Image-finite first-order structures

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One of the well known difficulties (see [4, p. 236] or [1]) when dealing with first-order many-valued logics is the requirement to consider *safe* structures, i.e., those ones which have the necessary infima and suprema for computing the values of all first-order formulas.

Besides trivial cases like witnessed structures (which include finite ones) [2], in general it is quite difficult to show that a particular structure is safe. Another quite simple case, and apparently not previously considered in the literature, of safe structures is the one provided by what we call here image-finite. By definition, an structure is *image-finite* when, for each one of the predicate symbols in the vocabulary, its interpretation only takes a finite number of values in the many-valued chain considered. As previously pointed, it is not difficult to prove that all image-finite structures are safe (indeed witnessed), and another straightforward result is the following one.

Lemma. Let us assume that K_1 and K_2 are two classes of MTL-chains generating the same variety. Then, image-finite structures over chains in K_1 and image-finite structures over chains in K_2 share the same 1-valid sentences.

The previous result can be used to prove the statements below.

Theorem (Łukasiewicz Standard Semantics). The following families of first-order structures have the same set of 1-valid sentences.

- image-finite structures over the class of all MV-chains.
- (image-finite) first-order structures over the standard MV-chain.
- (image-finite) first-order structures over the rational standard MV-chain.
- (image-finite) first-order structures over some subalgebra of the standard MV-chain generated by one irrational.

Theorem (Monadic vocabulary with just one variable). Let us assume that the vocabulary only consists on unary predicate symbols, and let us just consider formulas using only one variable. Then, the family of structures given in the previous theorem and sharing the very set of 1-valid sentences can be enlarged with:

- (finite) first-order structures over the class of all MV-chains.
- finite first-order structures over the standard MV-chain.

The statement in this last result was already obtained by a different (and very messy) method by Rutledge in his PhD dissertation [5, Chapter IV] (see also [3]), but up to now there were no alternative proofs available in the literature.

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References

[1] P. Hájek. *Metamathematics of fuzzy logic*, volume 4 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 1998.

[2] P. Hájek. On witnessed models in fuzzy logic. *Mathematical Logic Quarterly*, 53(1):66–77, 2007.

[3] P. Hájek. On fuzzy modal logics $S5(\mathcal{C})$. *Fuzzy Sets and Systems*, 161(18):2389–2396, 2010.

[4] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*. PWN—Polish Scientific Publishers, Warsaw, third edition, 1970. Monografie Matematyczne, Tom 41.

[5] J. D. Rutledge. *A preliminary investigation of the infinitely-many-valued predicate calculus*. Ph. D. Dissertation, Cornell University, 1959.