under permutation and isomorphism (cfr. [7]). In order to apply these criteria to abstraction principles, we can specify at least three different subjects to be examined: the whole abstraction principle [2], the abstraction

[1] We will describe a relation between the abstraction principles based on the *finesse* of their equivalence relations. Cfr. [1].

[2] Regarding the abstraction principle, the more informative criterion consists of *contextual invariance*: an abstraction principle AP is *contextually invariant* if and only if , for any abstraction function f_R: $D_2 \rightarrow D_1$ and permutation π, $\pi(f_R)$ satisfies AP whenever f_R does (cfr. [1]).

relation[3] and the abstraction operator[4]. Different results has been already proved (cf. [7], [6], [1], [4], [2], [5]) but a new dilemma appeared. More precisely, given a semantical definition of logicality as permutation and/or isomorphism invariance, we are able to prove that some abstraction principles (like Hume's Principle) are logical ([4])[6] but their implicit *definienda* are not ([1])[7] – so preventing a full achievement of Logicist goal.

My preliminary aim will consists in showing that this unfortunate situation closely depends on the (unjustified) adoption of a same notion of reference for all the expressions of a same syntactical category (e.g. singular terms as always referential and denoting singular, knowable and standard objects). On the contrary, a less demanding reading of the abstractionist vocabulary – namely, a reading that renounces to the semantical assumption mentioned above – is available; furthermore, such a reading, by admitting a different evaluation of primitive an defined expressions, is able to focus on the only information actually provided by the APs and turns out to be preferable because it is more faithful to the theory. Thus, chosen this reading of the APs and, particularly, an arbitrary interpretation (cf. [3]) of the abstractionist vocabulary, my main aim will consist in inquiring its consequences on the logicality of abstractionist theories.

Particularly, given such an interpretation of the APs, we can rephrase the main criterion of logicality for abstraction operators (*objectual invariance*, cf. [1]), obtaining a weaker one (*general objectual invariance*[8], GWI, cf. [8], [2]) and proving that it is satisfied not not only by cardinal operator but also by many other second-order ones, including those implicitly defined by consistent weakenings of Fregean Basic Law V. So, we will note that, given (what I

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[3]Regarding the abstraction relation, we can distinguish, at least, four kind of invariance: *weak invariance*, *double invariance*, *internal invariance* and *double weak invariance* (cfr. [1], [4], [6].).

[4]Regarding the canonical reading of the abstraction operator, logicality is usually spelled out in terms of *objectual invariance*[5] (cf. [1]).

[6]More precisely, some abstraction principles (like Hume's Principle) satisfy the criterion of *contextual invariance* and their abstraction relations (e.g. equinumerosity) satisfy many logicality criteria, like *weak invariance*, *internal invariance*, *double internal invariance*. Cf. [1], [6], [4].

[7]More precisely, the corresponding abstraction operators (e.g. cardinal operators) do not satisfy the criterion of *objectual invariance*. Furthermore, such criterion fails precisely in case of operators related to internal (and,*a fortiori* double internal) invariant relation (cfr. [1]). So, operators fail to be *logical* though – just in case – they are implicitly defined by *logical* AP.

[8]An expression ϕ is *generally weak invariant* just in case, for all domains D, D' and bijections ι from D to D', the set of candidate denotations of ϕ on D ($\phi^{*D}) = \{\gamma : \gamma$ is a candidate denotation for ϕ on $D\}$ is such that $\iota(\phi^{*D}) = \phi^{*D'} = \{\gamma : \gamma$ is a candidate denotation for ϕ on $D'\}$.

argued as) a preferable reading of the APs, both main strategies pursued in the last century to save Fregean project – Neologicism and consistent revisions of *Grundgesetze* – are able to achieve the desirable logicality objective. Further generalising, I will prove that the logicality criterion could be satisfied by a large range of APs and is apparently liable to a triviality objection – e.g. it is not able to distinguish between HP and some of its Bad Companions (like Nuisance Principle). I will answer to such a potential objection by showing that GWI however introduces interesting differences. More precisely, I will discuss the controversial case of Ordinal Abstraction and I will prove that GWI is not satisfied by any first-order abstraction principles (cf. [7], [8]). So, by comparing respective schemas of first-order and second-order APs[9], we will note that logicality (in the chosen meaning) mirrors a relevant distinction between same-order and different-order abstraction principles.

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[9]More precisely, a schematic second-order abstraction principle – of form $\S(RF) = \S(RG) \leftrightarrow R(F,G)$, where § is a binary abstraction operator and E any isomorphism invariant equivalence relation – defines an abstraction function from $\wp(\wp(D) \times \wp(D)) \times \wp(D) \to D$ that satisfies the criterion of GWI and – differently from the specific unary operators – is total ([8]). On the other side, a schematic first-order abstraction principle – of form $\S(Ra) = \S(Rb) \leftrightarrow R(a,b)$, where § is a binary abstraction operator and E any first-order equivalence relation – defines an abstraction function from $\wp(D \times D) \times D \to D$ that is – differently from the corresponding unary operators – total, but however does not satisfy GWI.

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Describable Nuclei, Subframe Logics and Negative Translations

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What do soundness/completeness of negative translations of intutionistic modal logics, extension stability of preservativity/provability logics and the use of nuclei on Heyting Algebra Expansions (HAEs) to form other HAEs have in common? As it turns out, in all those cases one deals with a certain kind of subframe property for a given logic, i.e., the question whether the logic in question survives a transition to at least some types of substructures of a specific (usually Kripke) semantics. The nucleic perspective on subframe logics has been introduced by Bezhanishvili and Ghilardi [2007] for the purely (super)intuitionistic syntax, *i.e.*, without modalities or other additional connectives. It has not been employed so much in the modal setting, mostly because the focus in the field tends to be on modal logics with classical propositional base, and nuclei are a rather trivial notion in the boolean setting. However, things are very different intuitionistically. Since the 1970's, nuclei have been studied in the context of point-free topologies (a.k.a. lattice-complete Heyting algebras), sheaves and Grothendieck topologies on toposes [Fourman and Scott, 1979], and finally arbitrary Heyting algebras [Macnab, 1981]. Other communities may know them as *lax modalities* [Fairtlough and Mendler, 1997] or *strong monads* (when algebras are understood as posets, and posets are understood as skeleton categories).

The presentation marries the nucleic view on subframe properties with the framework of *describable operations* introduced to study subframe logics in Wolter [1993]. Wolter's original setting was restricted to classical modal logics, but with minimal care his setup can be made to work intuitionistically and nuclei provide the missing ingredient to make it fruitful. From this perspective, we revisit our earlier syntactic studies of soundness and completeness of negative translations in modal logic [Litak et al., 2017] or *extension stability* for preservativity logics of Heyting Arithmetic (HA) based constructive strict implication $\prec\!\!\!\!\!3$ [Litak and Visser, 2022]. Various characterization and completeness results can be obtained in a generic way. Further applications in progress include, *e.g.*, a joint study with Georg Struth of a describable nucleus on (B)BI yielding the class of intuitionistic (affine) assertions of separation logic [Ishtiaq and O'Hearn, 2001, §9] or nucleic perspective on algebraic cut elimination and algebraic proof theory [Belardinelli et al., 2004].

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Basic analytic functions in VTC^0

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One of the basic themes in proof complexity is a loose correspondence between weak theories of arithmetic and computational complexity classes. If a theory T corresponds to a class C, it usually means that on the one hand, T can reason with C-concepts in the sense that it proves induction, comprehension, minimization, or similar schemata for formulas expressing predicates from C; on the other hand, provably total computable functions of T of suitable syntactic shape are C-functions. We may interpret this situation as a formalization of *feasible reasoning*. Here, we consider a natural concept X, and we ask what properties of X can be proved in an efficient manner while only using reasoning with concepts whose complexity does not exceed that of X itself; if C is a class that adequately describes the complexity of X, and T an arithmetical theory corresponding to C, we can approximate this form of feasible reasoning about X simply by provability in T. (This idea goes back to Parikh [9] and Cook [1].)

In this talk, we will be interested in feasible reasoning with the elementary integer arithmetic operations $+, \cdot, \leq$. Their computational complexity is captured by the class TC^0 (a small subclass of P): all the operations are computable in TC^0, and $\cdot$ is TC^0-complete under a suitable notion of reduction. Many other related functions are computable in TC^0 as well: iterated addition $\sum_{i<n} x_i$ and multiplication $\prod_{i<n} x_i$, division with remainder, the corresponding arithmetical operations in $\mathbb{Q}$, $\mathbb{Q}(i)$, number fields, or polynomial rings, and approximations of analytic functions such as log or sin defined by sufficiently nice power series. Here, the TC^0-computability of $\prod_{i<n} x_i$ and other above-mentioned functions that depend on it is a difficult result with a long history, finally settled by Hesse, Allender, and Barrington [2].

The theory of bounded arithmetic corresponding to TC^0 is the theory Δ^b_1-*CR* of Johannsen and Pollett [7], or equivalently (up to $RSUV$-isomorphism), the two-sorted theory VTC^0 introduced by Nguyen and Cook [8].

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This talk will showcase several exhibits of provability in VTC^0, based on [3, 4, 5, 6]:

- VTC^0 can do iterated multiplication by formalizing a variant of the algorithm from [2].
- VTC^0 proves induction for open formulas (*IOpen*), and even for translations of Σ^b_0 formulas of Buss, using a formalization of TC^0 root approximation algorithms for constant-degree polynomials.
- VTC^0 can formalize basic properties of approximations of elementary analytic functions (exp, log, trigonometric functions); in a more convenient setup, these functions can be defined on topological completions of models of VTC^0.

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