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On strong standard completeness in some MTL_Δ expansions

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Abstract In this paper, inspired by the previous work of Franco Mundici on infinitary axiomatizations for standard BL-algebras, we focus on a uniform approach to the following problem: given a left-continuous t-norm $*$, find an axiomatic system (possibly with infinitary rules) which is strongly complete with respect to the standard algebra $[\mathbf{0},\mathbf{1}]_*$. This system will be an expansion of MTL (Monoidal t-norm based logic). First, we introduce an infinitary axiomatic system L_*^∞, expanding the language with Δ and countably many truth-constants, and with only one infinitary inference rule, that is inspired in Takeuti-Titani density rule. Then we show that L_*^∞ is indeed strongly complete with respect to the standard algebra $[\mathbf{0},\mathbf{1}]_*$. Moreover, the approach is generalized to axiomatize expansions of these logics with additional operators whose intended semantics over $[0,1]$ satisfy some regularity conditions.

1 Introduction

By *t-norm based fuzzy logics* one usually refers to a broad class of residuated many-valued logics (usually presented in a Hilbert-style form) whose intended semantics are given by taking the real unit interval $[0,1]$ as set of truth-values, and using a (left-continuous) t-norm $*$ and its residuum $\Rightarrow_*$ to interpret the conjunction and implication connectives respectively. This family of logics can be cast under the umbrella of the so-called Mathematical Fuzzy logic (MFL), a subdiscipline of mathematical logic encompassing general classes of substructural logics whose intended semantics is typically based on algebras of linearly-ordered truth-values. Among distinguished t-norm based fuzzy logics one can find Łukasiewicz logic Ł, Gödel-Dumett logic G, Product logic Π, Hájek's Basic logic BL, and Monoidal t-norm logic MTL. See e.g. [7] for a deeper and up-to-date overview of the topic.

In fact, all these logics are algebraizable and can be presented as logics of classes of algebras (MV-algebras, Gödel algebras, Product algebras, BL-algebras, and MTL-algebras respectively). Namely, given a class K of algebras, the logic (as consequence relation) induced by K is defined as follows: for any subset of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_K \varphi \quad \text{iff} \quad \begin{array}{l}\text{for every algebra } \mathbf{A} \in K \\ \text{and every evaluation } e \text{ in } \mathbf{A}, \\ \text{if } e[\Gamma] \subseteq \{1\} \text{ then } e(\varphi) = 1.\end{array}$$

The logic $\models_K$ is called *semilinear* if $\models_K = \models_{chains(K)}$, where $chains(K)$ denotes the subclass of linearly ordered members of K.

Besides being algebraizable, and thus strongly complete with respect to their corresponding class (variety, in fact) of algebras, all the above mentioned logics are semilinear as well, and hence they also are strongly complete with respect to their corresponding classes of chains. A more interesting question is the *standard completeness* status of these logics, that is, completeness with respect to the so-called *standard semantics*, determined by their subclass of chains on the real unit interval $[0,1]$ (defined by left continuous t-norms and

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their residua), which is in fact their intended semantics. In particular, our focus in this work is on the logics of classes $K = \{[\mathbf{0},\mathbf{1}]_*\}$ of a single standard algebra $[\mathbf{0},\mathbf{1}]_*$ defined by a given left-continuous t-norm $*$ and its residuum $\Rightarrow_*$. In the case that K are families of standard BL-algebras, axiomatizations for the finitary companions of such logics can be found in [11, 14].

Three outstanding examples of these logics are the ones defined by the standard algebras $[\mathbf{0},\mathbf{1}]_*$ for $*$ being the min, Łukasiewicz and Product t-norms (we will denote them by $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$, $[\mathbf{0},\mathbf{1}]_{Ł}$ and $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ respectively). It turns out that, from the three above mentioned axiomatic systems (G, Ł, and Π), only Gödel logic G is strongly complete w.r.t. its standard semantics $[0,1]_G$, that is, $\vdash_{\mathsf{G}} = \models_G$, while Łukasiewicz logic Ł and Product logic Π are complete with respect to their standard algebras ($[\mathbf{0},\mathbf{1}]_{Ł}$ and $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ resp.) only for deductions from finite sets of premises, i.e. they are only finite strongly standard complete. In terms of traditional algebraic logic, we can rephrase these results by saying that the class of Gödel algebras coincides with the generalized quasi-variety generated by $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$, while the classes of MV-algebras and Product algebras do not coincide with the generalized quasi-varieties generated by $[\mathbf{0},\mathbf{1}]_{Ł}$ and $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ respectively.

In this paper, given a left-continuous t-norm $*$, we focus on devising a uniform approach to the problem of axiomatising the logic $\models_{[\mathbf{0},\mathbf{1}]_*}$. Since we have not succeeded using the usual propositional language of t-norm based logics, we turn our attention to (possibly infinitary) Hilbert-style axiomatic systems expanding MTL with suitable infinitary rules, with countably many truth-constants and with Monteiro-Baaz Δ connective. In other words, for a given left-continuous t-norm $*$, we aim at axiomatizing the generalized quasi-variety generated by the standard algebra $[\mathbf{0},\mathbf{1}]_*$ expanded with Δ and countably many truth-constants.

The paper is structured as follows. In Section 2 we present some preliminaries on left-continuous t-norm based logics necessary for the comprehension of the paper together with some previous related work. In Section 3 we characterize a large class of semilinear infinitary logics and prove a generalized version of [7, Th. 3.2.14]. Section 4 presents an (infinitary) axiomatic system strongly complete with respect to the logic arising from the standard algebra of an arbitrary left-continuous t-norm expanded with rational constants and the Monteiro-Baaz Δ connective. We do this by means of a unique infinitary inference rule denoted by *Density rule*. In Section 5 we present an example in order to show that some of the premises from the main result presented in Section 3 cannot be weakened. Along Section 6 we show how to rely on the previously introduced Density rule in order to axiomatize logics arising from an standard algebra as above, but further expanded with arbitrary operations that fulfil some regularity conditions. The paper finishes with a section detailing some open problems that have risen through this research.

2 Preliminaries and related work

The reader is expected to be familiar with the general terminology and basic facts and results about MFL and t-norm based logics. Further notions can be found in well-known monographs like [7, 13]. With the aim of being self-contained, we present in Sections 2.1 and 2.2 a brief refresher of the main notions and tools used in this paper. On the other hand, in Section 2.3 we go through previous results in the literature concerning the problem of finding strong complete axiomatizations for logics defined by t-norm based logics.

Before focusing on aspects of MFL more relevant to our setting, let us use $\mathbf{Fm}_L$ to denote for the algebra of formulas built over the language L. If the language is clear from the context, we will simply write $\mathbf{Fm}$.

2.1 T-norm based fuzzy logics

For a left-continuous t-norm $*$, we let $[\mathbf{0},\mathbf{1}]_*$ be the algebra $\langle[0,1], *, \Rightarrow_*, \wedge, 0\rangle$, and call it the *standard $*$-algebra*, where $\wedge$ stands for the minimum and $\Rightarrow_*$ is the residuum of $*$ (that always exists because $*$ is left-continuous):

$$x \Rightarrow_* y := \max\{z : z * x \leqslant y\}.$$

Each one of these algebras determines a unique logic over formulas in the language L with a countable set of variables, by considering the logical matrix $\langle[\mathbf{0},\mathbf{1}]_*, \{1\}\rangle$.

Following the notation introduced in the previous section for $K = \{[\mathbf{0},\mathbf{1}]_*\}$, we will write $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$ when φ follows from Γ in the above semantics, i.e., when for any evaluation e in $[\mathbf{0},\mathbf{1}]_*$, if $e[\Gamma] \subseteq \{1\}$ then $e(\varphi) = 1$.

Definition 2.1 **([10])** MTL *(Monoidal t-norm logic)* is the logic given by the Hilbert style calculus with *Modus Ponens* (i.e., $\mathsf{MP} : \varphi, \varphi \to \psi \vdash \psi$) as its only inference rule and the following axioms:

(MTL1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(MTL2)	$(\varphi \,\&\, \psi) \to \varphi$
(MTL3)	$(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$
(MTL4a)	$(\varphi \wedge \psi) \to \varphi$
(MTL4b)	$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(MTL4c)	$(\varphi \,\&\, (\varphi \to \psi)) \to (\varphi \wedge \psi)$
(MTL5a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$
(MTL5b)	$((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(MTL6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(MTL7)	$\overline{0} \to \varphi$

Other connectives can be defined from $\&$, $\wedge$ and $\to$ as follows.

$$\overline{1} := \varphi \to \varphi, \qquad \neg\varphi := \varphi \to \overline{0}$$
$$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$
$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \,\&\, (\psi \to \varphi)$$

The notion of (finitary) *derivation* or *proof* in MTL is the usual one defined from the above axioms and rule, and we will write $\Gamma \vdash_{\mathsf{MTL}} \varphi$ to denote that φ follows from a set of formulas Γ.

MTL is algebraizable in the sense of Blok and Pigozzi [2] and its corresponding algebraic counterpart is the class of the so-called MTL-*algebras*. This class coincides with the variety of prelinear (commutative, integral, bounded) residuated lattices. The algebras of this variety are subdirect products of the linearly-ordered members of the class. This gives completeness of MTL w.r.t. to the class of *linearly-ordered* MTL-algebras. When we restrict ourselves to MTL-algebras over the real unit interval (i.e. to standard MTL-algebras), as expected, the operations are given by left-continuous t-norms and their residua. Jenei and Montagna proved in [15] that MTL is in fact strongly complete with respect to the class of standard MTL-algebras.

Theorem 2.2 *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then the following are equivalent:*

1. $\Gamma \vdash_{\mathsf{MTL}} \varphi$,
2. $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$ *for all left-continuous t-norm* $*$.

Hájek's Basic logic BL ([13]) is the axiomatic extension of MTL with the divisibility axiom $(\varphi \wedge \psi) \to (\varphi \& (\varphi \to \psi))$. It is algebraizable and its equivalent algebraic semantics is given by the variety of BL-algebras, that is, the subvariety of MTL-algebras satisfying the equation $x \wedge y = x * (x \Rightarrow_* y)$. This condition implies that the lattice conjunction $\wedge$, which was not definable in MTL, is definable in BL from the $\&$ and $\to$ operations. Moreover, over an standard algebra $[\mathbf{0},\mathbf{1}]_*$, the divisibility condition characterizes the continuity of the t-norm $*$. Therefore, standard BL-algebras are univocally determined by continuous t-norms and their residua.

Actually, as Hájek conjectured, the variety of BL-algebras is generated by the class of all standard BL-algebras, that is, the theorems of BL captures the set of 1-tautologies common to all standard BL-algebras [4]. Moreover, the following (finite strong) standard completeness property holds: if Γ is a *finite* set of formulas, then $\Gamma \vdash_{\mathsf{BL}} \varphi$ if and only if $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*}$ for each continuous t-norm $*$. However, unlike MTL, BL is not standard complete for deductions from infinite sets of formulas Γ, i.e., BL is not strongly standard complete.

The three most well known fuzzy logics are in fact axiomatic extensions of BL: Gödel logic is the extension of BL with the axiom $\varphi \to (\varphi \,\&\, \varphi)$, Łukasiewicz logic is the extension of BL with the axiom $\neg\neg\varphi \to \varphi$, and Product logic is the extension of BL with the axioms $\neg\neg\chi \to (((\varphi \,\&\, \chi) \to (\psi \,\&\, \chi)) \to (\varphi \to \psi))$, and $\varphi \wedge \neg\varphi \to \bar{0}$. Each one of the Gödel, Łukasiewicz and Product logics enjoys completeness with respect to its corresponding *single* standard algebra $[\mathbf{0},\mathbf{1}]_*$, with $*$ being the minimum t-norm, the product t-norm and the Łukasiewicz t-norm, respectively. We will denote these standard algebras as $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$, $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ and $[\mathbf{0},\mathbf{1}]_{\text{Ł}}$ respectively. While Łukasiewicz and Product logics are finite standard complete, Gödel logic enjoys strong standard completeness with respect to $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$ due to the fact that the logic arising from this algebra is intrinsically finitary (while this is not the case for the logics arising from $[\mathbf{0},\mathbf{1}]_{\text{Ł}}$ and $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$).

It is also known that BL is finite standard complete with respect to one particular standard BL-algebra: the one whose t-norm is given by the ordinal sum of infinitely many Łukasiewicz components [1]. On the other hand, in [11] the authors provide a general method to get a finite axiomatization L_* of the set of valid equations in a standard algebra $[\mathbf{0},\mathbf{1}]_*$ of any continuous t-norm $*$.

2.2 Expansions of MTL-logics

In what follows, let L be an axiomatic extension of MTL. We will briefly comment on two expansions of L that will be considered in the following sections.

Monteiro-Baaz Δ is a unary connective that allows to express crisp notions on many-valued logics. Its interpretation over any standard algebra $[\mathbf{0},\mathbf{1}]_*$ is

$$\Delta a := \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

In order to maintain completeness with respect to standard MTL-algebras expanded with the previous operation, the axiomatic system of L shall be extended as follows.[1]

Definition 2.3 We let L_Δ be the axiomatic system of L extended with the Generalization Rule for Δ, (G_Δ) : $\varphi \vdash \Delta\varphi$, and the following axiom schemata:

$$\begin{array}{ll} (\Delta1) & \Delta\varphi \vee \neg\Delta\varphi \\ (\Delta2) & \Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi) \\ (\Delta3) & \Delta\varphi \to \varphi \\ (\Delta4) & \Delta\varphi \to \Delta\Delta\varphi \\ (\Delta5) & \Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi) \end{array}$$

It is known that L_Δ is strongly complete with respect to linearly-ordered L-algebras expanded with a unary connective Δ that behaves as in the standard algebras.

Another kind of expansions that have been considered in the framework of MFL is by extending the language with more truth constants than just $\{\overline{0}, \overline{1}\}$. The first attempt in this direction was done by Pavelka ([17, 18, 19]), who defined a logic (later discovered that was based on Łukasiewicz logic) with a language featuring a truth constant for each real in $[0, 1]$, and proved that his logic enjoyed a new kind of completeness (called Pavelka-completeness), heavily relying on these constants. Later, Hájek [13] showed that the logic RPL resulting from the expansion of Łukasiewicz logic with one constant only for each rational in $[0, 1]$, and further extended with the so called book-keeping axioms,

$$(\overline{c} \,\&\, \overline{d}) \leftrightarrow \overline{c *_{\mathrm{L}} d}, \qquad (\overline{c} \to \overline{d}) \leftrightarrow \overline{c \Rightarrow_{\mathrm{L}} d},$$

for each rational $c, d \in [0, 1]$, enjoys the same kind of Pavelka-completeness.

Roughly speaking, for a given left-continuous t-norm $*$, a logic L, with a consequence relation $\vdash_{\mathsf{L}}$ extending the finitary companion of $\models_{[\mathbf{0},\mathbf{1}]_*}$, in a language containing a set of truth constants $\{\overline{c} : c \in \mathcal{C}\}$ for a given subset of reals $\{0, 1\} \subseteq \mathcal{C} \subseteq [0, 1]$, is said to be Pavelka-complete for $*$ (over $\mathcal{C}$) when, for any $\Gamma \cup \{\varphi\} \subseteq Fm$, it holds that

$$|\varphi|^{\mathsf{L}}_{\Gamma} = \|\varphi\|^{\mathsf{L}}_{\Gamma},$$

where $|\varphi|^{\mathsf{L}}_{\Gamma} = \sup\{c \in \mathcal{C} : \Gamma \vdash_{\mathsf{L}} \overline{c} \to \varphi\}$ (the *proof degree* of φ) and $\|\varphi\|^{\mathsf{L}}_{\Gamma} = \inf\{e(\varphi) : e$ is a $[0, 1]_*$-model of $\Gamma\}$ (the *truth degree* of φ).

For a logic L as above, to be Pavelka-complete does not imply to fully capture $\models_{[\mathbf{0},\mathbf{1}]_*}$, the logic of the standard algebra $[\mathbf{0}, \mathbf{1}]_*$ (extended with truth-constants). Indeed, combining Propositions 16 and 17 of [6], we have the following result.

[1] As it is usual in the literature, the symbol Δ denotes both the connective in the language and the corresponding operation in the algebra, as it happens with $\wedge$ and $\vee$.

Lemma 2.4 *Let $*$ be left-continuous t-norm, let $\mathcal{C} = [0, 1] \cap \mathbb{Q} = [0, 1]_{\mathbb{Q}}$, and let L be a logic as above, with $\vdash_{\mathsf{L}}$ denoting its consequence relation. Then the following are equivalent:*

1. *L is Pavelka-complete and derives the inference rule*

$$\mathsf{R1}^{\uparrow} : \quad \frac{\{\overline{c} \to \varphi : c \in \mathcal{C} \setminus \{1\}\}}{\varphi}$$

2. *$\vdash_{\mathsf{L}}$ and $\models_{[\mathbf{0},\mathbf{1}]_*}$ coincide.*

Therefore, if the logic L is given by an axiomatic system that is Pavelka-complete, a direct way to get a strongly complete system with respect to the standard algebra $[\mathbf{0}, \mathbf{1}]_*$ is just by adding the infinitary rule $\mathsf{R1}^{\uparrow}$ to L. The presence of this rule is crucial. For instance, Hájek's logic RPL is Pavelka-complete (and finite standard complete) but not strongly standard complete.

On the other hand, the particular behavior of Łukasiewicz logic, where Pavelka-completeness can be obtained with truth-constants but without infinitary inference rules, is due to the fact that all the operations in the standard algebra are continuous. In fact, as already mentioned, it has been shown [6] that if this is not the case, an axiomatic system enjoying Pavelka-style completeness must necessarily include some infinitary inference rule.

2.3 Previous approaches

One can find in the literature several investigations concerning the problem of finding strong complete axiomatizations for t-norm based logics and their expansions. Montagna in [16] deals with the problem of (the lack of) strong completeness of the logic BL and some of its extensions. He resorts in his work to expand the logic with an storage operator *sto*, a unary operator such that, at the algebraic level, for any element a of a BL-algebra, $sto(a)$ is the greatest idempotent element b of the algebra such that $b \leqslant a$. Such an operation can be always defined on weakly saturated and weakly Archimedean BL-chains, and can be understood as a generalized version of the Monteiro-Baaz Δ connective. In the logic BL expanded with *sto*, he introduces the following infinitary inference rule:

$$\frac{\{\varphi \vee (\psi \to \chi^n) : n \in \mathbb{N}\}}{\varphi \vee (\psi \to sto(\chi))},$$

where, as usual, χ^n is a shorthand for $\chi \& \overset{n}{\ldots} \& \chi$. He shows this rule makes the logic strongly complete with respect to the class of standard BL chains expanded with the storage operator *sto*.

This rule also works particularly well for the cases of Product and Łukasiewicz logics, where the storage operator coincides with the Δ connective. That is to say, adding the previous infinitary rule to the logics Π_Δ and Ł$_\Delta$ results into axiomatic systems that are strongly complete with respect to their standard algebras $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi_\Delta}}$ and $[\mathbf{0},\mathbf{1}]_{\text{Ł}_\Delta}$ respectively.

A different approach consists of expanding the logic with additional truth constants and try to get a Pavelka-complete expansion. If this is accomplished then, according to Lemma 2.4, one simply needs to add the inference rule $\mathsf{R1}^\uparrow$ to come up with a strongly complete axiomatic system. However, as it has been proved by Cintula in [6], if L_* is an axiomatic system that is finite standard complete w.r.t. $[\mathbf{0},\mathbf{1}]_*$, with $*$ being different from Łukasiewicz t-norm, at least some additional infinitary inference rule is required to turn the system Pavelka-complete.

Actually, in [5, 6] Cintula deals indeed with the problem of getting a Pavelka-style complete axiomatization of the logic of a standard algebra specified by a left-continuous t-norm expanded with rational truth constants (i.e. taking $\mathcal{C} = [0,1]_\mathbb{Q}$), and possibly with an additional set of operations that are component-wisely increasing or decreasing . For this goal, a family of infinitary inference rules $\mathsf{R}^\mathsf{f}(\vec{x})$ is introduced for each n-ary operation f in the algebra and each $\vec{x} \in [0,1]^n$. In [6, Thm. 22] it is proven that if L is an axiomatic system such that:

- L extends the finitary companion of the logic $\models_{[\mathbf{0},\mathbf{1}]_*}$ of a left-continuous t-norm $*$,
- L derives the book-keeping axioms (for all the connectives in the language) for the truth constants in $[0,1]_\mathbb{Q}$ and the rules $\overline{c} \vdash \overline{0}$, for all $c < 1$.
- L derives, for each operation f from the algebra and each $\vec{x}$ discontinuity point of f, the rules in $\mathsf{R}^\mathsf{f}(\vec{x})$,
- L derives the infinitary rule $\mathsf{R1}^\uparrow$, and
- L is semilinear,

then L is Pavelka-complete.

In particular, for what concerns the t-norm and the residuum operations, the previous families of rules are the following ones (see [6, Prop. 19]):

- for every discontinuity point $\langle r_1, r_2\rangle \in [0,1]^2$ of the operation $*$ and every rational $c < r_1 * r_2$, the rule

$$\frac{\{\overline{c_1} \to \varphi : c_1 \in [0, r_1)_\mathbb{Q}\} \cup \{\overline{c_2} \to \psi : c_2 \in [0, r_2)_\mathbb{Q}\}}{\overline{c} \to (\varphi \,\&\, \psi)} \quad (1)$$

- for every discontinuity point $\langle r_1, r_2\rangle \in [0,1]^2$ of the operation $\Rightarrow_*$ and every rational $c < r_1 \Rightarrow_* r_2$, the rule

$$\frac{\{\varphi \to \overline{c_1} : c_1 \in (r_1, 1]_\mathbb{Q}\} \cup \{\overline{c_2} \to \psi : c_2 \in [0, r_2)_\mathbb{Q}\}}{\overline{c} \to (\varphi \to \psi)} \quad (2)$$

- for every discontinuity point $\langle r_1, r_2\rangle \in [0,1]^2$ of the operation $*$ and every rational $c > r_1 * r_2$, the rule

$$\frac{\{\varphi \to \overline{c_1} : c_1 \in (r_1, 1]_\mathbb{Q}\} \cup \{\psi \to \overline{c_2} : c_2 \in (r_2, 1]_\mathbb{Q}\}}{(\varphi \,\&\, \psi) \to \overline{c}} \quad (3)$$

- for every discontinuity point $\langle r_1, r_2\rangle \in [0,1]^2$ of the operation $\Rightarrow_*$ and every rational $c > r_1 \Rightarrow_* r_2$, the rule

$$\frac{\{\overline{c_1} \to \varphi : c_1 \in [0, r_1)_\mathbb{Q}\} \cup \{\psi \to \overline{c_2} : c_2 \in (r_2, 1]_\mathbb{Q}\}}{(\varphi \to \psi) \to \overline{c}} \quad (4)$$

Some logics, like Product and Łukasiewicz logics expanded with Δ, only have a finite number of discontinuity points for all their operations which are, moreover, rational elements of $[0,1]$, so it is only necessary to consider a finite number of the previous infinitary inference rules. The addition of these finite sets of infinitary inference rules (and the book-keeping axioms) to the usual Hilbert calculus of Łukasiewicz and Product logics with Δ allows to prove Pavelka-style completeness of these logics.[2]

However, this construction does not work in general to provide a Pavelka-complete axiomatic system with respect to the standard semantics based on an arbitrary left-continuous t-norm. The main drawback is that it is not known, in general, when the system resulting from the addition of all the infinitary rules associated to the discontinuity points of the operations to the (finitary) calculus of the t-norm is semilinear or not. It is worth noticing that, in many cases, the operations involved have an uncountable set of discontinuity points, like for instance in the Gödel case (whose residuum is discontinuous along the diagonal of the unit square), or in the case of any ordinal sum of more than one component (where the residuum is again discontinuous along the diagonal of all the components of the ordinal sum with the possible exception of the elements belonging to the the last component).

Motivated by the study of the modal expansions of product fuzzy logic, in [21] the authors provide an axiomatic system that is strongly complete with respect to the standard product algebra $[0,1]_\Pi$ with rational truth constants and the Δ connective. It expands Π_Δ with book-keeping axioms and the following two infinitary rules, that coincide with ones presented in [5]:

$$\mathsf{R0}^\downarrow : \frac{\{\varphi \to \overline{c} : c \in (0,1]_\mathbb{Q}\}}{\neg\varphi} \qquad \mathsf{R1}^\uparrow : \frac{\{\overline{c} \to \varphi : c \in [0,1)_\mathbb{Q}\}}{\varphi}$$

Strong standard completeness of this system is proved in an algebraic fashion, in contrast with the Pavelka-style completeness proof in [5, 6]. Actually, this algebraic

[2] The Δ connective is used to prove semilinearity of the axiomatic system.

approach is the one we have followed in this paper to uniformly deal with a larger class of logics, a task that is not clear how to achieve by following the Pavelka-style approach.

3 Infinitary logics and semilinearity

In this section we provide conditions under which a logic $\vdash_{\mathsf{L}}$ extending MTL_{Δ} is semilinear. From now on we will also sometimes use the symbol L (instead of $\vdash_{\mathsf{L}}$) to refer to the very logic.

While it is well-known that all axiomatic extensions of MTL (or MTL_{Δ}) are semilinear, the situation in the general setting is not so simple. Indeed, the main result of this section (Corollary 3.6) requires a cardinality assumption that cannot be deleted.

Next we remind some well-known notions that are used. A *theory* of a logic L is a set of formulas closed under $\vdash_{\mathsf{L}}$. A theory Σ is said to be *linear* when for any two formulas φ and ψ, either $\varphi \to \psi \in \Sigma$ or $\psi \to \varphi \in \Sigma$. And Σ is *prime* when for any two formulas φ and ψ such that $\varphi \vee \psi \in \Sigma$, either $\varphi \in \Sigma$ or $\psi \in \Sigma$. The fact that L is extending MTL_{Δ} guarantees that

$$\vdash_{\mathsf{L}} (\varphi \to \psi) \vee (\psi \to \varphi),$$

$$\varphi \vee \psi, \varphi \to \psi \vdash_{\mathsf{L}} \psi \quad \text{and} \quad \varphi \vee \psi, \psi \to \varphi \vdash_{\mathsf{L}} \varphi.$$

These three conditions obviously imply that in our setting linear and prime theories coincide.

Finally, we introduce a kind of proof by cases property (cf. [9] and Corollary 3.3); a logic L will be said to be $\vee$-*closed* when it holds that for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$,

$$\text{if } \Gamma \vdash_{\mathsf{L}} \varphi, \text{ then } \Gamma \vee \psi \vdash_{\mathsf{L}} \varphi \vee \psi,$$

where $\Gamma \vee \psi$ refers to the set $\{\gamma \vee \psi : \gamma \in \Gamma\}$. Analogously, we will also use the notation $\Gamma_1 \vee \Gamma_2$ to refer to $\{\gamma_1 \vee \gamma_2 : \gamma_1 \in \Gamma_1 \text{ and } \gamma_2 \in \Gamma_2\}$. Following the same spirit, the $\vee$-*closure of* an inference rule $\frac{\Gamma}{\varphi}$ will refer to the rule

$$\frac{\{\chi \vee \gamma : \gamma \in \Gamma\}}{\chi \vee \varphi}.$$

If the initial rule $\frac{\Gamma}{\varphi}$ is denoted by R, then its $\vee$-closure will be denoted by $\vee\mathsf{R}$.

Since not all logics (e.g., all non-finitary logics) can be syntactically axiomatized using the finitary syntactic proofs considered in Page 3, we consider the following enhanced notion of proof (common in the setting of infinitary proofs), which from now on will be the one used in this paper.

Definition 3.1 Given a Hilbert-style axiomatic system,[3] a *proof* of a formula φ from Γ is a sequence $\langle \varphi_i : i \leqslant \xi \rangle$ where:

- ξ is an ordinal (perhaps non-finite) number,
- for each $i \leqslant \xi$, either φ_i is an axiom, or φ_i belongs to Γ, or φ_i can be derived from some subset of $\{\varphi_j : j < i\}$ using some rule in the axiomatic system.
- $\varphi_\xi = \varphi$.

We write $\Gamma \vdash \varphi$ if there is a proof of φ from Γ; and such $\vdash$ is said to be the *logic associated* with the axiomatic system.

Notice that sometimes this definition is disguised in the literature using the more complex notion of (well-founded) trees instead of well-ordered sets (i.e., ordinals). For the sake of simplicity we have adopted the one involving well-ordered sets.

An obvious remark is that for checking that the logic of an axiomatic system is $\vee$-closed it is enough to deal with the $\vee$-closure of the rules in an axiomatic presentation of such logic.

Before going into the details behind the semilinearity problem for expansions of MTL_{Δ}, we analyse in this setting some metaproperties. Let us start with the local Deduction theorem (see [13, Theorem 2.2.18]), which holds for all logics $\models_K$ where K is a variety of MTL-algebras, and which is a powerful result for the basic language (without Δ). The reader must be careful because such result is not true for all extensions of MTL; a concrete counterexample is given by $\models_{[0,1]_*}$ where $*$ is a weak nilpotent minimum t-norm (see the details in [3, Example A4]).

When the Δ connective is present, then the local Deduction theorem fails for all logics $\models_{[0,1]_*}$ given by a left-continuous t-norm $*$; the counterexample $p \models_{[0,1]_*} \Delta p$ works for all of them. On other hand, the following Δ-Deduction theorem holds.

Proposition 3.2 (Δ-Deduction Theorem) *Let L be an expansion of MTL_{Δ} which is $\vee$-closed. Then, for any set $\Gamma \cup \{\varphi, \psi\}$ of formulas,*

$$\Gamma, \varphi \vdash_{\mathsf{L}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathsf{L}} \Delta\varphi \to \psi.$$

Proof The leftward direction is an immediate consequence of MP together with the fact that $\varphi \vdash \Delta\varphi$.

To show the other direction, let us assume that $\Gamma, \varphi \vdash_{\mathsf{L}} \psi$. Since $\psi \vee \neg\Delta\varphi$ is MTL_{Δ}-equivalent to $\Delta\varphi \to \psi$, it is enough checking that $\Gamma \vdash_{\mathsf{L}} \psi \vee \neg\Delta\varphi$. By the $\vee$-closed assumption, from $\Gamma, \varphi \vdash_{\mathsf{L}} \psi$ we deduce that $\Gamma \vee \neg\Delta\varphi, \varphi \vee \neg\Delta\varphi \vdash_{\mathsf{L}} \psi \vee \neg\Delta\varphi$. Using that $\varphi \vee \neg\Delta\varphi$ is an MTL theorem, we obtain that

[3] This is simply a subset, closed under substitutions, of $\{\langle \Gamma, \varphi \rangle : \Gamma \cup \{\varphi\} \subseteq Fm\}$. Notice that we allow rules with infinite premises Γ.

$$\Gamma \quad \vdash_{\mathsf{L}} \quad \Gamma \vee \neg\Delta\varphi, \varphi \vee \neg\Delta\varphi \quad \vdash_{\mathsf{L}} \quad \psi \vee \neg\Delta\varphi,$$

which finishes the proof. □

Since $\varphi \vee \psi$ is MTL_Δ-interderivable with $\neg\Delta\varphi \to \psi$, the previous result also tells us that

$$\Gamma \vdash_{\mathsf{L}} \varphi \vee \psi \qquad \text{iff} \qquad \Gamma, \neg\Delta\varphi \vdash_{\mathsf{L}} \psi. \tag{5}$$

Corollary 3.3 (Proof by Cases Property, cf. [9]) *Let* L *be an expansion of* MTL_Δ *which is* $\vee$*-closed. Then, for any set* $\Gamma \cup \Gamma_1 \cup \Gamma_2 \cup \{\varphi\}$ *of formulas,*

$$\Gamma, \Gamma_1 \vdash_{\mathsf{L}} \varphi \text{ and } \Gamma, \Gamma_2 \vdash_{\mathsf{L}} \varphi \qquad \text{iff} \qquad \Gamma, \Gamma_1 \vee \Gamma_2 \vdash_{\mathsf{L}} \varphi.$$

Proof The leftward direction is trivial. For the other direction let us assume that $\Gamma, \Gamma_1 \vdash_{\mathsf{L}} \varphi$ and $\Gamma, \Gamma_2 \vdash_{\mathsf{L}} \varphi$. In the following the set Γ_2' refers to $\Gamma \cup \Gamma_2$. Using $\Gamma, \Gamma_1 \vdash_{\mathsf{L}} \varphi$ together with the $\vee$-closed property we obtain that $\Gamma \vee \Gamma_2', \Gamma_1 \vee \Gamma_2' \vdash_{\mathsf{L}} \varphi \vee \Gamma_2'$. Then, by (5) it follows that $\Gamma \vee \Gamma_2', \Gamma_1 \vee \Gamma_2', \neg\Delta\varphi \vdash_{\mathsf{L}} \Gamma_2'$. Using this last fact together with $\Gamma_2' \vdash_{\mathsf{L}} \varphi$ we get that $\Gamma \vee \Gamma_2', \Gamma_1 \vee \Gamma_2', \neg\Delta\varphi \vdash_{\mathsf{L}} \varphi$; and so by (5) we know that $\Gamma \vee \Gamma_2', \Gamma_1 \vee \Gamma_2' \vdash_{\mathsf{L}} \varphi$. Thus, to finish the proof it is enough to check both that $\Gamma, \Gamma_1 \vee \Gamma_2 \vdash_{\mathsf{L}} \Gamma \vee \Gamma_2'$ and $\Gamma, \Gamma_1 \vee \Gamma_2 \vdash_{\mathsf{L}} \Gamma_1 \vee \Gamma_2'$, and these two facts are trivial. □

In the rest of the section we will consider the semilinearity problem. The usual method in the literature (e.g., [13, 10]) to show that a logic L is semilinear consists on proving the Prime Theory Extension Property, and this is the method that we will adopt. We remark that in [8, Theorem 16] it is proved that this strategy is indeed equivalent to proving the semilinearity.

An interesting remark is that when the Δ connective is in the language, the prime theories are maximal (and so corresponds to ultrafilters); indeed, prime theories satisfy that for every formula φ, either φ or $\neg\Delta\varphi$ belong to the theory.

It is worth pointing out that the same collapse of prime and maximal theories happens when there are rational truth constants (even without Δ in the language) and the rule $\mathsf{R1}^{\uparrow}$ is derivable; and next we show it. Let us assume that Σ is a prime theory of L and that $\varphi \notin \Sigma$. Our aim is proving that $\Sigma, \varphi \vdash_{\mathsf{L}} \overline{0}$. Using that $\mathsf{R1}^{\uparrow}$ is derivable and $\varphi \notin \Sigma$ we get that $\Sigma \nvdash_{\mathsf{L}} \overline{c} \to \varphi$ for some $c \in \mathcal{C} \setminus \{1\}$. Therefore, using the primality condition together with the fact that $(\varphi \to \overline{c}) \vee (\overline{c} \to \varphi)$ is a theorem, we deduce that $\Sigma \vdash_{\mathsf{L}} \varphi \to \overline{c}$. Since $\varphi, \varphi \to \overline{c} \vdash_{\mathsf{L}} \overline{0}$ (because $c \neq 1$) we obtain that $\Sigma, \varphi \vdash_{\mathsf{L}} \overline{0}$.

In the finitary case, semilinearity proofs are usually based on using Zorn's Lemma through a union of theories construction. Unfortunalety, this approach cannot be used in the non-finitary case and this was already pointed out by Montagna in [16, Page 261]. The alternative construction that we develop in Theorem 3.5 will require the following technical lemma.

Lemma 3.4 *Let* L *be an expansion of* MTL_Δ *which is* $\vee$*-closed and let* $\Sigma \cup \Phi \cup \{\varphi, \xi\}$ *be a set of formulas such that* $\Sigma, \neg\Delta\xi \nvdash_{\mathsf{L}} \varphi$ *and* $\Phi \vdash_{\mathsf{L}} \xi$*. Then,* $\Sigma, \neg\Delta\xi, \neg\Delta\psi \nvdash_{\mathsf{L}} \varphi$ *for some* $\psi \in \Phi$*.*

Proof Let us proceed assuming that the conclusion is false, i.e., $\Sigma, \neg\Delta\xi, \neg\Delta\psi \vdash_{\mathsf{L}} \varphi$ for all $\psi \in \Phi$. Then, by (5) it follows that $\Sigma \vdash_{\mathsf{L}} \Phi \vee \varphi \vee \xi$.

On the other hand, since $\Phi \vdash_{\mathsf{L}} \xi$ it follows that $\Phi \vdash_{\mathsf{L}} \varphi \vee \xi$; and then by the $\vee$-closed assumption we can obtain that $\Phi \vee \varphi \vee \xi \vdash_{\mathsf{L}} \varphi \vee \xi$.

Using the final statements in the two last paragraphs we obviously get that $\Sigma \vdash_{\mathsf{L}} \varphi \vee \xi$; and so by (5) we obtain that $\Sigma, \neg\Delta\xi \vdash_{\mathsf{L}} \varphi$, which is the contradiction we were looking for. □

Theorem 3.5 (Prime Theory Extension Property) *Let* L *be an expansion of* MTL_Δ *which is* $\vee$*-closed and such that* $\vdash_{\mathsf{L}}$ *is the smallest consequence operator*[4] *extending a countable set. For every set* $\Gamma \cup \{\varphi\}$ *of formulas, if* $\Gamma \nvdash_{\mathsf{L}} \varphi$*, then there is a prime theory* Σ *such that* $\Gamma \subseteq \Sigma$ *and* $\varphi \notin \Sigma$*.*

Proof Let us start by fixing a countable enumeration $\langle (\Phi_n, \xi_n) : n \in \mathbb{N} \rangle$ such that

- $\vdash_{\mathsf{L}}$ is the smallest consequence operator such that $\Phi_n \vdash \xi_n$ for every $n \in \mathbb{N}$, and
- the sequence $\langle \xi_n : n \in \mathbb{N} \rangle$ is an enumeration of all formulas.

Notice that such enumeration exists thanks to the assumptions of this theorem. In particular, $\Phi_n \vdash_{\mathsf{L}} \xi_n$ for every $n \in \mathbb{N}$,

Before proving the two claims that allow us to finish the proof, let us devote some time to explaining the idea behind the construction. The idea is to build the desired prime theory Σ using a countable number of approximation steps. At every step $n \in \mathbb{N}$, the intuition is that Σ_n is capturing a partial approximation (of Σ) satisfying that

- the elements of Σ_n will belong to Σ,
- the elements of $\{\psi \in Fm : \neg\Delta\psi \in \Sigma_n\}$ will belong to $Fm \setminus \Sigma$.

In other words, at every step of our construction we are both selecting and discarding some elements for Σ.

Claim 1: There is a sequence $\langle \Sigma_n : n \in \mathbb{N} \rangle$ which satisfies that for every $n \in \mathbb{N}$,

1. $\Gamma \subseteq \Sigma_n \subseteq \Sigma_{n+1}$,
2. $\Sigma_n \nvdash \varphi$,
3. it holds that either[5]

[4] Let us stress that we do not consider the smallest consequence operator closed under substitutions.

[5] Let us point out that indeed this condition implies that for every natural number $m \leqslant n$, it holds that either

- $\xi_n \in \Sigma_{n+1}$, or
- $\neg\Delta\xi_n \in \Sigma_{n+1}$ and there is some $\psi \in \Phi_n$ such that $\neg\Delta\psi \in \Sigma_{n+1}$.

Proof of Claim 1: Let us consider the sequence defined by (for every $n \in \mathbb{N}$):

- $\Sigma_0 := \Gamma$,
- if $\Sigma_n, \neg\Delta\xi_n \vdash_{\mathsf{L}} \varphi$, then $\Sigma_{n+1} := \Sigma_n \cup \{\xi_n\}$,
- if $\Sigma_n, \neg\Delta\xi_n \nvdash_{\mathsf{L}} \varphi$, then $\Sigma_{n+1} := \Sigma_n \cup \{\neg\Delta\xi_n, \neg\Delta\psi\}$ where ψ is an arbitrary formula of Φ_n such that $\Sigma_n, \neg\Delta\xi_n, \neg\Delta\psi \nvdash_{\mathsf{L}} \varphi$.

First of all let us point out that this definition is meaningful thanks to Lemma 3.4, which guarantees that if $\Sigma_n, \neg\Delta\xi_n \nvdash_{\mathsf{L}} \varphi$ then there is a formula $\psi \in \Phi_n$ such that $\Sigma_n, \neg\Delta\xi_n, \neg\Delta\psi \nvdash_{\mathsf{L}} \varphi$.

The proof, by induction in the construction, that this sequence fulfills the three conditions of the claim is rather trivial (thanks to the chosen definition). The only non-trivial part is checking that if $\Sigma_n \nvdash_{\mathsf{L}} \varphi$ and $\Sigma_n, \neg\Delta\xi_n \vdash_{\mathsf{L}} \varphi$ then $\Sigma_n, \xi_n \nvdash_{\mathsf{L}} \varphi$, which follows from the Proof by Cases Property (remember that $\xi_n \vee \neg\Delta\xi_n$ is a theorem).

Claim 2: For every sequence $\langle \Sigma_n : n \in \mathbb{N}\rangle$ satisfying the three properties in the previous claim, it holds that the set $\Sigma := \bigcup\{\Sigma_n : n \in \mathbb{N}\}$ is a prime theory of $\vdash_{\mathsf{L}}$ extending Γ and such that $\Sigma \nvdash_{\mathsf{L}} \varphi$.

Proof of Claim 2: To proceed with the proof let us consider, for every $n \in \mathbb{N}$, the set $I_n := \{\psi \in Fm : \neg\Delta\psi \in \Sigma_n\}$. The strategy of the proof of this result is based on the following steps.

- First of all we notice that $(\Sigma, \bigcup\{I_n : n \in \mathbb{N}\})$ is a partition of Fm. Disjointness follows from the second condition in the previous claim which guarantees that $\{\psi, \neg\Delta\psi\} \not\subseteq \Sigma_n$ for every formula ψ and every natural number n. On the other hand, the covering of all formulas is a consequence of third condition in the previous claim, condition that in particular says that for every natural number n, either $\xi_n \in \Sigma_{n+1}$ or $\xi_n \in I_{n+1}$. Notice that here it is crucial that $\langle \xi_n : n \in \mathbb{N}\rangle$ is an enumeration of all formulas.
- Now it is time to check that Σ is a theory of $\vdash_{\mathsf{L}}$. Thanks to the fixed enumeration we only need to check that for every $n \in \mathbb{N}$, if $\Phi_n \subseteq \Sigma$ then $\xi_n \in \Sigma$. This is obvious by the third condition in the Claim 1.

- $\xi_m \in \Sigma_{n+1}$, or
- $\neg\Delta\xi_m \in \Sigma_{n+1}$ and there is some $\psi \in \Phi_m$ such that $\neg\Delta\psi \in \Sigma_{n+1}$.

Notice that this last disjunction is capturing the intuition that Σ_{n+1} (and all its extensions) is a model of all the derivations in $\langle(\Phi_m, \xi_m) : m \leqslant n\rangle$.

- It is obvious, by the second condition in Claim 1, that Σ is a theory extending Γ and such that $\varphi \notin \Sigma$ (and so Σ is consistent).
- The partition previously considered guarantees that for every formula ψ, it holds that $\psi \notin \Sigma$ iff $\neg\Delta\psi \in \Sigma$. Using this together with $\neg\Delta\psi_1, \neg\Delta\psi_2, \psi_1 \vee \psi_2 \vdash_{\mathsf{L}} \overline{0}$ it follows that it cannot happen that $\psi_1 \vee \psi_2 \in \Sigma$ while $\psi_1 \notin \Sigma$ and $\psi_2 \notin \Sigma$. Therefore, we have just proved that Σ is prime. □

If we further require that the logic L enjoys completeness with respect to its class of algebras in the usual sense (i.e., with respect to the logical matrices whose filter is $\{1\}$), the previous result has as a corollary the semilinearity of L. Let L be an expansion of MTL_{Δ} such that for each new n-ary connective λ with $n \geqslant 1$ the following congruence conditions hold:

$$\{\, \varphi_1 \leftrightarrow \psi_1, \ldots, \varphi_n \leftrightarrow \psi_n \,\} \vdash_{\mathsf{L}} \lambda(\varphi_1, \ldots, \varphi_n) \to \lambda(\psi_1, \ldots, \psi_n).$$

It follows that L is a (Rasiowa) implicative logic [2, Section 5.2], and thus it enjoys completeness with respect to the class of L-algebras in the above sense. That is to say, the following conditions are equivalent:

1. $\Gamma \vdash_{\mathsf{L}} \varphi$.
2. for every $h \in Hom(\mathbf{Fm}, \mathbf{A})$ and every L-algebra $\mathbf{A}$, if $h[\Gamma] \subseteq \{\overline{1}^{\mathbf{A}}\}$, then $h(\varphi) = \overline{1}^{\mathbf{A}}$.

Corollary 3.6 (Semilinearity) *Let L be an implicative expansion of MTL_{Δ} which is $\vee$-closed and such that $\vdash_{\mathsf{L}}$ is the smallest consequence operator extending a countable set. Then L is semilinear, i.e. L is (strongly) complete with respect to the class of linearly-ordered L-algebras.*

Proof This is a well-known consequence of the Prime Theory Extension Property using the MTL_{Δ}-chain obtained as a quotient of the free MTL_{Δ}-algebra by a prime theory. The interested reader can get the details, among other places, in [8, Theorem 16]. □

Notice that if a logic as above is finitary then it is a Δ-core fuzzy logic (see [7, Ch. I, Def. 3.2.6]). In this case, the previous result coincides with Theorem [7, Ch. I,Th. 3.2.14]).

The previous corollary opens the door to a systematic and uniform study of (perhaps non-finitary) extensions of MTL_{Δ}, which is done in Section 4.

Let us remark that the countability assumption in Corollary 3.6 is necessary; indeed, a non-semilinear logic (expanding Gödel logic) fulfilling all premises of the corollary except the countability one is later explicitly given in Corollary 5.5. It is also worth pointing out that the countability condition in Corollary 3.6 trivially holds for the logics associated with an axiomatic system involving at most a countable number of rules and such that each of the rules only uses a finite number of variables.[6] Indeed, an example of such axiomatic systems will be the one considered in Definition 4.4.

We point out that the content of Theorem 3.5 has been lately improved by Cintula (personal communication) showing that the same result holds without having the Δ operator in the language; in this improvement, the technical result employed by Cintula in the construction is the analogous of Lemma 3.4 (following the guidelines of (5)) .

To finish this section let us remark that, in contrast with finitary logics, the separation property given in Theorem 3.5 cannot be improved replacing the principal ideal $\{\psi \in Fm : \psi \vdash_{\mathsf{L}} \varphi\}$ with an arbitrary ideal. In other words, while the assumptions of such theorem allow us to separate theories and principal ideals using prime theories, they do not allow to separate theories and arbitrary ideals using prime theories as the following counterexamples shows. Let us take L as the logic of some left-continuous t-norm $*$, the principal theory $\Gamma := \{\psi \in Fm : \neg\Delta\neg p \vdash_{\mathsf{L}} \psi\}$ and the ideal $I := \{\psi \in Fm : \psi \vdash_{\mathsf{L}} \overline{c} \to p \text{ for some } c \in (0,1]_{\mathbb{Q}}\}$. In this setting, Γ and I are disjoint, but using the derivation

$$\{\neg\Delta\neg p\} \cup \{p \to \overline{c} : c \in (0,1]_{\mathbb{Q}}\} \vdash_{\mathsf{L}} \overline{0}$$

it is very easy to check that there is no prime theory of L extending Γ that is disjoint with I. The main reason is that in prime theories it holds that for every rational c, either $p \to \overline{c}$ or $\overline{c} \to p$ must belong to the theory.

4 Axiomatizing the logic of $[\mathbf{0},\mathbf{1}]_*$ using the Density rule

Let us formally define the standard algebras whose logic we are aiming to axiomatize. We are considering the logic with the Monteiro-Baaz Δ operator and a set of truth constants isomorphic to the rational numbers from $[0,1]$ (for our purposes, we could equivalently consider any subset dense on $[0,1]$). It is remarkable that, in order to be able to express the book-keeping axioms, we need to close this set of constants by just one level of application of the operations, and this suffices to prove all further results. However, to avoid unnecessary complexity in the notation, we will consider the subalgebra generated by the basic set of truth constants ($[0,1] \cap \mathbb{Q}$). In any case, the set of truth constants is countable. For simplicity on the notation, all along the rest of the paper we will use the name $[\mathbf{0},\mathbf{1}]_*$ to denote the expansion of the standard algebra of the t-norm $*$ with Δ and the previous set of truth constants (in contrast to its meaning along the preliminaries section, where it just referred to the standard algebra).

[6] This finiteness is employed to show that for each rule in the axiomatic system, there is at most a countable number of substitutions.

Definition 4.1 Let $*$ be a left-continuous t-norm. We let

$$[\mathbf{0},\mathbf{1}]_* := \langle [0,1], *, \Rightarrow_*, \wedge, \vee, \Delta, \{c\}_{c\in\mathbb{Q}_*}\rangle,$$

where $\mathbb{Q}_*$ is the subalgebra of $\langle [0,1], *, \Rightarrow_*, \wedge, \vee, \Delta\rangle$ generated by the rational numbers in $[0,1]$.

The logic arising from the previous algebra is formally defined as follows.

Definition 4.2 (Logic of $[\mathbf{0},\mathbf{1}]_*$) Let $\Gamma \cup \{\varphi\}$ be formulas in the language of MTL_Δ expanded by the constant symbols $\{\overline{c} : c \in \mathbb{Q}_*\}$. We write

$$\Gamma \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$$

when $e(\varphi) = 1$ for all $e \in Hom(\mathbf{Fm}, [\mathbf{0},\mathbf{1}]_*)$ such that $e([\Gamma]) \subseteq \{1\}$.

Aiming towards an axiomatic system strongly complete with respect to $\models_{[\mathbf{0},\mathbf{1}]_*}$ let us first define the following finitary logic.

Definition 4.3 L_* is the logic associated with the axiomatic system of MTL_Δ expanded with the book-keeping axioms for the constants in $\mathbb{Q}_*$, i.e.,

$$(\overline{c} \,\&\, \overline{d}) \leftrightarrow \overline{c * d}\ ,\ (\overline{c} \to \overline{d}) \leftrightarrow \overline{c \Rightarrow_* d}, \qquad \text{for all } c,d \in \mathbb{Q}_*$$
$$\neg\Delta\overline{c}, \qquad \text{for all } c \in \mathbb{Q}_* \setminus \{1\}$$

Observing the previous works on the topic, an axiomatic system strongly complete with respect to $\models_{[\mathbf{0},\mathbf{1}]_*}$ for some left-continuous t-norm $*$ has to deal with the discontinuity points of the operations. In the approach presented by Cintula in [6], this is done pointwisely, which may have some drawbacks (as we said, it is not clear when such an axiomatic system is semilinear). On the other hand, the approach followed by Montagna for BL (without constants but with the storage operator) [16] successfully used a unique infinitary inference rule. In a similar way, in this work we propose an axiomatization of the logic $\models_{[\mathbf{0},\mathbf{1}]_*}$ with only one infinitary rule, inspired in one rule used by Takeuti and Titani to axiomatize Intuitionistic predicate logic in [20], namely:

$$\frac{(\varphi \to p) \vee (p \to \psi) \vee \chi}{(\varphi \to \psi) \vee \chi}$$

where p is a propositional variable not occurring in φ, ψ, χ. Indeed, this rule exploits the concept of free variable from first order logics, and its validity in a given linearly-ordered algebra forces its universe to be dense (in the usual sense that in between two different elements there is always a third one). For this reason, this rule is known as Takeuti-Titani density rule. In our framework, we propose a similar rule, with an infinite number of premises, in order to enforce the density of the constants in the universes of the linearly-ordered algebras of our logic.

Definition 4.4 We define L_*^∞ as the extension of L_* with the infinitary inference rule

$$\mathsf{D}^\infty : \frac{\{(\varphi \to \overline{c}) \vee (\overline{c} \to \psi) : c \in \mathbb{Q}_*\}}{\varphi \to \psi}.$$

We will call D^∞ *density rule.*

To avoid misunderstandings, let us remark that the finitary companion of L_*^∞ is not L_*.

It is easy to see that L_*^∞ is sound with respect to $\models_{[\mathbf{0},\mathbf{1}]_*}$ since D^∞ is sound in this semantics. Towards the proof of completeness of L_*^∞ we first show that $\vee\mathsf{D}^\infty$ is provable in L_*^∞; and so L_*^∞ is $\vee$-closed.

Lemma 4.5 *The rule*

$$\vee\mathsf{D}^\infty : \frac{\{\chi \vee (\varphi \to \overline{c}) \vee (\overline{c} \to \psi) : c \in \mathbb{Q}_*\}}{\chi \vee (\varphi \to \psi)}$$

is provable in L_*^∞.

Proof We begin by noticing that for all $c \in \mathbb{Q}_*$ it holds that $((\varphi \wedge (\neg\Delta\chi)) \to \overline{c}) \vee (\overline{c} \to (\psi \vee \Delta\chi)$ can be derived from $\chi \vee (\varphi \to \overline{c}) \vee (\overline{c} \to \psi)$ in L_*^∞. This follows from the fact that finitary deductions of L_*^∞ coincide with those of L_*. Since this is true for each $\overline{c}$, we can apply the infinitary inference rule D^∞ and so get $(\varphi \wedge (\neg\Delta\chi)) \to (\psi \vee \Delta\chi)$. Using again that the theorems of L_*^∞ coincide with those of L_* we have that $(\varphi \to \psi) \vee (\varphi \to \Delta\chi) \vee ((\neg\Delta\chi) \to \psi) \vee ((\neg\Delta\chi) \to \Delta\chi)$. From here, since $\varphi \to \Delta\chi \vdash_{\mathsf{L}_*} \Delta\chi \vee \neg\varphi$, $\neg\Delta\chi \to \psi \vdash_{\mathsf{L}_*} \Delta\chi \vee \psi$ and $\neg\Delta\chi \to \Delta\chi \vdash_{\mathsf{L}_*} \Delta\chi$ we can conclude that $(\varphi \to \psi) \vee \Delta\chi$, and so, $(\varphi \to \psi) \vee \chi$. □

It is now easy to see that L_*^∞ validates all the premises of Corollary 3.6: it is an $\vee$-closed implicative expansion of MTL_Δ and has a finite number of inference rules, each one of them using a finite number of variables (e.g., the density rule only uses two variables). It then follows that L_*^∞ is semilinear. That is to say, it is complete with respect to the linearly-ordered L_*^∞-algebras, that is, algebras of the form $\mathbf{A} = \langle A, \odot, \Rightarrow, \wedge, \Delta^{\mathbf{A}}, \{\overline{c}^{\mathbf{A}}\}_{c\in\mathbb{Q}_*}\rangle$ that validate all the equations arising from the axioms of L_*^∞ and all the quasi-equations and generalized quasi-equations associated to the inference rules. Observe that in particular, any L_*^∞-algebra $\mathbf{A}$ validates the following generalized quasi-equation

$$\mathcal{D}^\infty : \bigwedge_{c\in\mathbb{Q}_*} [(x \to \overline{c}) \vee (\overline{c} \to y)] \approx \overline{1} \quad \Longrightarrow \quad (x \to y) \approx \overline{1}.$$

Concerning the linearly-ordered algebras of the previous class, the following result shows that the name *density rule* was properly chosen.

Lemma 4.6 *Let* $\mathbf{A}$ *be a* L_*^∞*-chain and take* $a < b$ *in* A. *Then there is* $c \in \mathbb{Q}_*$ *such that* $a < \overline{c}^{\mathbf{A}} < b$.

Proof Towards a contradiction, suppose that there is no such c. Then, since $\mathbf{A}$ is linearly-ordered we have that for all $c \in \mathbb{Q}_*$, either $b \leqslant \overline{c}^{\mathbf{A}}$ or $\overline{c}^{\mathbf{A}} \leqslant a$. Thus for all $c \in \mathbb{Q}_*$, $[(b \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow a)] = 1$, which means that the premises of the generalized quasi-equation $\mathcal{D}^\infty$ are true and so it can be applied. The consequence of this instantiation of $\mathcal{D}^\infty$ is that $b \leqslant a$, which contradicts the assumptions of the lemma. □

As a consequence, a natural mapping from any linearly-ordered L_*^∞-algebra into $[\mathbf{0},\mathbf{1}]_*$ can be defined. Given a linearly-ordered L_*^∞-algebra $\mathbf{A}$ and an element $a \in A$, we consider the following subsets of $[0,1]$:

$$\mathcal{C}_a^+ := \{c \in \mathbb{Q}_* : a \leqslant_{\mathbf{A}} \overline{c}^{\mathbf{A}}\}, \quad \mathcal{C}_a^- := \{c \in \mathbb{Q}_* : \overline{c}^{\mathbf{A}} \leqslant_{\mathbf{A}} a\}.$$

Clearly, for each $a \in A$ the set $\mathcal{C}_a^-$ is downward closed and $\mathcal{C}_a^+$ is upward closed. Moreover, it also holds that $\sup \mathcal{C}_a^- = \inf \mathcal{C}_a^+$ for any $a \in A$, where the supremum and infimum are considered in the complete real unit interval. Indeed, these two values cannot be different, since if that was the case the previous Lemma would imply the existence of a constant d between them (i.e., $d \in \mathbb{Q}_*$ such that $d \notin \mathcal{C}_a^-$ and $d \notin \mathcal{C}_a^+$). However, $\mathbf{A}$ is linearly-ordered so we have that either $a \leqslant \overline{d}^{\mathbf{A}}$ or $\overline{d}^{\mathbf{A}} \leqslant a$, which contradicts the previous statement.

Lemma 4.7 *Let* $\mathbf{A}$ *be a linearly-ordered* L_*^∞*-algebra. The map* $\rho : A \to [0,1]$ *defined by*

$$\rho(a) := \sup \mathcal{C}_a^- = \inf \mathcal{C}_a^+$$

is an embedding from $\mathbf{A}$ *into* $[\mathbf{0},\mathbf{1}]_*$.

Proof First note that for any constant $\overline{d}$, $d = \min \mathcal{C}^+_{\overline{c}^{\mathbf{A}}} = \max \mathcal{C}^-_{\overline{c}^{\mathbf{A}}}$ and so $\rho(\overline{d}^{\mathbf{A}}) = d = \overline{d}^{[\mathbf{0},\mathbf{1}]_*}$.

As for the operations, we can resort to the density of the constants in $\mathbf{A}$ and in $[\mathbf{0},\mathbf{1}]_*$. This means that in order to check that two elements $a, b \in [0,1]$ coincide, it is enough to check that for each constant c, if $a < c$ then $b \leqslant c$ and that if $c < a$ then $c \leqslant b$.

We can prove the homomorphism conditions just for the $\odot, \Rightarrow, \wedge$ and $\Delta^{\mathbf{A}}$ operations, since the rest are definable from these ones.

– First of all, the case of Δ is trivial, since $\Delta^{\mathbf{A}} x = 1$ if and only if $x = 1$ in the algebra. Then $\rho(\Delta^{\mathbf{A}} a) = \inf \mathcal{C}^+_{\Delta^{\mathbf{A}} a} = 1$ if and only if $\Delta^{\mathbf{A}} a = 1$ in A, i.e., if and only if $a = 1$ in A (by the definition of Δ over a chain). Then, this happens if and only if $\Delta\rho(a) = 1$. On the other hand, if $a < 1$ in A then $\Delta^{\mathbf{A}} a = 0$ and thus, $\rho(\Delta^{\mathbf{A}} a) = 0$. Since $a < 1$, there is c such that $a < \overline{c}^{\mathbf{A}} < 1$, so $\rho(a) < 1$ and thus, $\Delta\rho(a) = 0$ too.

– Concerning $\odot$, observe that for any $a, b \in A$, since $\odot$ is an increasing function in both components it holds that $c \in \mathcal{C}_a^-$ and $d \in \mathcal{C}_b^-$ implies that $c * d \in \mathcal{C}_{a\odot b}^-$, and similarly, $c \in \mathcal{C}_a^+$ and $d \in \mathcal{C}_b^+$ implies that $c * d \in \mathcal{C}_{a\odot b}^+$.
  Let $c \in \mathbb{Q}_*$ be such that $c < \rho(a) * \rho(b) = \sup \mathcal{C}_a^- * \sup \mathcal{C}_b^-$. Using that $*$ has a residuum that coincides with the order operation in the algebra it follows that there exist $d_1 \in \mathcal{C}_a^-$ and $d_2 \in \mathcal{C}_b^-$ such that $c < d_1 * d_2$. [7] Then, from the previous observation and given that $\mathcal{C}_{a\odot b}^-$ is downward closed, $c \in \mathcal{C}_{a\odot b}^-$ too and thus, $c \leqslant \sup \mathcal{C}_{a\odot b}^- = \rho(a \odot b)$.
  For the other direction, let us first prove an auxiliary claim:

  Claim: Let $\mathbf{A}$ be a Ł_*^∞-chain. Then,

$$a \odot b = \sup\{\overline{c * d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a, \overline{d}^{\mathbf{A}} \leqslant b\}.$$

  Proof of Claim: First we check that $a \odot b \geqslant \sup\{\overline{c * d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a, \overline{d}^{\mathbf{A}} \leqslant b\}$. For any $\overline{r}^{\mathbf{A}} \leqslant \sup\{\overline{c * d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a, \overline{d}^{\mathbf{A}} \leqslant b\}$, by definition there exist c, d with $\overline{c}^{\mathbf{A}} \leqslant a, \overline{d}^{\mathbf{A}} \leqslant b$ such that $\overline{r}^{\mathbf{A}} \leqslant \overline{c * d}^{\mathbf{A}}$. By the monotonicity of $\odot$, $\overline{c}^{\mathbf{A}} \odot \overline{d}^{\mathbf{A}} \leqslant a \odot b$ and from the book-keeping axioms, $\overline{r}^{\mathbf{A}} \leqslant a \odot b$.
  To see that $a \odot b \leqslant \sup\{\overline{c * d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a, \overline{d}^{\mathbf{A}} \leqslant b\}$, observe that, for any $\overline{r}^{\mathbf{A}} \geqslant \sup\{\overline{c * d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a, \overline{d}^{\mathbf{A}} \leqslant b\}$, by definition it holds that $\overline{r}^{\mathbf{A}} \geqslant \overline{c * d}^{\mathbf{A}}$ for any c, d like in the formula. Then, by the book-keeping axioms, $\overline{r}^{\mathbf{A}} \geqslant \overline{c}^{\mathbf{A}} \odot \overline{d}^{\mathbf{A}}$ for such c, d. Applying that $\odot$ is a residuated operation we have that $\overline{c}^{\mathbf{A}} \leqslant \overline{d}^{\mathbf{A}} \Rightarrow \overline{r}^{\mathbf{A}}$. We can now take the supremum at the left side, so $a = \sup\{\overline{c}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a\} \leqslant \overline{d}^{\mathbf{A}} \Rightarrow^{\mathbf{A}} \overline{r}^{\mathbf{A}}$ (since the constants in $\mathbf{A}$ are dense by Lemma 4.6). Proceeding similarly for the other component, we get that $a *^{\mathbf{A}} b = \sup\{\overline{c}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a\} \odot \sup\{\overline{d}^{\mathbf{A}} : \overline{d}^{\mathbf{A}} \leqslant b\}$, concluding the proof of the claim.

  Let now $c \in \mathbb{Q}_*$ be such that $c \leqslant \rho(a \odot b)$. By definition, $\overline{c}^{\mathbf{A}} \leqslant a \odot b$ and by the previous claim, $\overline{c}^{\mathbf{A}} \leqslant \sup\{\overline{c * d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leqslant a, \overline{d}^{\mathbf{A}} \leqslant b\}$. Then, there exist $c_0, c_1 \in \mathbb{Q}_*$ with $\overline{c_0}^{\mathbf{A}} \leqslant a$ and $\overline{c_1}^{\mathbf{A}} \leqslant b$ such that $\overline{c}^{\mathbf{A}} \leqslant \overline{c_0 * c_1}^{\mathbf{A}} = \overline{c_0}^{\mathbf{A}} \odot \overline{c_1}^{\mathbf{A}}$. Then, $c \leqslant c_0 * c_1$. Given that $c_0 \in \mathcal{C}_a^-$ and $c_1 \in \mathcal{C}_b^-$, then from the first remark we have that $c_0 * c_1 \in \mathcal{C}_{a\odot b}^-$ and given that this set is downwards closed, $c \in \mathcal{C}_{a\odot b}^-$ too, concluding the proof of the case.

– The reasoning for $\wedge$ is exactly the same done for $\odot$, since it is also true that $\wedge$ is an increasing function in both components which is, moreover, continuous.

– The case of the $\Rightarrow$ connective can be approached in a simpler way, using that it is the residuum of $\odot$. Indeed, first consider $c \in \mathbb{Q}_*$ such that $c \leqslant e(a) \to e(b)$. By residuation (on $[0, 1]$), $c * e(a) \leqslant e(b)$. By definition of e over the constants, this is the same that $e(\overline{c}^{\mathbf{A}}) * e(a) \leqslant e(b)$. Then, from the previous point of the proof, $e(\overline{c}^{\mathbf{A}} \odot a) \leqslant e(b)$. Given that from the definition of e it is immediate that it is order-preserving, we have that $\overline{c}^{\mathbf{A}} \odot a \leqslant b$. Applying now residuation of $\odot$, $\overline{c}^{\mathbf{A}} \leqslant a \Rightarrow b$. Then, again by the definition of e, $c = e(\overline{c}^{\mathbf{A}}) \leqslant e(a \Rightarrow b)$.
  For the other direction, let $c \in \mathbb{Q}_*$ such that $c \leqslant e(a \Rightarrow b)$. By definition, $\overline{c}^{\mathbf{A}} \leqslant a \Rightarrow b$. By residuation of $\odot$, it follows that $\overline{c}^{\mathbf{A}} \odot \leqslant b$. Then, $e(\overline{c}^{\mathbf{A}} \odot a) \leqslant e(b)$ and from the previous point of the proof, $c * e(a) = e(\overline{c}^{\mathbf{A}}) * e(a) = e(\overline{c}^{\mathbf{A}} \odot a) \leqslant e(b)$. By residuation of $*$, $c \leqslant e(a) \to e(b)$, that concludes the proof.

On the other hand, we know that for any two elements a, b of a linearly-ordered Ł_*^∞-algebra,

$$\neg\Delta^{\mathbf{A}}(a \Rightarrow b) = \begin{cases} 1 & \text{if } b < a, \\ 0 & \text{if } a \leqslant b. \end{cases}$$

From here, it is immediate to see that any homomorphism between two different Ł_*^∞-chains $\mathbf{A}$ and $\mathbf{A}'$ is injective. Indeed, if $b < a$ in A, then under a homomorphism h we have that $h(\neg\Delta^{\mathbf{A}}(a \Rightarrow b)) = 1$ and thus, being a homomorphism, that $\neg\delta^{\mathbf{A}'}(h(a) \Rightarrow' h(b)) = 1$, so $h(b) < h(a)$ in $\mathbf{A}'$.[8] This concludes the proof. □

In the next lemma we point out some results about Ł_*^∞-algebras. It is worth saying that the countability assumption in the last item is crucial in our proof; this is so because the set of propositional variables used for the proof of Theorem 3.5 is countable. It is not clear whether this assumption might be avoided; indeed, the construction used in Theorem 3.5 does not seem possible to be generalized for larger sets of propositional variables (which obviously do not allow countable enumerations).

Lemma 4.8

1. *$\mathbf{A}$ is a linearly-ordered Ł_*^∞-algebra if and only if $\mathbf{A}$ is, up to isomorphism, a subalgebra of $[\mathbf{0}, \mathbf{1}]_*$.*
2. *$\mathbb{Q}_*$ is embeddable in any linearly-ordered Ł_*^∞-algebra.*

[7] If for all two constants like above $d_1 * d_2 \leqslant c$, applying residuation $d_1 \leqslant d_2 \to c$ and so the supremum can be taken in the left side. Similarly, we get that $\sup \mathcal{C}_a^- * \sup \mathcal{C}_b^- \leqslant c$ which contradicts the assumptions.

[8] This can be also seen as a direct consequence of the fact that all L_*^∞-chains are relatively simple.

3. $[\mathbf{0},\mathbf{1}]_*$ *is the unique, up to isomorphism, linearly-ordered* L_*^∞*-algebra which is lattice complete.*
4. *Any countable* L_*^∞*-algebra is a subalgebra of a direct product of* $[\mathbf{0},\mathbf{1}]_*$.

Proof The first item is a direct consequence of Lemma 4.7, the second one is a trivial consequence of the book-keeping axioms, and the third one follows from the fact that $[0,1]$ is the topological closure of $[0,1]_{\mathbb{Q}}$. Finally, the fourth item is proved using that Theorem 3.5 tells us, in particular, that every countable L_*^∞-algebra is subdirect product of countable L_*^∞-chains. □

Moreover, strong standard completeness of L_*^∞ follows now easily.

Theorem 4.9 (Strong Standard Completeness of L_*^∞) *Let* $\Gamma \cup \{\varphi\} \subseteq Fm$. *Then the following are equivalent:*

1. $\Gamma \vdash_{\mathsf{L}_*^\infty} \varphi$,
2. $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$,
3. $\Gamma \models_{\mathbf{A}} \varphi$ *for every* L_*^∞*-algebra* $\mathbf{A}$.

Proof Observe that $1 \Leftrightarrow 3$ follows from the general theory of algebraization of logics. $1 \Rightarrow 2$ comes from the soundness of L_*^∞ with respect to $[\mathbf{0},\mathbf{1}]_*$, which is routine. As for the other direction, suppose that $\Gamma \nvdash_{\mathsf{L}_*^\infty} \varphi$. Then, by Corollary 3.6, there is a linearly-ordered L_*^∞-algebra $\mathbf{A}$ and a homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h([\Gamma]) \subseteq \{1\}$ and $h(\varphi) < 1$. It is immediate that $h([\mathbf{Fm}])$ is a subalgebra of $\mathbf{A}$ (thus linearly-ordered) and so it can be embedded into $[\mathbf{0},\mathbf{1}]_*$ by the embedding e built in Lemma 4.7. Then, it is clear that $e \circ h : \mathbf{Fm} \to [\mathbf{0},\mathbf{1}]_*$ is a homomorphism such that $e \circ h(\Gamma) \subseteq \{1\}$ and $e \circ h(\varphi) < 1$. □

We have just proved that the density rule is enough to provide a strongly complete axiomatization of $\models_{[\mathbf{0},\mathbf{1}]_*}$, for any left-continuous t-norm $*$. It is remarkable that the only thing that distinguishes the axiomatic systems associated with $\mathsf{L}_{*_1}^\infty$ and $\mathsf{L}_{*_2}^\infty$ (for two left-continuous t-norms $*_1$ and $*_2$) are the book-keeping axioms. Notice also that even when $*_1$ and $*_2$ are two different isomorphic operations, the logics $\mathsf{L}_{*_1}^\infty$ and $\mathsf{L}_{*_2}^\infty$ are not comparable.

5 A non-semilinear infinitary logic

In the first part of this section we show there are counterexamples to Corollary 3.6 (and thus to Theorem 3.5 as well) when the countability assumption is removed. We show it by considering a logic $\mathsf{L}_{\mathsf{G}}^+$ (later introduced) extending the Gödel logic G.

To this purpose we first consider the following family of infinitary inference rules (one for each $x \in [0,1)$)

$$\mathsf{R}_x^\infty : \frac{\{(\varphi \to \overline{c}) \wedge (\overline{d} \to \psi) : c \in (x,1] \cap \mathbb{Q}_*, d \in [0,x) \cap \mathbb{Q}_*\}}{\varphi \to \psi}.$$

Clearly, each one of these rules only involves a finite number of variables, but there is a continuum of such rules. In what follows, we will denote the generalized quasi-equation associated with R_x^∞ by $\mathcal{R}_x^\infty$.

When $*$ is the minimum t-norm, we consider the logic $\mathsf{L}_{\mathsf{G}}^+$ as the extension of the finitary logic L_* (see Definition 4.3) with the axioms of the (finitary) Gödel logic G (see [13, Section 4.2]) together with all the rules $\{\mathsf{R}_x^\infty : x \in [0,1]\}$.

Lemma 5.1 *The logic* $\mathsf{L}_{\mathsf{G}}^+$ *is* $\vee$*-closed. That is, for all* $x \in [0,1]$, *the rule*

$$\vee\mathsf{R}_x^\infty : \frac{\{\chi \vee ((\varphi \to \overline{c}) \wedge (\overline{d} \to \psi)) : c \in (x,1] \cap \mathbb{Q}_*, d \in [0,x) \cap \mathbb{Q}_*\}}{\chi \vee (\varphi \to \psi)}$$

is provable in $\mathsf{L}_{\mathsf{G}}^+$.

Proof In $\mathsf{L}_{\mathsf{G}}^+$, from $\chi \vee ((\varphi \to \overline{c}) \wedge (\overline{d} \to \psi))$ it can be deduced $(((\varphi \wedge \neg\Delta\chi) \to \overline{c}) \wedge (\overline{d} \to (\psi \vee \Delta\chi))$. Applying the corresponding infinitary rule (R_x^∞) over this latter set of formulas we have that $(\varphi \wedge \neg\Delta\chi) \to (\psi \vee \Delta\chi)$, and from here it follows that $\chi \vee (\varphi \to \psi)$. □

In the following, we will see that the logic $\mathsf{L}_{\mathsf{G}}^+$, although being $\vee$-closed, is not semilinear. Let us point out that this does not contradict Corollary 3.6 because in our definition of $\mathsf{L}_{\mathsf{G}}^+$ we are employing a continuum number of rules.

We can first observe that, over linearly-ordered algebras, validating all the generalized quasi-equations arising from $\{\mathsf{R}_x^\infty\}_{x \in [0,1]}$ amounts to validating the one arising from density rule.

Lemma 5.2 *Let us assume that* $\mathbf{A}$ *is a* L_**-chain. Then, the following conditions are equivalent:*

1. $\mathbf{A}$ *validates* $\mathcal{D}^\infty$.
2. $\mathbf{A}$ *validates* $\mathcal{R}_x^\infty$ *for all* $x \in [0,1]$.

Proof To check $1 \Rightarrow 2$, let us assume $\mathbf{A}$ validates $\mathcal{D}^\infty$, and let $a, b \in A$ and $x \in [0,1]$ be such that $(a \Rightarrow \overline{c}^{\mathbf{A}}) \approx \overline{1}^{\mathbf{A}}$ for all $c \in [x,1] \cap \mathbb{Q}_*$ and $(\overline{d}^{\mathbf{A}} \Rightarrow b) \approx \overline{1}^{\mathbf{A}}$ for all $d \in [0,x] \cap \mathbb{Q}_*$. Since for any $e \in \mathbb{Q}_*$, either $e \leqslant x$ or $x < e$, it follows that either $(a \Rightarrow \overline{e}^{\mathbf{A}}) \approx \overline{1}^{\mathbf{A}}$ or $(\overline{e}^{\mathbf{A}} \Rightarrow b) \approx \overline{1}^{\mathbf{A}}$. Thus, for all $e \in \mathbb{Q}_*$, it holds that $(a \Rightarrow \overline{e}^{\mathbf{A}}) \vee (\overline{e}^{\mathbf{A}} \Rightarrow b) \approx \overline{1}^{\mathbf{A}}$. With this, the premises of the quasi-equation $\mathcal{D}^\infty$ are met, so we can conclude that $(a \Rightarrow b) \approx \overline{1}^{\mathbf{A}}$.

To check $2 \Rightarrow 1$, let us assume that $\mathbf{A} \not\models \mathcal{D}^\infty$, and let us prove that $\mathbf{A} \not\models \mathcal{R}_x^\infty$ for some $x \in [0,1]$.

Since $\mathbf{A} \not\models \mathcal{D}^\infty$ and $\mathbf{A}$ is linearly-ordered, there are $a, b \in A$ such that $a > b$, but such that for each $c \in \mathbb{Q}_*$ either $a \leqslant \overline{c}^{\mathbf{A}}$ or $\overline{c}^{\mathbf{A}} \leqslant b$. This implies that $\inf\{c \in \mathbb{Q}_* : a \leqslant \overline{c}^{\mathbf{A}}\} = \sup\{c \in \mathbb{Q}_* : \overline{c}^{\mathbf{A}} \leqslant b\}$. Let x be this value, and observe that for each $c \in (x,1] \cap \mathbb{Q}_*$, (i.e., with $c > x = \inf\{c \in \mathbb{Q}_* : a \leqslant \overline{c}^{\mathbf{A}}\}$) it holds that $a \leqslant \overline{c}^{\mathbf{A}}$, so $a \Rightarrow \overline{c}^{\mathbf{A}} \approx \overline{1}^{\mathbf{A}}$. Similarly, for each $d \in [0,x) \cap \mathbb{Q}_*$ (i.e., with $d < x = \sup\{c \in \mathbb{Q}_* : \overline{c}^{\mathbf{A}} \leqslant b\}$) it holds that $\overline{d}^{\mathbf{A}} \Rightarrow b \approx \overline{1}^{\mathbf{A}}$. Therefore, $\mathbf{A} \not\models \mathsf{R}^\infty_x$. □

Next lemma shows a case where, for arbitrary algebras, the system resulting from the addition of D^∞ is equivalent to the one obtained using all the previous inference rules. The assumptions of this lemma will provide us with some constraints that will help in our search for a counterexample to the equivalence between D^∞ and $\{\mathsf{R}^\infty_x\}_{x\in[0,1]}$.

Lemma 5.3 *Let $\mathbf{A}$ be a L_*-algebra (not necessarily a chain), and let us assume that $\mathbf{A}$ has a subdirect representation as a subalgebra of a direct product $\prod_{i\in I} \mathbf{A_i}$ of subdirectly irreducible (and thus linearly-ordered) algebras $\mathbf{A_i}$, such that for every $i \in I$, the element $\vec{e_i}$ given by*

$$\vec{e_i}[j] = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{otherwise} \end{cases}$$

belongs to A. Then, the following statements are equivalent:

1. *$\mathbf{A}$ validates $\mathcal{D}^\infty$*
2. *For all $x \in [0,1]$, $\mathbf{A}$ validates R^∞_x.*

Proof It is enough to prove that $2 \Rightarrow 1$ because in Section 4 we already showed that the density rule provides an strongly complete axiomatization. Assume that $\mathbf{A} \not\models \mathcal{D}^\infty$. Then, there is some $i \in I$ such that $\mathbf{A_i} \not\models \mathcal{D}^\infty$, because the class of algebras is a generalised quasi-variety and the generalized quasi-equations are preserved under direct products and subalgebras. By linearity and Lemma 5.1, if $\mathbf{A_i} \not\models \mathcal{D}^\infty$ then $\mathbf{A_i} \not\models \mathcal{R}^\infty_x$ for some $x \in [0,1]$ (we have seen that, over the linearly-ordered algebras, these two sets of rules are interderivable). Then, since $\mathbf{A_i} \not\models \mathcal{R}^\infty_x$ and $\vec{e_i} \in A$, then it is clear that there exists a substitution σ such that $\vec{e_i} \to \sigma(\gamma) = 1$ for each γ in the premises of $\mathcal{R}^\infty_x$, while $\vec{e_i} \to \sigma(\delta)$ for δ being the consequence of the same rule. But this means that $\mathbf{A} \models (\neg\Delta^{\mathbf{A}}\vec{e_i} \vee \sigma(\gamma)) \approx \overline{1}^{\mathbf{A}}$, but $\mathbf{A} \not\models (\neg\neg\Delta^{\mathbf{A}}\vec{e_i} \vee \sigma(\delta)) \approx \overline{1}^{\mathbf{A}}$, having that $\mathcal{R}^\infty_x$ does not hold in $\mathbf{A}$. □

We can now provide the counterexample to the equivalence between D^∞ and $\{\mathsf{R}^\infty_x\}_{x\in[0,1]}$. As expected, such counterexample will not satisfy the assumptions in Lemma 5.3. Indeed, the construction we do is inspired by the construction of the unique, up to isomorphism, countable atomless Boolean algebra (see for instance [12, Chapter 16]).

Lemma 5.4 *Let $*$ be the Gödel t-norm. Then, there is a L_*-algebra $\mathbf{A}$ such that $\mathbf{A} \models \mathcal{R}^\infty_x$ for all $x \in [0,1]$ while $\mathbf{A} \not\models \mathcal{D}^\infty$.*

Proof Let I be the interval $[0,1)_\mathbb{Q}$. For every $q \in I$ we consider the Gödel-chain $\mathbf{A_q}$ defined by:

- the universe is $[0,1]_\mathbb{Q}$ enlarged with a new element $\widetilde{q}$.
- the universe is linearly-ordered with the expansion of the linear order among rational numbers such that $\widetilde{q}$ is strictly between the elements in $[0,q]_\mathbb{Q}$ and $(q,1]_\mathbb{Q}$. In other words, $\widetilde{q}$ is the sucessor (next element) of q. We emphasize that we do not consider $\widetilde{q}$ as a rational element.
- the operations of the Δ-Gödel chain $\mathbf{A_q}$ are the ones determined by the linear order in the previous item.
- for every $c \in \mathbb{Q}_*$, the interpretation of the constant $\bar{c}$ in $\mathbf{A_q}$ is the rational number c.

It is worth noticing that all chains $\mathbf{A_q}$ (with $q \in I$) satisfy that

- for every $x \in [0,1]_\mathbb{R} \setminus \{q\}$ there are no elements $a, b \in A_q$ which simultaneously satisfy 1) $a \leqslant c$ for every $c \in (x,1]_\mathbb{Q}$ and 2) $c \leqslant b$ for all $c \in [0,x)_\mathbb{Q}$ and 3) $b < a$. In other words, for every $x \in [0,1] \setminus \mathbb{Q}_*$, it holds that $\mathbf{A_q} \models \mathcal{R}^\infty_x$.
- $\mathbf{A_q} \not\models \mathcal{R}_q$. Indeed, there is only one pair of elements $a, b \in A_q$ which simultaneosly satisfy 1) $a \leqslant c$ for every $c \in (x,1]_\mathbb{Q}$ and 2) $c \leqslant b$ for all $c \in [0,x)_\mathbb{Q}$ and 3) $b < a$. Such a pair is the one given by $x := \widetilde{q}$ and $y := q$.

By Lemma 5.3 it is obvious that the direct product $\mathbf{B} := \prod_{q\in I} \mathbf{A_q}$ is not an algebra such that $\mathbf{B} \models \mathcal{R}^\infty_x$ for all $x \in [0,1]$ and $\mathbf{B} \not\models \mathcal{D}^\infty$.

Next we define $\mathbf{A}$ as the subalgebra of $\mathbf{B}$ whose universe is given by the elements $f \in B$ (seen as maps from I) such that there is a finite sequence $q_0 < q_1 < q_2 < \ldots < q_{n+1}$ of rational numbers with

- $q_0 := 0$ and $q_{n+1} := 1$ (and $n \in \mathbb{N}$),
- for $0 \leqslant i \leqslant n$, $f \restriction [q_i, q_{i+1})$ is either a constant function given by a rational number or the function given by $f(q) = \widetilde{q}$ or the function given by $f(q) = q$.

It is quite simple[9] to verify that such set A is closed under all operations, and so A is the support of a MTL_Δ-chain $\mathbf{A}$.

[9] For the reader interested in checking the details we suggest to start considering the following three elements in $\mathbf{A}$:

$$t_1 := (\tfrac{1}{2})_{q\in[0,1)_\mathbb{Q}} \quad t_2 := (\widetilde{q})_{q\in[0,1)_\mathbb{Q}} \quad t_3 := (q)_{q\in[0,1)_\mathbb{Q}}$$

It is worth noticing here that such an algebra $\mathbf{A}$ has the subdirect product representation given by $\mathbf{A} \subseteq \prod_{q\in I} \mathbf{A_q}$ (i.e., all projections are surjective), and that for every $q \in I$ the element e_i considered in Lemma 5.3 does not belong to $\mathbf{A}$. Thus, the assumptions in Lemma 5.3 do not hold for this particular algebra $\mathbf{A}$.

Next we check the following claims.

– $\mathbf{A}$ *is not a model of* $\mathcal{D}^\infty$. To show this, let us consider the elements $s, t \in A$ defined by $s \coloneqq (\widetilde{q})_{q\in I}$ and $t \coloneqq (q)_{q\in I}$. It is obvious that $s \Rightarrow t = t \neq \overline{1}^{\mathbf{A}}$. Moreover, for every $c \in \mathbb{Q}_*$ it holds that $s \Rightarrow \overline{c}^{\mathbf{A}}$ and $\overline{c}^{\mathbf{A}} \Rightarrow t$ are the elements given respectively by

$$(s \Rightarrow \overline{c}^{\mathbf{A}})(q) \coloneqq \begin{cases} 1, & \text{if } q \in [0,c)_{\mathbb{Q}} \\ c, & \text{if } q \in [c,1)_{\mathbb{Q}} \end{cases}$$

$$(\overline{c}^{\mathbf{A}} \Rightarrow t)(q) \coloneqq \begin{cases} q, & \text{if } q \in [0,c)_{\mathbb{Q}} \\ 1, & \text{if } q \in [c,1)_{\mathbb{Q}}. \end{cases}$$

Therefore, for every $c \in \mathbb{Q}_*$ it holds that $(s \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow t) = \overline{1}^{\mathbf{A}}$. Thus, $\mathbf{A} \not\models \mathcal{D}^\infty$ under the interpretation sending φ to the element s and ψ to the element t.

– *For every* $x \in [0,1] \setminus \mathbb{Q}_*$*, it holds that* $\mathbf{A} \models \mathcal{R}^\infty_x$. This is trivial because all algebras in $\{\mathbf{A_q} : q \in I\}$ validate such generalized quasi-equation $\mathcal{R}^\infty_x$.

– *For every* $r \in \mathbb{Q}_*$*, it holds that* $\mathbf{A} \models \mathcal{R}^\infty_r$. Instead of directly proving $\mathbf{A} \models \mathcal{R}^\infty_r$ we will focus on proving $\mathbf{A} \models \mathcal{R}^1_r$ and $\mathbf{A} \models \mathcal{R}^2_r$, where

$$\mathcal{R}^1_r \coloneqq [\bigwedge_{c\in(r,1]_{\mathbb{Q}}} (x \to \overline{c}) \approx \overline{1}\,] \quad \Longrightarrow \quad [(x \to \overline{r}) \approx \overline{1}]$$

$$\mathcal{R}^2_r \coloneqq [\bigwedge_{c\in[0,r)_{\mathbb{Q}}} (\overline{c} \to x) \approx \overline{1}\,] \quad \Longrightarrow \quad [(\overline{r} \to x) \approx \overline{1}].$$

This is enough because it is very easy (by a trivial combinatorial argumentation) to show that if $\mathbf{A} \models \mathcal{R}^1_r$ and $\mathbf{A} \models \mathcal{R}^2_r$, then $\mathbf{A}$ is also a model of

$$[\textstyle\bigwedge_{c\in(r,1]_{\mathbb{Q}}}(x \to \overline{c}) \approx \overline{1}] \wedge [\bigwedge_{c\in[0,r)_{\mathbb{Q}}}(\overline{c} \to y) \approx \overline{1}]$$

$$\Longrightarrow \quad [(x \to \overline{r}) \approx \overline{1}] \wedge [(\overline{r} \to y) \approx \overline{1}],$$

and so $\mathbf{A} \models \mathcal{R}^\infty_r$.

Let us fix a rational number $r \in [0,1]_{\mathbb{Q}}$, and next we prove that $\mathbf{A} \models \mathcal{R}^1_r$ and $\mathbf{A} \models \mathcal{R}^2_r$ by cases.

Case $x \in A$ such that $x \leqslant r\uparrow$: [10] We need to show that $\mathbf{A} \models \mathcal{R}^1_r$, i.e., that $x \leqslant \overline{r}$. We will check this showing that for each one of the rational intervals $[q_i, q_{i+1})$ determined by the element $x \in A$, it holds that $x \restriction [q_i, q_{i+1})$ is less or equal than $\overline{r} \restriction [q_i, q_{i+1})$. The fact that $x \leqslant r\uparrow$ tells us that in each one of the intervals $[q_i, q_{i+1})$ one of the following conditions hold:
  – $x \restriction [q_i, q_{i+1})$ is a rational constant function whose values is $\leqslant r$,
  – $x \restriction [q_i, q_{i+1})$ is a function given by $q \mapsto \widetilde{q}$, and moreover $q_{i+1} \leqslant r$
  – $x \restriction [q_i, q_{i+1})$ is a function given by $q \mapsto q$, and moreover $q_{i+1} \leqslant r$.

In all three cases, using that q_{i+1} is not an element of the interval $[q_i, q_{i+1})$, it follows that $x \restriction [q_i, q_{i+1})$ is less or equal than $\overline{r} \restriction [q_i, q_{i+1})$.

Case $x \in A$ such that $r \downarrow\leqslant x$: We need to show that $\mathbf{A} \models \mathcal{R}^2_r$, that is to say, that $\overline{r} \leqslant x$. We will do this showing that for each one of the rational intervals $[q_i, q_{i+1})$ determined by the element $x \in A$, it holds that $\overline{r} \restriction [q_i, q_{i+1})$ is less or equal than $x \restriction [q_i, q_{i+1})$. The fact that $r \downarrow\leqslant x$ tells us that in each one of the intervals $[q_i, q_{i+1})$ one of the following conditions hold:
  – $x \restriction [q_i, q_{i+1})$ is a rational constant function whose values is $\geqslant r$,
  – $x \restriction [q_i, q_{i+1})$ is a function given by $q \mapsto \widetilde{q}$, and moreover $q_i \geqslant r$
  – $x \restriction [q_i, q_{i+1})$ is a function given by $q \mapsto q$, and moreover $q_i \geqslant r$.

In all three cases it holds that $x \restriction [q_i, q_{i+1})$ is greater or equal than $\overline{r} \restriction [q_i, q_{i+1})$.

This finishes the proof that $\mathbf{A} \models \mathcal{R}^\infty_r$ for the case that r is rational.

Therefore, we have just seen that $\mathbf{A} \not\models \mathcal{D}^\infty$ while $\mathbf{A} \models \mathcal{R}^\infty_x$ for all $x \in [0,1]$. □

Now, we are ready to provide the promised counterexample.

Corollary 5.5 *The logic* $\mathsf{L}^+_{\mathsf{G}}$ *is* $\vee$*-closed and not semilinear.*

Proof The failure of semilinearity is obtained from the previous lemma, noticing that D^∞ is not derivable in $\mathsf{L}^+_{\mathsf{G}}$, while the quasi-equation $\mathcal{D}^\infty$ is valid in all $\mathsf{L}^+_{\mathsf{G}}$-chains. □

Therefore, $\mathsf{L}^+_{\mathsf{G}}$ is different from than the logic L^∞_* where $*$ is the minimum t-norm. Although $\mathsf{L}^+_{\mathsf{G}}$ does not axiomatize the logic arising from the standard Gödel algebra with constants using inference rules from $\{\mathsf{R}^\infty_x\}_{x\in[0,1]}$, these rules are enough in order to axiomatize some logics of left-continuous t-norms. Let us notice that these rules have a quite different structure from the

and checking that all possible combinations of these three elements under $\wedge, \vee, \to, \Delta$ are also elements in our universe A. Indeed, all difficulties to provide a general proof that A is closed under the operations are illustrated in the previous particular case.

[10] With the notation $x \leqslant r\uparrow$ we mean that $x \leqslant \overline{c}^{\mathbf{A}}$ for every $c \in (r,1]_{\mathbb{Q}}$. In an analogous way, $r \downarrow\leqslant x$ stands for $\overline{c}^{\mathbf{A}} \leqslant x$ for all $c \in [0,r)_{\mathbb{Q}}$.

density rule (they are based on the conjunction operation instead of in the disjunction), and next we provide some results for these rules in the case of an arbitrary left continuous t-norm. First, it is easy to obtain the following consequence of Lemma 5.2.

Lemma 5.6 *Let $*$ be a left-continuous t-norm whose residuum has up to a countable amount of discontinuity points on the diagonal. Then, $\models_{[\mathbf{0},\mathbf{1}]_*}$ can be axiomatized by adding to the finitary companion of L_*^∞ a countable subset of rules from $\{\mathsf{R}_x^\infty\}_{x\in[0,1]}$.*

Proof It can be shown equivalently proven that for any linearly-ordered L_*-algebra $\mathbf{A}$, if $\mathbf{A}\models\mathcal{R}_1$ and $\mathbf{A}\models\mathcal{R}_x^\infty$ for all x such that $\langle x,x\rangle$ is a discontinuity point of $\Rightarrow_*$, then $\mathbf{A}\models\mathcal{R}_y$ for all $y\in[0,1]$.

Observe first that $\varphi\to\psi,\chi\to\delta\vdash_{\mathsf{MTL}}(\psi\to\chi)\to(\varphi\to\psi)$, so it is also true in any of its expansions and thus in $\mathbf{A}$. Let $u\in(0,1]$ and take $a,b\in A$ such that $(a\Rightarrow\overline{c}^{\mathbf{A}})=\overline{1}^{\mathbf{A}}$ for all $c>u$ and $(\overline{d}^{\mathbf{A}}\Rightarrow b)=\overline{1}^{\mathbf{A}}$ in $\mathbf{A}$ for all $d<u$. If u was a discontinuity point for $\Rightarrow_*$ the rule was satisfied by assumption. Otherwise, from the previous observation we have that $((\overline{c}^{\mathbf{A}}\Rightarrow\overline{d}^{\mathbf{A}})\Rightarrow(a\Rightarrow b))=\overline{1}^{\mathbf{A}}$. But then, since u was not a discontinuity point of $\Rightarrow_*$, and given that $(u\Rightarrow_* u)=1$, then it holds that $\sup\{c\Rightarrow_* d: c>u>d\}=1$. That is to say, for each $r<1$, there are $c>u>d$ such that $r<c\Rightarrow_* d$. Using the book-keeping axioms, we get that $(\overline{r}^{\mathbf{A}}\Rightarrow(a\Rightarrow b))=\overline{1}^{\mathbf{A}}$ for all $r<1$. Then, using $\mathcal{R}_1$, we have that $(a\Rightarrow b)=\overline{1}^{\mathbf{A}}$.
If $u=0$, and $\langle 0,0\rangle$ is not a discontinuity point. Take $a\in A$ such that $a\Rightarrow\overline{c}^{\mathbf{A}}=\overline{1}^{\mathbf{A}}$ for all $c\in\mathbb{Q}_*\setminus\{0\}$. From the previous observation, and using that $\overline{0}\to\overline{0}$ is a theorem, we have that $((\overline{c}^{\mathbf{A}}\Rightarrow\overline{0}^{\mathbf{A}})\Rightarrow(a\Rightarrow\overline{0}^{\mathbf{A}}))=\overline{1}^{\mathbf{A}}$ for all $c\in\mathbb{Q}_*\setminus\{0\}$. Since $\langle 0,0\rangle$ is not a discontinuity point we know that $\sup\{c\Rightarrow_* 0: c\in\mathbb{Q}_*\setminus\{0\}\}=\overline{1}^{\mathbf{A}}$. Then, we have that $(\overline{r}^{\mathbf{A}}\Rightarrow(a\Rightarrow\overline{0}^{\mathbf{A}}))=\overline{1}^{\mathbf{A}}$ for all $r<1$, and again by $\mathcal{R}_1$ we can conclude that $\neg a=\overline{1}^{\mathbf{A}}$. □

While the only two continuous t-norms that belong to the previous class are the Łukasiewicz and the Product t-norms, we do not know whether there are left-continuous (non-continuous) t-norms that verify the previous statement. A different class of left-continuous t-norms that can be axiomatized using rules from L_*, that for what concerns the continuous t-norms is larger than the previous one, is formed by all the ordinal sums of Product and Łukasiewicz components.

Lemma 5.7 *Let $*$ be a continuous t-norm. Then $\models_{[\mathbf{0},\mathbf{1}]_*}$ can be axiomatized by adding to the finitary companion of L_*^∞ a countable subset of rules from $\{\mathsf{R}_x^\infty\}_{x\in[0,1]}$ if and only if it is an ordinal sum of only Łukasiewicz and Product t-norms.*

Proof One direction follows from Lemma 5.6. For the other direction, let L be the finitary companion of L_*^∞. It is worth pointing out that a concrete axiomatization of L can be easily obtained from [11].

If there is some Gödel component in the construction of $*$, then it is possible to build a counterexample like Lemma 5.4. It can be done analogously by choosing the interval I from the beginning of the proof to be inside the Gödel component. Let us consider the axiomatic system given by the expansion of L with the following axioms and rules, associated to the idempotent elements of the standard algebra:

- For each $b\in\{x\in[0,1]: x*x=x\}$ the axiom
$$(\Delta(\overline{b}\to(\varphi\to\psi))\wedge\Delta(\varphi\to\overline{b}))\to(\varphi\to\psi),$$
- For each $b\in\{x\in[0,1]: x*x=x\}$, the rule $\mathsf{R_b}$, i.e.
$$\frac{\{(\varphi\to\overline{c})\wedge(\overline{d}\to\psi): c\in(b,1]\cap\mathbb{Q}_*, d\in[0,b)\cap\mathbb{Q}_*\}}{\varphi\to\psi}$$

We know there are countably many components by definition of ordinal sum, and since the only idempotent elements of such a t-norm are the top and bottom elements of each component, using Corollary 3.6 we get that the previous axiomatic systems are strongly complete with respect to the linearly-ordered algebras of their corresponding algebraic companion. On the other hand, we can prove that any of these linearly-ordered algebras validates all the rules in $\{\mathsf{R}_x^\infty\}_{x\in[0,1]}$, which is enough to prove the lemma.

Let $\mathbf{A}$ be a linearly-ordered L-algebra that satisfies the previous axioms and rules schemata, and let $a,b\in A$ and $x\in[0,1]$ such that $((a\Rightarrow\overline{c}^{\mathbf{A}})\wedge(\overline{d}^{\mathbf{A}}\Rightarrow b))=\overline{1}^{\mathbf{A}}$ for all $c\in(x,1]\cap\mathbb{Q}_*, d\in[0,x)\cap\mathbb{Q}_*$. It holds that $b\leqslant x\leqslant t$ where b is the maximum idempotent element below x and t is the minimum idempotent element above x (which always exist in an ordinal sum of Łukasiewicz and Product components).

We are only interested in the case when x belongs to the interior of a component, since if it is an end point of a component, the corresponding infinitary rule holds by definition, so let $b<x<t$. From $((a\Rightarrow\overline{c}^{\mathbf{A}})\wedge(\overline{d}^{\mathbf{A}}\Rightarrow b))=\overline{1}^{\mathbf{A}}$ it follows that $((\overline{c}^{\mathbf{A}}\Rightarrow\overline{d}^{\mathbf{A}})\Rightarrow(a\Rightarrow b))=\overline{1}^{\mathbf{A}}$. From the behaviour of the residuum in an ordinal sum of Łukasiewicz and Product components, we know that, on the standard algebra it holds that
$$\sup\{c\Rightarrow_* d: c\in(x,1]\cap\mathbb{Q}_*, d\in[0,x)\cap\mathbb{Q}_*\}=t.$$
Then, from the book-keeping axioms, we get that $(\overline{r}^{\mathbf{A}}\Rightarrow(a\Rightarrow b))=\overline{1}^{\mathbf{A}}$ for all $r\in[0,t)\cap\mathbb{Q}_*$. Applying the

generalised quasi-equation arising from the inference rule $\mathsf{R_t}$ we get that $(\overline{t}^{\mathbf{A}} \Rightarrow (a \Rightarrow b)) = \overline{1}^{\mathbf{A}}$.

On the other hand, since $x < t$ and $(a \Rightarrow \overline{c}^{\mathbf{A}}) = \overline{1}^{\mathbf{A}}$ for all $c > x$, in particular $(a \Rightarrow \overline{t}^{\mathbf{A}}) = \overline{1}^{\mathbf{A}}$. Then, applying MP and the axiom we originally added to the system, we get that $(a \Rightarrow b) = \overline{1}^{\mathbf{A}}$. □

6 Expanding L_*^{∞} with representable operations

In this section we show that the approach developed in the previous sections to axiomatize the logics $\models_{[\mathbf{0},\mathbf{1}]_*}$ is applicable as well to the more general case of logics defined by standard algebras $[\mathbf{0}, \mathbf{1}]_*$ expanded with *an arbitrary set of operations* obeying some -but not very strict- regularity conditions. In some sense, in doing this, we follow the path already introduced by Pavelka in one of his three foundational papers [18] when he extended his (Łukasiewicz logic-based) formalism to account for additional *logically fitting* operations, i.e. operations in $[0, 1]$ satisfying some congruence-like conditions, also related to Lipschitz continuity conditions. Also, the approach we develop in this section complements the one by Cintula in [6], showing an alternative way of axiomatizing Pavelka-complete logics with an extended set of operations.

In what follows, given a set OP of operations in $[0, 1]$ we will consider a new language $L(OP)$ expanding the one of L_*^{∞} with a connective $\overline{f}$ for each operation $f \in OP$, with the corresponding arity, and with the necessary truth constants, i.e., the countable set of constants $\mathbb{Q}_*^{OP}$ defined by the subalgebra generated by $[0, 1]_{\mathbb{Q}}$ using the operations $*, \Rightarrow_*, \Delta$ and each $f \in OP$.

Accordingly, we consider the standard algebra

$$[\mathbf{0}, \mathbf{1}]_*^{\mathbf{OP}} = \langle [0, 1], *, \Rightarrow_*, \wedge, \Delta, \{f\}_{f \in OP}, \{c\}_{c \in \mathbb{Q}_*^{OP}} \rangle.$$

Our goal is to axiomatize $\models_{[\mathbf{0},\mathbf{1}]_*^{\mathbf{OP}}}$, that is defined analogously to $\models_{[\mathbf{0},\mathbf{1}]_*}$ in Definition 4.2. The first task is determining when this can be done using the tools we have developed up to now, that is to say, determining which kind of operations can be included in OP.

6.1 Representable operations

In [6] Cintula studies logics of standard MTL algebras with rational truth constants extended by an abitrary set of argument-wise monotonic operations, i.e. operations that, fixing all variables but one, result in monotonically increasing or decreasing one-place operations. Our approach allows us to partially generalize Cintula's approach, working with a family of operations with different restrictions. Namely, we rely on the density rule D^{∞} and the book-keeping axioms to fully determine the new additional operations on the whole real interval $[0, 1]$. Thus we will restrict ourselves to operations for which this approach is feasible, i.e. those operations whose images can be reached as limits of the values taken by the constants. These turn to be operations whose domain can be decomposed into a set of regions where the operation is argument-wise monotonic and either left or right continuous, and moreover these regions are such that can be determined with the language of the logic. However, in our approach we lose the capacity to work with some operations that can be dealt in [6]: for instance, operations that have jump-type discontinuity points for which, for some argument, the value of the function coincides neither with the left nor with the right limit. The reason is that we cannot deal with functions whose limit points cannot be reached through the rationals using the density rule D^{∞}. For instance, the unary operation $f : [0, 1] \to [0, 1]$ given by

$$f(x) := \begin{cases} x & \text{if } x \leqslant 0.5 \\ 1 - x & \text{otherwise} \end{cases}$$

can be considered in ours but not in Cintula's approach, while the unary operation f' defined as

$$f'(x) := \begin{cases} x/2 & \text{if } x < 0.5 \\ 0.5 & \text{if } x = 0.5 \\ 2x/3 & \text{otherwise} \end{cases}$$

can be considered using Cintula's formalism but not with the methods presented below.

Notation For the sake of a simpler notation, for any n-tuple $\overrightarrow{a} = \langle a_1, \ldots, a_n \rangle \in [0, 1]^n$ and value $b \in [0, 1]$, we will denote by $\langle \overrightarrow{a}, b \rangle_k$ the n-tuple

$$\langle a_1, \ldots, a_{k-1}, b, a_{k+1}, \ldots a_n \rangle$$

obtained by replacing a_k by b. We will also use this notation when referring to tuples of propositional variables or formulas, e.g. $\overrightarrow{\varphi}$ will denote a tuple of formulas $\langle \varphi_1, \ldots, \varphi_n \rangle$. Moreover, for $R \subseteq [0, 1]^n$, $\overrightarrow{a} \in [0, 1]^n$ and $1 \leqslant k \leqslant n$ we will denote by $\Pi_{\langle \overrightarrow{a}, k \rangle}(R)$ the projection over the k-th component of R fixing the other components to the values in $\overrightarrow{a}$, i.e.,

$$\Pi_{\langle \overrightarrow{a}, k \rangle}(R) := \{b \in [0, 1] : \langle \overrightarrow{a}, b \rangle_k \in R\}$$

Similarly given an n-ary operation $f : [0, 1]^n \to [0, 1]$, a tuple $\overrightarrow{a} \in [0, 1]^n$ and an index $1 \leqslant k \leqslant n$, we will denote by $f_{\langle \overrightarrow{a}, k \rangle}$ the unary function resulting from fixing all variables to the values in $\overrightarrow{a}$ except for the k-th variable, i.e.,

$$f_{\langle \overrightarrow{a}, k \rangle}(x) := f(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n).$$

Definition 6.1 A subset $R \subseteq [0,1]^n$ is called a *region* of a n-ary operation $f : [0,1]^n \to [0,1]$ if the following conditions hold:

1. R is *convex*, i.e., for any tuple $\vec{a} \in [0,1]^n$, for any values $0 \leqslant b_1 < b_2 \leqslant 1$ and for any index k, the following condition holds: if $\langle \vec{a}, b_1\rangle_k, \langle \vec{a}, b_2\rangle_k \in R$, then $\langle \vec{a}, b\rangle_k \in R$ for any $b_1 \leqslant b \leqslant b_2$.
2. For each $1 \leqslant k \leqslant n$ and any $\vec{a} \in R$, the function $f_{\langle \vec{a},k\rangle}(x)$ is *left or right continuous* (or completely continuous) in $\Pi_{\langle \vec{a},k\rangle}(R)$. We encode this information by letting

$$\delta^f_{\langle R,k\rangle} \coloneqq \begin{cases} 0 & \text{if } f_{\langle \vec{a},k\rangle}(x) \text{ is continuous } \forall \vec{a} \in R, \\ 1 & \text{else if } f_{\langle \vec{a},k\rangle}(x) \text{ is left-cont. } \forall \vec{a} \in R, \\ -1 & \text{else if } f_{\langle \vec{a},k\rangle}(x) \text{ is right-cont. } \forall \vec{a} \in R. \end{cases}$$

The component-wise continuity of f allows us to refer to the set of *interior* (with respect to the left and right continuity conditions) points of R as:

$$Int_f(R) \coloneqq R \setminus \bigcup_{1 \leqslant j \leqslant n} extr(R,j)$$

where $extr(R,j) \coloneqq$

$$\begin{cases} \emptyset & \text{if } \delta^f_{\langle R,j\rangle} = 0 \\ \{\langle \vec{a}, b\rangle_j \in R : b = \inf\{z : \langle \vec{a}, z\rangle_j \in R\}\} & \text{if } \delta^f_{\langle R,j\rangle} = 1 \\ \{\langle \vec{a}, b\rangle_j \in R : b = \sup\{z : \langle \vec{a}, z\rangle_j \in R\}\} & \text{if } \delta^f_{\langle R,j\rangle} = -1 \end{cases}$$

3. For each $1 \leqslant k \leqslant n$ and any $\vec{a} \in R$, the function $f_{\langle \vec{a},k\rangle}(x)$ is *monotonically increasing or decreasing* (or constant) in $\Pi_{\langle \vec{a},k\rangle}(Int_f(R))$. We encode this information by letting

$$\eta^f_{\langle R,k\rangle} \coloneqq \begin{cases} 0 & \text{if } f_{\langle \vec{a},k\rangle}(x) \text{ is constant } \forall \vec{a} \in R, \\ 1 & \text{else if } f_{\langle \vec{a},k\rangle}(x) \text{ is incr. } \forall \vec{a} \in R, \\ -1 & \text{else if } f_{\langle \vec{a},k\rangle}(x) \text{ is decr. } \forall \vec{a} \in R. \end{cases}$$

For future uses of the previous definitions, for any L^∞_*-formulas φ, ψ, we let $\varphi \to^1 \psi = \varphi \to^0 \psi = \varphi \to \psi$, while we let $\varphi \to^{-1} \psi = \psi \to \varphi$. This will allow us to propose a uniform definition of the axiomatic systems associated to expanded standard MTL-algebras.

The operations that we will be able to consider are those whose universe can be split into at most countably-many regions expressible (by means of a set of formulas) in the non-expanded logic L^∞_*. We recall that in the following definition **Fm** stands for the algebra of formulas of this logic.

Definition 6.2 Let $*$ be a left-continuous t-norm. We say that a n-ary function $f\colon [0,1]^n \to [0,1]$ is $*$-*representable* (or *representable in* L^∞_*) when there is an at most countable set of regions $\{R_i\}_{i\in I}$ of f such that:

1. The set $\{Int_f(R_i)\}_{i\in I}$ covers the non-rational elements of $[0,1]^n$, i.e., for all $\vec{x} \in [0,1]^n \setminus (\mathbb{Q}_*)^n$ there is $i \in I$ such that $\vec{x} \in Int_f(R_i)$.
2. One-dimensional components of the regions are rationals, i.e., for any R_i and $1 \leqslant k \leqslant n$ such that $\{y : \exists \vec{x} \text{ s.t } \langle \vec{x}, y\rangle_k \in R_i\} = \{y_0\}$ then $y_0 \in \mathbb{Q}_*$.
3. For each $i \in I$, there is a (possibly infinite) set of n-ary (characteristic) formulas $\Upsilon^f_i(p_1, \ldots, p_n) \subseteq Fm$ such that, for any n-tuple of formulas $\vec{\varphi} = \langle \varphi_1, \ldots, \varphi_n\rangle$ and any $e \in Hom(\mathbf{Fm}, [\mathbf{0}, \mathbf{1}]_*)$,

$$e(\Upsilon^f_i(\vec{\varphi})) \subseteq \{1\} \iff \overrightarrow{e(\varphi)} \in Int_f(R_i).^{11}$$

In such a case, we say that $\{R_i\}_{i\in I}$ is a *representable universe of* f.

If f has multiple representable universes, we will arbitrarily fix one of them and refer to it as *the* representable universe of f, since different representable universes would simply result in different but equivalent axiomatic systems.

An example of a family of representable operations is given by those mappings that have up to a countable set of regions $\{R_i\}_{i\in I}$ covering $[0,1]^n$ such that each R_i is a product of n intervals in $[0,1]$. In this case, if the bounds of each interval are elements from $\mathbb{Q}^{OP}_*$ (and so, they have a corresponding truth constant in the language), it is immediate to characterize the region, using that the well-known facts that $e(\varphi \to \psi) = 1$ if and only if $e(\varphi) \leqslant e(\psi)$, and that $e(\neg\Delta(\varphi \to \psi)) = 1$ if and only if $e(\psi) < e(\varphi)$. On the other hand, for an interval with any of its bounds not in $\mathbb{Q}^{OP}_*$ the interval can be characterized using an infinite set of terms, using the fact that the set of constants is dense in $[0,1]$. For instance, the condition $z \leqslant e(\varphi)$ can be equivalently expressed by the set of conditions $\{e(\overline{c} \to \varphi) = 1 : c \in [0,z] \cap \mathbb{Q}^{OP}_*\}$.

However, not only this kind of functions is representable. For instance, any binary operation that has as regions the sets $\{\langle x,y\rangle : x \leqslant y\}$ and $\{\langle x,y\rangle : x > y\}$ is clearly representable, it is enough to consider the set of characteristic functions $\{\varphi_1 \to \varphi_2, \neg\Delta(\varphi_1 \to \varphi_2)\}$ and adjust the boundaries depending on the continuity type. Operations whose universe can be expressed by combining these kind of regions with intervals also yield representable functions.

Aiming towards a better comprehension of further definitions and results, we will focus on an example of a quite simple representable operation, that nevertheless covers different cases without adding too many unnecessary complications. In the following, in order to simplify the notation, if f is $*$-representable we will write $\delta^f_{\langle i,k\rangle}$ ($\eta^f_{\langle i,k\rangle}$ resp.) instead of $\delta^f_{\langle R_i,k\rangle}$ ($\eta^f_{\langle R_i,k\rangle}$ resp.).

[11] As expected, $\Upsilon^f_i(\vec{\varphi})$ stands for $\{\lambda(\vec{\varphi})\}_{\lambda\in\Upsilon_i}$, and $\overrightarrow{e(\varphi)} = \langle e(\varphi_1), \ldots e(\varphi_n)\rangle$.

Example 6.3 Let g be a binary representable operation whose representable universe is given by the set of regions $\{U_1, U_2\}$, with

$$U_1 = [0,1] \times [0,b] \text{ and } U_2 = [0,1] \times [b,1]$$

and with

$$\begin{cases}\delta^g_{\langle 1,1\rangle} = 1, & \eta^g_{\langle 1,1\rangle} = 1 \\ \delta^g_{\langle 1,2\rangle} = 0, & \eta^g_{\langle 1,2\rangle} = -1\end{cases} \text{ and } \begin{cases}\delta^g_{\langle 2,1\rangle} = 1, & \eta^g_{\langle 2,1\rangle} = -1 \\ \delta^g_{\langle 2,2\rangle} = -1, & \eta^g_{\langle 2,2\rangle} = -1\end{cases}$$

Observe that the interior points of each region are

$$Int_g(U_1) = (0,1] \times [0,b] \qquad Int_g(U_2) = (0,1] \times [b,1),$$

and so, the characteristic functions for this operation (the sets Υ^f_i from Definition 6.2) are given by:

$\Upsilon^g_1(\varphi_1, \varphi_2) \coloneqq \{\neg\Delta\neg\varphi_1, \varphi_2 \to \bar{b}\}$,
$\Upsilon^g_2(\varphi_1, \varphi_2) \coloneqq \{\neg\Delta\neg\varphi_1, \bar{b} \to \varphi_2, \neg\Delta\varphi_2\}$. □

6.2 Inference rules for representable operations

We shall now study which inference rules have to added to $\mathrm{Ł}^\infty_*$ in order to axiomatize $\models_{[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_*}$, with OP being a countable set of representable operations. We first consider three types of inference rules and later we will use their $\vee$-closures for the definition of an axiomatic system (so we will be able to easily resort to Theorem 3.6).

First of all, since $\models_{[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_*}$ is an implicative logic, for every $f \in OP$, we need to add to our axiomatic system, the following congruence rule for the corresponding connective $\overline{f}$:

$$\mathsf{CONG^f} : \frac{\varphi_1 \leftrightarrow \psi_1, \ldots, \varphi_n \leftrightarrow \psi_n}{\overline{f}(\varphi_1, \ldots, \varphi_n) \to \overline{f}(\psi_1, \ldots, \psi_n)}.$$

Besides these rules, for each operation $f \in OP$, we need to consider two new families of rules in order to control the behaviour of the operation f on the 'non-rational" elements of $[0,1]$ (i.e. elements that do not have a corresponding truth-constant in $\mathbb{Q}^{OP}_*$). One family of rules will cope with the monotonicity properties of the operation and the other with the continuity conditions. While the latter ones are intrinsically infinitary, the infinitary status of the former ones depends on the cardinality of the sets Υ^f_i of formulas that define the regions of the operation f.

Formally, the rules that characterize the monotonicity of a n-ary operation f are of the following form: for each region R_i of its representable universe, with defining formulas Υ^f_i, and each index $1 \leqslant k \leqslant n$, if we let $\eta = \eta^f_{\langle i,k\rangle}$, we introduce the following rule:

$$\mathsf{M^f_{\langle i,k\rangle}} : \frac{\Upsilon^f_i(\overrightarrow{\varphi}), \Upsilon^f_i(\langle \overrightarrow{\varphi}, \psi\rangle_k), \varphi_k \to^\eta \psi}{\overline{f}(\overrightarrow{\varphi}) \to \overline{f}(\langle \overrightarrow{\varphi}, \psi\rangle_k)}$$

This rule expresses the increasing or decreasing behaviour of the function in the region R_i along the component k. This is determined by stating that, if the function is increasing (so $\to^\eta = \to$) then, for two elements $\overrightarrow{v_1}, \overrightarrow{v_2}$ in the region (that is, satisfying Υ^f_i) that are unequal only in the k-th component with $\overrightarrow{v_1}[k] \leqslant \overrightarrow{v_2}[k]$, we have $\overline{f}(\overrightarrow{v_1}) \leqslant \overline{f}(\overrightarrow{v_2})$ as well (and vice-versa if f is decreasing).

Example 6.4 [*Monotonicity rules*] Following Example 6.3, we have the following four monotonicity rules for g (one for each pair region-component):

$$\mathsf{M^g_{\langle 1,1\rangle}} : \frac{\Upsilon^g_1(\varphi_1, \varphi_2), \Upsilon^g_1(\psi, \varphi_2), \psi \to \varphi_1}{\overline{g}(\psi, \varphi_2) \to \overline{g}(\varphi_1, \varphi_2)}$$

$$\mathsf{M^g_{\langle 1,2\rangle}} : \frac{\Upsilon^g_1(\varphi_1, \varphi_2), \Upsilon^g_1(\varphi_1, \psi), \varphi_2 \to \psi}{\overline{g}(\varphi_1, \psi) \to \overline{g}(\varphi_1, \varphi_2)}$$

$$\mathsf{M^g_{\langle 2,1\rangle}} : \frac{\Upsilon^g_2(\varphi_1, \varphi_2), \Upsilon^g_2(\psi, \varphi_2), \varphi_1 \to \psi}{\overline{g}(\psi, \varphi_2) \to \overline{g}(\varphi_1, \varphi_2)}$$

$$\mathsf{M^g_{\langle 2,2\rangle}} : \frac{\Upsilon^g_2(\varphi_1, \varphi_2), \Upsilon^g_2(\varphi_1, \psi), \varphi_2 \to \psi}{\overline{g}(\varphi_1, \psi) \to \overline{g}(\varphi_1, \varphi_2)}$$

□

On the other hand, we also need rules that account for the continuity properties of every function $f \in OP$ in the regions of its representable universe. This can be done by encoding into the axiomatic system some of the information about the operation, namely its behaviour on the limit points. For this we need to resort to infinitary rules. The intuitive meaning of the rules we introduce below is to capture the fact that, in the interior of a region, the value of a function f in a given point can be determined by the values on rational points in the vicinity (along the direction in which the function is continuous).

Formally, for each region R_i of f, with defining formulas Υ^f_i, and each component $1 \leqslant k \leqslant n$, we let $\eta = \eta^f_{\langle i,k\rangle}$ and $\delta = \delta^f_{\langle i,k\rangle}$, and introduce the following rule:

$$\mathsf{C^f_{\langle i,k\rangle}} : \frac{\begin{array}{c}\Upsilon^f_i(\overrightarrow{\varphi}), \Upsilon^f_i(\langle \overrightarrow{\varphi}, \beta\rangle_k), \Upsilon^f_i(\langle \overrightarrow{\varphi}, \gamma\rangle_k), \\ \beta \to \varphi_k, \varphi_k \to \gamma, \neg\Delta(\gamma \to \beta), \\ \psi \to^{\eta\cdot\delta} \overline{f}(\overrightarrow{\varphi}), \\ \{\Delta((\beta \to \overline{c}) \wedge (\overline{c} \to \gamma)) \to \\ ((\overline{f}(\langle \overrightarrow{\varphi}, \overline{c}\rangle_k) \to^{\eta\cdot\delta} \psi) \vee (\varphi_k \to^\delta \overline{c})) : c \in \mathbb{Q}_*\}\end{array}}{\overline{f}(\overrightarrow{\varphi}) \to^{\eta\cdot\delta} \psi}$$

Let us explain what the premises of the previous rule stand for when we interpret them in an standard algebra. $\overrightarrow{\varphi}$ represents the point where we want to approximate the value of f. The premises in the first line are just stating that $\overrightarrow{\varphi}, \langle \overrightarrow{\varphi}, \beta\rangle_k$ and $\langle \overrightarrow{\varphi}, \gamma\rangle_k$ belong to $Int_f(R_i)$. The second line places β and γ around φ_k in such a way that $\beta \leqslant \varphi_k \leqslant \gamma$ and $\beta < \gamma$.[12] The third line simply states that ψ represents a value that is ordered with respect to $f(\vec{\varphi})$ depending on the monotonicity and continuity properties of f on that region. For instance, for a left-continuous and increasing component (and for a right-continuous and decreasing one), it states a $\leqslant$ relation, while if we consider a decreasing left-continuous component (increasing right-continuous), the relation will change to $\geqslant$. Observe that, in doing so, we have that ψ is somehow horizontally *cutting* the function along the component k (if ψ is near enough to $f(\vec{\varphi})$).

Finally, the last two lines of the premises of the rule represent an infinitary condition (quantified over the truth constants). It encodes that for every constant $\bar{c}$ between β and γ, then either $\bar{c}$ is above φ_k in the continuity sense or $f(\langle\vec{\varphi}, \bar{c}\rangle)$ is below ψ in the monotony-continuity sense used before. The conclusion of the rule is therefore that that the value of the function f at the point $\vec{\varphi}$ can be approximated by the values on rational constants near the point. Figure 1 shows the intuition behind the premises of the previous rule for two simple examples over 1-dimensional functions.

Example 6.5 [*Continuity rules*] Following Examples 6.3 and 6.4, the following are the continuity rules for the operation g, one for each pair region-component:

$$\mathsf{C}^{\mathsf{g}}_{\langle 1,1\rangle}: \frac{\begin{array}{c}\Upsilon^g_1(\varphi_1,\varphi_2), \Upsilon^g_1(\beta,\varphi_2), \Upsilon^g_1(\gamma,\varphi_2),\\ \beta \to \varphi_1, \varphi_1 \to \gamma, \neg\Delta(\gamma \to \beta),\\ \psi \to \overline{g}(\varphi_1,\varphi_2),\\ \{\Delta((\beta \to \bar{c}) \wedge (\bar{c} \to \gamma)) \to\\ ((\overline{g}(\bar{c},\varphi_2) \to \psi) \vee (\varphi_1 \to \bar{c})) : c \in \mathbb{Q}_*\}\end{array}}{\overline{g}(\varphi_1,\varphi_2) \to \psi}$$

$$\mathsf{C}^{\mathsf{g}}_{\langle 1,2\rangle}: \frac{\begin{array}{c}\Upsilon^g_1(\varphi_1,\varphi_2), \Upsilon^g_1(\varphi_1,\beta,), \Upsilon^g_1(\varphi_1,\gamma),\\ \beta \to \varphi_2, \varphi_2 \to \gamma, \neg\Delta(\gamma \to \beta),\\ \overline{g}(\varphi_1,\varphi_2) \to \psi,\\ \{\Delta((\beta \to \bar{c}) \wedge (\bar{c} \to \gamma)) \to\\ ((\psi \to \overline{g}(\varphi_1,\bar{c})) \vee (\varphi_2 \to \bar{c})) : c \in \mathbb{Q}_*\}\end{array}}{\psi \to \overline{g}(\varphi_1,\varphi_2)}$$

$$\mathsf{C}^{\mathsf{g}}_{\langle 2,1\rangle}: \frac{\begin{array}{c}\Upsilon^g_1(\varphi_1,\varphi_2), \Upsilon^g_2(\beta,\varphi_2), \Upsilon^g_2(\gamma,\varphi_2),\\ \beta \to \varphi_1, \varphi_1 \to \gamma, \neg\Delta(\gamma \to \beta),\\ \overline{g}(\varphi_1,\varphi_2) \to \psi,\\ \{\Delta((\beta \to \bar{c}) \wedge (\bar{c} \to \gamma)) \to\\ ((\psi \to \overline{g}(\bar{c},\varphi_2)) \vee (\varphi_1 \to \bar{c})) : c \in \mathbb{Q}_*\}\end{array}}{\psi \to \overline{g}(\varphi_1,\varphi_2)}$$

$$\mathsf{C}^{\mathsf{g}}_{\langle 2,2\rangle}: \frac{\begin{array}{c}\Upsilon^g_1(\varphi_1,\varphi_2), \Upsilon^g_2(\varphi_1,\beta), \Upsilon^g_2(\varphi_1,\gamma),\\ \beta \to \varphi_2, \varphi_2 \to \gamma, \neg\Delta(\gamma \to \beta),\\ \psi \to \overline{g}(\varphi_1,\varphi_2),\\ \{\Delta((\beta \to \bar{c}) \wedge (\bar{c} \to \gamma)) \to\\ ((\overline{g}(\varphi_1,\bar{c}) \to \psi) \vee (\bar{c} \to \varphi_2)) : c \in \mathbb{Q}_*\}\end{array}}{\overline{g}(\varphi_1,\varphi_2) \to \psi}$$

□

[12] Observe that if component k of region R_i has only one point, these premises are never met. Thus, the rule associated to that pair of region and component is equivalent to the rule $\bar{0} \vdash \vartheta$, and so, it will not be used in order to "compute" the value of the function in that point (this will be done with a different component or a book-keeping axiom, depending on the case).

6.3 Strong standard completeness

In the previous section, for a given set OP of representable operations, we have introduced a set of congruence, monotonicity and continuity rules for each operation in OP that will used to define next an axiomatic system that will be shown to be strongly complete with respect to $[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_*$.

Definition 6.6 Let $*$ be a left-continuous t-norm and let OP be a set of $*$-representable operations. Then the axiomatic system $\mathsf{L}^\infty_*(\mathsf{OP})$ in the language $L(OP)$ is defined by adding to L^∞_* the following axioms and rules:

- For each $f \in OP$, book-keeping axioms:

$$(\text{Book-}f) \quad \overline{f}(\bar{c}_1,\ldots,\bar{c}_n) \leftrightarrow \overline{f(c_1,\ldots,c_n)},$$

- For each $f \in OP$, the rule $\vee\mathsf{CONG}^{\mathsf{f}}$,
- For each $f \in OP$, each region R_i of its universe and each component k, the rule $\vee\mathsf{M}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$,
- For each $f \in OP$, each region R_i of its universe and each component k, the rule $\vee\mathsf{C}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$,

where, as usual, $\vee\mathsf{R}$ denotes the $\vee$-closure of the rule R.

First, we check that all the new rules, and in particular the ones arising from the the regularity conditions, are sound. Indeed, the only case that could be somewhat not obvious is the family of continuity rules $\vee\mathsf{C}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$, but observe that they hold in the standard algebra with the corresponding operations $[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_*$. For this, letting again $\eta = \eta^f_{\langle i,k\rangle}$ and $\delta = \delta^f_{\langle i,k\rangle}$, it is enough to check that, if

- $\vec{a} \in [0,1]^n$ is such that $\vec{a} \in Int_f(R_i)$, i.e. it validates the condition in line 1 of the premises of $\vee\mathsf{C}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$, and
- $d \in [0,1]$ is such that $d <^{\eta\cdot\delta} f(\vec{a})$, where $<^1$ is $<$ and $<^{-1}$ is $>$, i.e. it validates the condition in line 3 of the premises,

then at least one of the following conditions holds (by definition of representable operation):

- $\vec{a} \in (Q_*)^n$, and so its value is determined by some book-keeping axiom;
- there is no c such that $\langle\vec{a}, c\rangle_k \in Int_f(R_i)$ and $c \neq a_k$ (and so, line 2 of the premises of rule $\vee\mathsf{C}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$ is not satisfied for any $b, t \in [0,1]$);

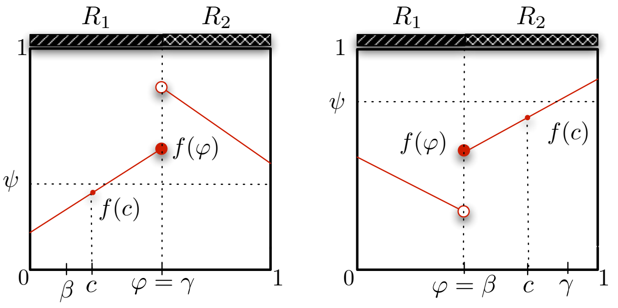

Fig. 1: Examples of the meaning of rule $\mathsf{C}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$.

– for $b, t \in [0,1]$ such that $\langle \vec{a}, b\rangle_k, \langle \vec{a}, t\rangle_k \in Int_f(R_i)$ with $b \leqslant x_k \leqslant t$ and $b < t$ (i.e. satisfying lines 1 and 2 of the premises of $\vee\mathsf{C}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$), there exists $c \in \mathbb{Q}_* \cap [b,t]$ such that $c <^{\delta} a_k$ and $d <^{\eta\cdot\delta} f(\langle \vec{a}, c\rangle_k) <^{\eta\cdot\delta} f(\vec{a})$ (and so, the last two lines from the premises of $\vee\mathsf{C}^{\mathsf{f}}_{\langle \mathsf{i},\mathsf{k}\rangle}$ do not hold).

We refer the reader again to Figure 1 to get a visual representation of the previous observations.

In order to prove standard completeness of $\mathsf{L}^{\infty}_{*}(\mathsf{OP})$, we begin by checking its completeness with respect to the class of linearly-ordered $\mathsf{L}^{\infty}_{*}(\mathsf{OP})$-algebras. For this, we just need to check that it fulfills the requirements of Theorem 3.6.

But this can be easily checked. First, observe that we have added a rule for each new operation in order to get an implicative logic (thus, finitely many congruence rules). Moreover, representable universes are limited to have a countable amount of regions, so there is a countable amount of inference rules concerning the regularity conditions. On the other hand, all the rules are $\vee$-closed operation (by definition) and have a finite number of variables appearing in them, and thus their closure under substitutions are also countable. Therefore, as corollary of Theorem 3.6, we then obtain completeness of $\mathsf{L}^{\infty}_{*}(\mathsf{OP})$ with respect to the linearly-ordered algebras of the class.

Theorem 6.7 *For any set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, the following are equivalent:*

1. $\Gamma \vdash_{\mathsf{L}^{\infty}_{*}(\mathsf{OP})} \varphi$
2. $\Gamma \models_{\mathbf{C}} \varphi$ *for all $\mathsf{L}^{\infty}_{*}(OP)$-chain $\mathbf{C}$.*

Moreover, it is clear that Lemma 4.6 keeps being valid in this more general context, so truth-constants in $\mathsf{L}^{\infty}_{*}(OP)$-chains are dense.

What remains then is to study the relationship of the linearly-ordered $\mathsf{L}^{\infty}_{*}(\mathsf{OP})$-algebras to the standard algebra $[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_{*}$. To show that $\mathsf{L}^{\infty}_{*}(\mathsf{OP})$ enjoys the strong standard completeness we can resort to the same method used for L^{∞}_{*}: to show that any linearly-ordered $\mathsf{L}^{\infty}_{*}(\mathsf{OP})$ is embeddable into $[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_{*}$. The natural approach is to consider again the mapping defined in Lemma 4.7 and prove it is an embedding in the new algebraic setting, with the new operations from OP. To this end, first observe that the required regularity conditions of the operations from OP in $[0,1]$ are properly translated to linearly-ordered $\mathsf{L}^{\infty}_{*}(OP)$-algebras.

Lemma 6.8 *Let OP be a set of $*$-representable operations in $[0,1]$ and let $\mathbf{A}$ be a linearly-ordered $\mathsf{L}^{\infty}_{*}(OP)$-algebra. Let $f \in OP$ be a n-ary operation with representable universe $R = \bigcup_{i\in I} R_i \subseteq [0,1]^n$, and for a given $i \in I$, let $\vec{a} \in A^n$ such that $(\Upsilon^f_i)^{\mathbf{A}}(\vec{a}) = \{\overline{1}^{\mathbf{A}}\}$. Then, for any $1 \leqslant k \leqslant n$, we have*

$$\overline{f}^{\mathbf{A}}(\vec{a}) = Agg_k\{\overline{f}^{\mathbf{A}}(\langle \vec{a}, \overline{c}^{\mathbf{A}}\rangle_k) : c_k \in C_k\},$$

where

$$Agg_k \text{ is } \begin{cases} \sup, & \text{if } \eta = \delta = 1 \\ \inf, & \text{otherwise,} \end{cases}$$

and

$$C_k := \{c \in \mathbb{Q}^{OP}_{*} : (\Upsilon^f_i)^{\mathbf{A}}(\langle \vec{a}, \overline{c}^{\mathbf{A}}\rangle_k) = \{\overline{1}^{\mathbf{A}}\} \text{ and } \overline{c}^{\mathbf{A}} \to^{\delta} a_k = 1\},$$

with the usual notation $\eta = \eta^f_{\langle i,k\rangle}$ and $\delta = \delta^f_{\langle i,k\rangle}$.

Proof The proof is done by considering the particular regularity conditions of the operation f in each region and each component. For the sake of readability, we detail here the case of f being the operation g used in the running example within this section (starting at Example 6.3), and in particular, we consider the first component of its first region, where $\delta^g_{\langle 1,1\rangle} = \eta^g_{\langle 1,1\rangle} = 1$. Other cases can be proven analogously, taking into account that proof is simplified in the case of regions with some one-dimensional component (i.e., such that for some k we have $\{y : \exists \vec{x} \text{ s.t } \langle \vec{x}, y\rangle_k \in R_i\} = \{y_0\}$). In these cases, the result is immediate by applying that, by definition, $y_0 \in \mathbb{Q}_*$ (and so, in C_k).

In the case we are going to prove, the statement of the lemma is translated to

$$\overline{f}^{\mathbf{A}}(a_1, a_2) = \sup\{\overline{f}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, a_2) : c \in [0,b]_{\mathbb{Q}}, \overline{c}^{\mathbf{A}} \leqslant a_1\}.$$

First, observe that for any $d \in A$ such that $d < \sup\{\overline{f}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, a_2) : c \in [0,b]_{\mathbb{Q}}, \overline{c}^{\mathbf{A}} \leqslant a_1\}$ we know there is $c_0 \in [0,b]_{\mathbb{Q}}$ with $\overline{c_0}^{\mathbf{A}} \leqslant a_1$ such that $d < \overline{f}^{\mathbf{A}}(\overline{c_0}^{\mathbf{A}}, a_2)$. From $\overline{c_0}^{\mathbf{A}} \leqslant a_1$ and using of the quasi-equation corresponding to the monotonociy rule $\vee\mathsf{M}^{\mathsf{f}}_{\langle 1,1\rangle}$, we have that $\overline{f}^{\mathbf{A}}(\overline{c_0}^{\mathbf{A}}, a_2) \leqslant \overline{f}^{\mathbf{A}}(a_1, a_2)$. But then $d < \overline{f}^{\mathbf{A}}(a_1, a_2)$, which allows us to derive

$$\sup\{\overline{f}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, a_2) : c \in [0,b]_{\mathbb{Q}}, \overline{c}^{\mathbf{A}} \leqslant a_1\} \leqslant \overline{f}^{\mathbf{A}}(a_1, a_2).$$

On the other hand,[13] suppose towards a contradiction that there is $d \in A$ such that

$$\sup\{\overline{f}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, a_2) : c \in [0,b]_{\mathbb{Q}}, \overline{c}^{\mathbf{A}} \leqslant a_1\} < d < \overline{f}^{\mathbf{A}}(a_1, a_2).$$

Since $d < \overline{f}^{\mathbf{A}}(a_1, a_2)$, we know that the quasi-equation arising from the continuity rule $\vee\mathsf{C}^{\mathsf{f}}_{\langle 1,1\rangle}$ cannot be applied, so some of its premises have to be false. The assignment that maps β to any value $s \in (0, x_1) \cap A$ [14] and γ to $\overline{b}^{\mathbf{A}}$ satisfies all the other premises of the rule, so there must exist $c_0 \in [0,b]_{\mathbb{Q}^{OP}_*}$ with $s \leqslant \overline{c_0}^{\mathbf{A}}$ such that

$$\overline{f}^{\mathbf{A}}(\overline{c_0}^{\mathbf{A}}, a_2) \to^{\mathbf{A}} d < 1 \quad \text{and} \quad a_1 \to^{\mathbf{A}} \overline{c_0}^{\mathbf{A}} < 1.$$

Since $\mathbf{A}$ is linearly-ordered, the previous condition is equivalent to $\overline{f}^{\mathbf{A}}(\overline{c_0}^{\mathbf{A}}, a_2) > d$ and $a_1 > \overline{c_0}^{\mathbf{A}}$.

On the other hand, the condition $\sup\{\overline{f}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, a_2) : c \in [0,b]_{\mathbb{Q}}, \overline{c}^{\mathbf{A}} \leqslant a_1\} < d$ implies there is $e < d \in A$ such that, for all $c \in [0,b]_{\mathbb{Q}}$ such that $\overline{c}^{\mathbf{A}} \leqslant a_1$, $\overline{f}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, a_2) < e$. In particular, this holds for the previous c_0, since $c_0 \in [0,b]_{\mathbb{Q}}$ and $a_1 > \overline{c_0}^{\mathbf{A}}$. Moreover, we know that $\overline{f}^{\mathbf{A}}(\overline{c_0}^{\mathbf{A}}, a_2) > a$, but this leads to have $d < \overline{f}^{\mathbf{A}}(\overline{c_0}^{\mathbf{A}}, a_2) < e < d$, which is a contradiction. This allows us to conclude the proof. □

We can iterate the previous result in order to check the following corollary.

Corollary 6.9 *Let OP be a set of $*$-representable operations in $[0,1]$ and let $\mathbf{A}$ be a linearly-ordered $\mathsf{L}^\infty_*(OP)$-algebra. Let $f \in OP$ be a n-ary operation with representable universe $R = \bigcup_{i\in I} R_i \subseteq [0,1]^n$, $i \in I$ and $\overrightarrow{a} \in A^n$ such that $(\Upsilon^f_i)^{\mathbf{A}}(\overrightarrow{a}) = \{\overline{1}^{\mathbf{A}}\}$. Then for any permutation $\langle k_1, \dots, k_n\rangle$ of $\{1, \dots, n\}$ it holds that $\overline{f}^{\mathbf{A}}(\overrightarrow{a})$ equals*

$$Agg_{k_1}\left\{\dots Agg_{k_n}\{\overline{f}^{\mathbf{A}}(\langle\overline{c_1}^{\mathbf{A}}, \dots, \overline{c_n}^{\mathbf{A}}\rangle) : c_{k_n} \in C_{k_n}\}\dots : c_{k_1} \in C_{k_1}\}\right\}$$

where Agg_i and C_i are defined as in Lemma 6.8.

Proof For a permutation $\langle k_1, \dots, k_n\rangle$ of $\{1, \dots, n\}$, we know that

$$\overline{f}^{\mathbf{A}}(a_1, \dots, a_n) = Agg_{k_1}\{\overline{f}^{\mathbf{A}}(\langle\overrightarrow{a}, \overline{c_{k_1}}^{\mathbf{A}}\rangle_{k_1}) : c_{k_1} \in C_{k_1}\}$$

by Lemma 6.8. Similarly,

$$\overline{f}^{\mathbf{A}}(\langle\overrightarrow{a}, \overline{c_{k_1}}^{\mathbf{A}}\rangle_{k_1}) = Agg_{k_2}\{\overline{f}^{\mathbf{A}}(\langle\langle\overrightarrow{a}, \overline{c_{k_1}}^{\mathbf{A}}\rangle_{k_1}, \overline{c_{k_2}}^{\mathbf{A}}\rangle_{k_2}) : c_{k_2} \in C_{k_2}\}$$

and so for each k_i. Iterating this process from k_1 to k_n, we get that $\overline{f}^{\mathbf{A}}(\langle x_1, \dots, x_n\rangle_k)$ equals

$$Agg_{k_1}\{\dots Agg_{k_n}\{\overline{f}^{\mathbf{A}}(\langle\langle\overrightarrow{a}, \overline{c_{k_1}}^{\mathbf{A}}\rangle_{k_1}, \dots, \ \overline{c_{k_n}}^{\mathbf{A}}\rangle_{k_n}) : c_{k_n} \in C_{k_n}\} \\ \dots : c_{k_1} \in C_{k_1}\}$$

Since $\langle k_1, \dots, k_n\rangle$ is a permutation (and thus, it contains all elements) of $\{1, \dots, n\}$, this concludes the proof. □

Using the previous result we can easily prove that the map ρ defined as in Lemma 4.7 is also an embedding from any linearly-ordered $\mathsf{L}^\infty_*(\mathsf{OP})$-algebra into $[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_*$.

Lemma 6.10 *Let $\mathbf{A}$ be a $\mathsf{L}^\infty_*(OP)$-chain. Then, the function $\rho\colon A \to [0,1]$ given by $\rho(a) = \sup\mathcal{C}^-_a = \inf\mathcal{C}^+_a$ is an embedding from $\mathbf{A}$ into $[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_*$.*

Proof First note that for any constant $\overline{d}$,

$$d = \min\{c \in \mathbb{Q}^{OP}_* : \overline{c}^{\mathbf{A}} \geqslant \overline{d}^{\mathbf{A}}\} = \max\{c \in \mathbb{Q}^{OP}_* : \overline{c}^{\mathbf{A}} \leqslant \overline{d}^{\mathbf{A}}\},$$

and so $\rho(\overline{d}^{\mathbf{A}}) = d = \overline{d}^{[\mathbf{0},\mathbf{1}]^{\mathbf{OP}}_*}$. On the other hand, it is immediate to see that ρ is a mapping strictly order preserving: if $a, b \in A$ are such that $a < b$ then there exists $c \in \mathbb{Q}^{OP}_*$ such that $a < \overline{c}^{\mathbf{A}} < b$ and thus, $\rho(a) < c < \rho(b)$. This shows that ρ is one-to-one.

To prove the homomorphic conditions for the operations, i.e., that for any operation $f \in OP$ it holds $\rho(\overline{f}^{\mathbf{A}}(a_1, \dots, a_n)) = f(\rho a_1, \dots, \rho a_n)$, we make use of the density of the constants in $\mathbf{A}$. Observe that the left-continuous t-norm and its residuum are, by definition, $*$-representable operations, so the proof can be done in general for any $*$-representable operation f.[15]

In order to prove that

$$\rho(\overline{f}^{\mathbf{A}}(a_1, \dots, a_n)) \leqslant f(\rho a_1, \dots, \rho a_n)$$

let $c \in \mathbb{Q}^{OP}_*$ such that $c < \rho(\overline{f}^{\mathbf{A}}(a_1, \dots, a_n))$. By definition and given that ρ preserves the order, $\overline{c}^{\mathbf{A}} \leqslant \overline{f}^{\mathbf{A}}(a_1, \dots, a_n)$. By the previous corollary, it follows that $\overline{c}^{\mathbf{A}} \leqslant Agg_1\{\dots Agg_n\{\overline{f}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \dots, \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}\dots : c_1 \in C_1\}$. Then, $\overline{c}^{\mathbf{A}} \leqslant \overline{f}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \dots, \overline{c_n}^{\mathbf{A}})$ for some $c_i \in C_i$ if $Agg_i = \sup$ and for all $c_i \in C_i$ if $Agg_i = \inf$ (for each $1 \leqslant i \leqslant n$).

[13] As already mentioned before concerning operations with regions with smaller dimensions, at this point of the proof, if f was an operation such that $\{d \in A : (\Upsilon^f_i)^{\mathbf{A}}(\langle\overrightarrow{b}, d\rangle_k \in Int_f(R_i)$ for some $\overrightarrow{b} \in A^n\} = \{\overline{c_0}^{\mathbf{A}}\}$ for some $c_0 \in \mathbb{Q}^{OP}_*$, the proof would finish here. Indeed, c_0 in C_1 by definition, and so trivially $\overline{f}^{\mathbf{A}}(\overline{c_0}^{\mathbf{A}}, a_2) \leqslant \sup\{\overline{f}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, a_2) : c \in [0,b]_{\mathbb{Q}}, \overline{c}^{\mathbf{A}} \leqslant a_1\}$.

[14] This set is non-empty: $x_1 > 0$ because $(\Upsilon^g_1)^{\mathbf{A}}(x_1, x_2) = \{\overline{1}^{\mathbf{A}}\}$ and the constants are dense in A.

[15] Nevertheless, the case of the left-continuous t-norm operation $*$ has a more direct proof, not needing any of the $\vee\mathrm{CONG}^{\mathrm{f}}$, $\vee\mathrm{M}^{\mathrm{fU}}_{\langle\mathrm{i,k}\rangle}$ nor $\vee\mathrm{C}^{\mathrm{fU}}_{\langle\mathrm{i,k}\rangle}$ rules, relying on the MTL-axiomatization of a residuated operation, as we saw in Lemma 4.7.

We can use the book-keeping axioms to get that $c \leqslant f(c_1, \ldots, c_n)$ for c_i as above. It is possible to resort now to the properties of f in $[0,1]$ (monotonicity and left/right continuity), take the corresponding limits (ranging over the C_i's) and conclude that $c \leqslant f(\rho a_1, \ldots, \rho a_n)$. Since this holds for each $c < \rho(\overline{f}^{\mathbf{A}}(x_1, \ldots, x_n))$, it follows that

$$\rho(\overline{f}^{\mathbf{A}}(a_1, \ldots, a_n)) \leqslant f(\rho a_1, \ldots, \rho a_n).$$

To prove that $\rho(\overline{f}^{\mathbf{A}}(a_1, \ldots, a_n)) \geqslant f(\rho a_1, \ldots, \rho a_n)$, let $c \in \mathbb{Q}_*$ be such that $f(\rho a_1, \ldots, \rho a_n) < c$. Then, as before (since $[\mathbf{0},\mathbf{1}]_*^{\mathbf{OP}}$ is linearly-ordered), from the previous corollary we get $Agg_1\{\ldots\{Agg_n\{f(c_1, \ldots, c_n) : c_n \in C_n\}\ldots\} : c_1 \in C_1\} < c$. Then, $f(c_1, \ldots, c_n) < c$ for some $c_i \in C_i$ if $Agg_i = \inf$ and for all $c_i \in C_i$ if $Agg_i = \sup$.

From the book-keeping axioms we have that

$$\overline{f}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \ldots, \overline{c_n}^{\mathbf{A}}) < \overline{c}^{\mathbf{A}}$$

for c_i as above. We can now take suprema and infima again to get $Agg_1\{\ldots Agg_n\{\overline{f}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \ldots, \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}\ldots : c_1 \in C_1\} \leqslant \overline{c}^{\mathbf{A}}$. Again from the previous corollary, it follows that $\overline{f}^{\mathbf{A}}(a_1, \ldots, a_n) \leqslant \overline{c}^{\mathbf{A}}$. Since ρ is order preserving, we finally have $\rho(\overline{f}^{\mathbf{A}}(a_1, \ldots, a_n)) \leqslant \rho(\overline{c}^{\mathbf{A}}) = c$, which concludes the proof. □

Strong standard completeness of $\mathsf{L}_*^\infty(\mathsf{OP})$ then readily follows.

Theorem 6.11 (Strong Standard Completeness of $\mathsf{L}_*^\infty(\mathsf{OP})$) *For any set of formulas* $\Gamma \cup \{\varphi\}$

$$\Gamma \vdash_{\mathsf{L}_*^\infty(\mathsf{OP})} \varphi \quad \textit{iff} \quad \Gamma \models_{[\mathbf{0},\mathbf{1}]_*^{\mathbf{OP}}} \varphi.$$

Proof As usual, one direction is soundness, that is easy to prove. As for the converse implication, suppose that $\Gamma \nvdash_{\mathsf{L}_*^\infty(\mathsf{OP})} \varphi$. Then, by Theorem 6.7 there is a linearly-ordered $\mathsf{L}_*^\infty(\mathsf{OP})$-algebra $\mathbf{A}$ and an $\mathbf{A}$-evaluation h such that $h[\Gamma] \subseteq \{1\}$ and $h(\varphi) < 1$. But the chain $\mathbf{A}$ can be embedded into the standard algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbf{OP}}$ by the embedding ρ from the previous lemma. Then, it is clear that $\rho \circ h$ is a $[\mathbf{0},\mathbf{1}]_*^{\mathbf{OP}}$-evaluation such that $\rho \circ h[\Gamma] \subseteq \{1\}$ and $\rho \circ h(\varphi) < 1$. This ends the proof. □

7 Conclusions and open problems

In this paper we have been concerned with the problem of devising a uniform approach to get a strongly complete axiomatization of the logic induced by an arbitrary standard MTL-algebra $[0,1]_*$. In particular, we have solved this problem when the algebra $[0,1]_*$ is expanded with Monteiro-Baaz Δ operator and with countably many truth-constants. In order to do this, we have first shown the semilinearity of a large family of infinitary axiomatic systems expanding MTL_Δ. We have then adapted Takeuti-Titani density rule of first order intuitionistic logic for our purposes, and have presented a uniform procedure to get, for an arbitrary left-continuous t-norm $*$, a recursively enumerable axiomatic system L_*^∞, with just one infinitary inference rule, that is strongly complete with respect to the standard algebra $[\mathbf{0},\mathbf{1}]_*$. Moreover, we have finally shown how this uniform approach can be extended to get strongly complete axiomatic systems with respect to $[\mathbf{0},\mathbf{1}]_*$ expanded with an at most countable set of operations in $[0,1]$ satisfying a few regularity conditions.

Some interesting problems that remain open are listed below:

- can the cardinality constraint in our Prime Theory Extension Property result be avoided when there are no truth-constants in the language? Notice that the construction provided in Section 5 is based on Gödel logic with rational truth constants.
- are there (uniform) axiomatizations in the language without truth-constants for the logic of an arbitrary $[\mathbf{0},\mathbf{1}]_*$?
- which axiomatic systems given by Cintula in [6] are semilinear? From his results, it would follow they would be Pavelka-complete.
- is it possible to axiomatize the first-order version of the logic of an arbitrary $[\mathbf{0},\mathbf{1}]_*$? Related to this question, does the embedding given in Lemma 4.6 preserve arbitrary suprema and infima?

Dedication This paper is dedicated to the memory of Professor Franco Montagna, an excellent researcher and better person to whom these authors will always be indebted to. He has been a pioneer in many research areas (not only in MFL), and among such pioneering contributions we can find his work on infinitary axiomatic systems for standard BL-algebras. The present paper has been inspired by the novel ideas introduced by Franco on that topic.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

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