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Axiomatising a fuzzy modal logic over the standard product algebra.
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Modal extensions of main systems of mathematical fuzzy logic are a family of logics that are still under research. Several papers have been published on this topic treating different aspects, see for instance [9] for modal extension of Łukasiewicz logic, [3, 4, 2] for modal extensions of Gödel fuzzy logic or [1] for modal logics over finite residuated lattices. However, the study of modal extensions over the product fuzzy logic Π, with semantics based on Kripke structures where both worlds and accessibility relations are evaluated over the standard product algebra, has remained open. We present here some results that partially fill this gap for the case of Kripke semantics with crisp accessibility relations and when the underlying product fuzzy logic is expanded with truth-constants, Δ operator and with two infinitary inference rules. We also explore the algebraic semantics for this modal logic.

§1. Enforcing propositional strong completeness of product logic. Propositional product logic Π is known to be finitely strong complete but not strongly complete with respect to the standard product chain $[0,1]_\Pi$, i.e. the product algebra over the real unit interval with the usual product of reals as monoidal operation, see [8]. In [11], Montagna defined an expansion of the BL logic with a storage operator $*$ and an infinitary rule

$$(R_M)\ \frac{\chi \vee (\varphi \to \psi^k) \text{ for all } k \in \omega}{\chi \vee (\varphi \to \psi^*)},$$

where, as usual ψ^k denotes $\psi \& \overset{k}{\ldots} \& \psi$. This expansion was proved to be strongly complete (for infinite theories) with respect to the corresponding class of expanded standard BL-chains. In particular, for Product logic the $*$ operator coincides with the Monteiro-Baaz operator Δ in $[0,1]$.

On the other hand, in [12] the addition of rational truth constants to product logic was studied, and it was proven that the extension of product logic with the Δ and natural axioms for the constants was finitely strong complete with respect to the *canonical* standard product algebra $[0,1]_{\Pi^c_\Delta}$ (where the rational constants are interpreted by its name). Moreover, in the frame of rational Pavelka-like logics, Cintula in [7] had already proven that the addition of two infinitary inference rules made this logic to be strongly complete.

Let Π^c_Δ be the infinitary logic defined by the following axioms and rules:

- Axioms of Π (propositional product logic) (see for instance [8]);
- Axioms referring to rational constants over product logic [12];
- Axioms of the Δ operator ([8]) plus $\neg\Delta\overline{c}$ for each $c \in (0,1)_\mathbb{Q}$;
- Rules of Modus Ponens and Necessitation for Δ: $\varphi \vdash \Delta\varphi$;
- The infinitary rules

$$(\mathbf{R_1})\ \frac{\overline{c} \to \varphi, \text{ for all } c \in (0,1)_\mathbb{Q}}{\varphi} \qquad (\mathbf{R_2})\ \frac{\varphi \to \overline{c}, \text{ for all } c \in (0,1)_\mathbb{Q}}{\neg\varphi};$$

It is clear that Π^c_Δ is algebraizable and that its algebraic semantics is given by the class $\mathcal{P}^c_\Delta$ of algebras $\mathbf{A} = \langle A, \odot, \to, \Delta, \{c^\mathbf{A}\}_{c\in[0,1]_\mathbb{Q}}\rangle$ where

- $\langle A, \odot, \to, \Delta, \mathbf{0}^\mathbf{A}\rangle$ is a Π_Δ-algebra.
- The rational constants $\{c^\mathbf{A}\}_{c\in[0,1]_\mathbb{Q}}$ form a subalgebra isomorphic to $[0,1]_\mathbb{Q}$ (as Π_Δ-algebras) such that for each $c, d \in (0,1)_\mathbb{Q}$ the following equations and generalised quasi-equations hold:

$$d^{\mathbf{A}} \odot c^{\mathbf{A}} = (d \cdot c)^{\mathbf{A}}, \qquad d^{\mathbf{A}} \to c^{\mathbf{A}} = \min\{1, (c/d)^{\mathbf{A}}\}, \qquad \Delta c^{\mathbf{A}} = \mathbf{0};$$
$$\text{If } x \geq c^{\mathbf{A}} \text{ for all } c \in (0,1)_{\mathbb{Q}} \text{ then } x = 1,$$
$$\text{If } x \leq c^{\mathbf{A}} \text{ for all } c \in (0,1)_{\mathbb{Q}} \text{ then } x = \mathbf{0}.$$

Due to the above two generalised quasi-equations, [11, Lemma 10] yields that any $\mathcal{P}^c_\Delta$-chain is archimedean. Now, following similar arguments from [11], one can prove that any consistent set of formulas can be extended to a complete theory over Π^c_Δ (closed under R_1 and R_2). It is then routine to show that the Lindenbaum sentence algebra of this complete theory is a $\mathcal{P}^c_\Delta$-chain, and hence archimedean. Finally, using results about product algebras from [6], one can also prove that for any countable archimedean chain from $\mathcal{P}^c_\Delta$ there is a complete embedding (i.e. preserving sups and infs) of that chain into the canonical standard product algebra $[0,1]^c_\Delta$. This gives the following completeness results.

Theorem 1 (Strong Completeness of Π^c_Δ). *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then the following conditions are equivalent:*

- $\Gamma \vdash_{\Pi^c_\Delta} \varphi$;
- $\Gamma \models_{\mathcal{P}^c_\Delta} \varphi$;
- $\Gamma \models_{C\mathcal{P}^c_\Delta} \varphi$, *where $C\mathcal{P}^c_\Delta$ is the class the linearly ordered algebras in $\mathcal{P}^c_\Delta$;*
- $\Gamma \models_{[0,1]_{\Pi^c_\Delta}} \varphi$.

§2. Expanding product fuzzy logic with $\Box$ and $\Diamond$. In this section we expand the logic Π^c_Δ with the two usual modalities $\Box$ and $\Diamond$, we define a Kripke semantics for them and show an adequate complete axiomatization.

We start with the semantics. The notion of Kripke frame is as usual: a frame is a pair $\mathfrak{F} = \langle W, R\rangle$ with $W \neq \emptyset$ and $R \subseteq W \times W$. Given a product algebra $\mathbf{A} \in \mathcal{P}^c_\Delta$, an **A-Kripke model** $\mathsf{M} = (W, R, e)$ is just a Kripke frame $\langle W, R\rangle$ endowed with an evaluation of variables in $\mathbf{A}$ for each world $e\colon W \times \mathcal{V} \to A$ This evaluation is extended to non-modal formulas by its corresponding operations in $\mathbf{A}$, i.e. fulfilling $e(w, \varphi\&\psi) = e(w,\varphi) \odot e(w,\psi)$, $e(w, \varphi \to \psi) = e(w,\varphi) \to e(w,\psi)$, $e(w, \Delta\varphi) = \Delta(e(w,\varphi))$ and $e(w, \overline{c}) = c^{\mathbf{A}}$ and to modal formulas by:

$$e(w, \Box\varphi) := \inf\{e(v,\varphi) : \ Rwv = 1\}; \qquad e(w, \Diamond\varphi) := \sup\{e(v,\varphi) : \ Rwv = 1\};$$

A model $\mathsf{M} = (W, R, e)$ where these two values are defined for each $w \in W$ will be called *safe*, and we will denote the class of safe models by PK. For $\mathsf{M} = (W, R, e) \in \mathsf{PK}$ and $w \in W$ we write $\mathsf{M} \models_w \varphi$ whenever $e(w,\varphi) = 1$, and $\mathsf{M} \models \varphi$ whenever $\mathsf{M} \models_w \varphi$ for all $w \in W$.

Then, as usual in modal logics, two notions of logical consequence can be defined, a local and a global one. They are respectively defined as follows:

- $\Gamma \models^l_{\mathsf{PK}} \varphi$ if for any $\mathsf{M} = (W, R, e) \in \mathsf{PK}$ and any $w \in W$, if $\mathsf{M} \models_w \Gamma$ then $\mathsf{M} \models_w \varphi$;
- $\Gamma \models^g_{\mathsf{PK}} \varphi$ if for any $\mathsf{M} \in \mathsf{PK}$, if $\mathsf{M} \models \Gamma$ then $\mathsf{M} \models \varphi$.

A proposed axiomatization for the local consequence $\models^l_{\mathsf{PK}}$ is the following. Let $\mathbf{K}_\Pi$ be the logic defined by the following axioms and rules:

Π^c_Δ: Axioms and rules from Π^c_Δ
($\mathbf{K}$): $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
($\mathbf{A}_\Box 1$): $(\overline{c} \to \Box\varphi) \leftrightarrow \Box(\overline{c} \to \varphi)$
($\mathbf{A}_\Box 2$): $\Delta\Box\varphi \leftrightarrow \Box\Delta\varphi$
($\mathbf{A}_\Diamond 1$): $\Box(\varphi \to \overline{c}) \leftrightarrow (\Diamond\varphi \to \overline{c})$
($\mathbf{N}_\Box$): if φ is a theorem, then $\Box\varphi$ is a theorem as well.

The corresponding axiomatization for the global consequence $\models^g_{\mathsf{PK}}$ will be as above just replacing the necessitation rule ($\mathbf{N}_\Box$) by the more general rule

$(\mathbf{N}_\Box^G)$: from φ derive $\Box\varphi$

We will denote this latter logic by $\mathbf{K}_\Pi^g$. There are two interesting observations about the modal logic $\mathbf{K}_\Pi$. First, it holds that, for an arbitrary theory Γ and any formula φ, $\Gamma \vdash_{\mathbf{K}_\Pi} \varphi$ implies that $\Box\Gamma \vdash_{\mathbf{K}_\Pi} \Box\varphi$, where $\Box\Gamma = \{\Box\psi : \psi \in \Gamma\}$. Second, since the necessitation rule $(\mathbf{N}_\Box)$ only affects theorems, it also holds

$$\Gamma \vdash_{\mathbf{K}_\Pi} \varphi \text{ iff } \Gamma \cup Th_{\mathbf{K}_\Pi} \vdash_{\Pi_\Delta^c} \varphi,$$

where $Th_{\mathbf{K}_\Pi}$ stands for the set of theorems of $\mathbf{K}_\Pi$, and where in the right-hand deduction formulas starting by a modal symbol are understood as new propositional variables.

Then, a natural procedure to check that the logic $\mathbf{K}_\Pi$ indeed axiomatizes the local consequence $\models_{\mathsf{PK}}$ is through the usual canonical model construction. In what follows we denote by $Fm^\star$ the algebra of propositional formulas built from the extended set of variables $\mathcal{V}^\star = \mathcal{V} \cup \{(\Box\varphi)^*, (\Diamond\varphi)^* \mid \varphi \text{ is a modal formula}\}$, that is, we introduce a new propositional variable for each formula starting with a modal operator.

Definition 2. The **canonical model** is the $[0,1]_{\Pi_\Delta^c}$-model $\mathbf{M}_c = (W_c, R_c, e)$ where:
– $W_c := \{w \in Hom(Fm^\star, [0,1]_{\Pi_\Delta^c}) : \ w([Th_{\mathbf{K}_\Pi}]) \subseteq \{1\}\}$;
– $R_c := \{(w,v) \in W_c \times W_c : \text{for any } \varphi \in Fm^\star, \text{if } w((\Box\varphi)^*)) = 1 \text{ then } v(\varphi) = 1\}$;
– $e : W \times \mathcal{V}^* \to [0,1]$ such that $e(w,x) := w(x)$ for all $x \in \mathcal{V}^*$.

Next step is to check that the so-called *Truth Lemma* holds true, i.e. for any φ we have $e(w,\Box\varphi) = w((\Box\varphi)^*)$ and $e(w,\Diamond\varphi) = w((\Diamond\varphi)^*)$. This directly gives the following completeness theorem.

Theorem 3 (Kripke Completeness). *For any set of modal formulas* $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathbf{K}_\Pi} \varphi \textit{ iff } \Gamma \models_{\mathsf{PK}^l} \varphi.$$

§3. Algebraic semantics. In this section we study the algebraic semantics of the modal systems $\mathbf{K}_\Pi$ and $\mathbf{K}_\Pi^g$. We begin by classifying these logics in the Leibniz hierarchy of Abstract Algebraic Logic. It turns out that $\mathbf{K}_\Pi^g$ is algebraizable and that $\mathbf{K}_\Pi$ is not (even if it is still equivalential). Nevertheless, it turns out that the classes of algebras associated with these two logics coincide, and are given by the generalized quasi-variety $\mathcal{MP}_\Delta^c$ of modal product algebras $\mathbf{A} = \langle A, \odot, \to, \Delta, \Box, \Diamond, \{c\}_{c\in[0,1]_\mathbb{Q}}\rangle$ where

- $\langle A, \odot, \to, \Delta, \{c\}_{c\in[0,1]_\mathbb{Q}}\rangle \in \mathcal{P}_\Delta^c$;
- For every $x, y \in A$, $\Box(x \to y) \leq \Box x \to \Box y$;
- For every $x \in A$, $c \in [0,1]_\mathbb{Q}$ $\Box(x \to c^\mathbf{A}) = \Diamond x \to c^\mathbf{A}$ and $\Box(c^\mathbf{A} \to x) = c^\mathbf{A} \to \Box x$;
- For every $x \in A$, $\Box\Delta x = \Delta\Box x$;
- $\Box 1 = 1$.

One can check that the reduced filters of the global modal logic coincide with just $\{1\}$, and thus we obtain the following **completeness** result for any set of modal formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathbf{K}_\Pi^g} \varphi \text{ iff } \Gamma \models_{\mathcal{MP}_\Delta^c} \varphi.$$

However, the study of the local modal logic is not so neat. It is a general fact that any logic is strongly complete with respect to its class of reduced models, but for non-algebraizable logics these do not need to form a well-behaved class. Nevertheless, gaining inspiration from [10], we can provide a nice characterization of the reduced models of $\mathbf{K}_\Pi$ and thus a more concrete algebraic completeness result.

More precisely, it can be proven that the deductive filters of $\mathbf{K}_\Pi$ over a modal product algebra $\mathbf{A}$, in symbols $\mathcal{F}i_{\mathbf{K}_\Pi}\mathbf{A}$, coincide with those of the non-modal logic Π_Δ^c over the non-modal reduct of $\mathbf{A}$. Then, the reduced filters can be characterized using the concept of *open* filter of $\mathbf{A}$, i.e., the ones closed under the $\Box$ operator.

THEOREM 4. *$\langle \mathbf{A}, F\rangle$ is a reduced model of $\mathbf{K}_\Pi$ if and only if $\mathbf{A} \in \mathcal{MP}^c_\Delta$, $F \in \mathcal{F}i_{\mathbf{K}_\Pi}\mathbf{A}$ and $\{1\}$ is the maximum open filter in $\mathcal{F}i_{\mathbf{K}_\Pi}\mathbf{A}$ such that $\{1\} \subseteq F$.*

As we developed two semantics for our modal logics, namely the Kripke and the algebraic ones, it is natural to study their relationship. We describe a way of translating the Kripke semantics into the algebraic one by associating a modal product algebra to each safe $\mathbf{A}$-Kripke model (see for instance [5] for the classical case). More precisely, let $\mathbf{A} \in \mathcal{P}^c_\Delta$ and a safe $\mathbf{A}$-Kripke model $\mathbf{M}$. We say that $\mathbf{M}^+ = \langle \mathbf{A}^W, \odot, \to, \Delta, \Box, \Diamond, \{c\}_{c\in[0,1]_\mathbb{Q}}\rangle$ is the *dual algebra of* $\mathbf{M}$, where

$$f \odot g := [v \mapsto f(v) \odot g(v)]; \quad \Box f := [v \mapsto \inf\{f(w) : Rvw\}]; \quad c^{\mathbf{M}^+} := [v \mapsto c^{\mathbf{A}}].$$
$$f \to g := [v \mapsto f(v) \to g(v)]; \quad \Diamond f := [v \mapsto \sup\{f(w) : Rvw\}];$$

The *dual evaluation* $e^+ : Fm \to \mathbf{M}^+$ is given by $e^+(\varphi) = [v \mapsto e(v, \varphi)]$. It turns out that $\mathbf{M}^+ \in \mathcal{MP}^c_\Delta$, and applying this translation to the Canonical Model it is possible to obtain a second completeness result of $\mathbf{K}_\Pi$ with respect to $\mathcal{MP}^c_\Delta$.

THEOREM 5 (Algebraic completeness). *For any set of modal formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{K}_\Pi} \varphi \textit{ iff } \Delta\Gamma \models^{\leq}_{\mathcal{MP}^c_\Delta} \varphi,$$

where $\Theta \models^{\leq}_{\mathcal{MP}^c_\Delta} \chi$ means that for any $\mathbf{A} \in \mathcal{MP}^c_\Delta$, h homomorphism from the algebra of modal formulas into $\mathbf{A}$ and $a \in \mathbf{A}$, if $a \leq h(\theta)$ for all $\theta \in \Theta$, then $a \leq h(\chi)$.

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